CHAPTER II
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COMPLEMENTARY CONNECTED DOMINATION NUMBER

Introduction

Kulli and Janakiram have defined the parameter called “Non-split domination number” [16]. They obtained many bounds and some relations with other domination parameters. But we independently defined this concept as Complementary Connected Domination Number. Even though the two definitions viz., non-split domination number and complementary connected domination number are one and the same, the results obtained in this chapter are not found in [16]. More specially we obtain new bounds and relations with other domination related parameters. The contents have been published as a paper in the “International Journal of Management and Systems, Vol.18 No.2. p. 147-154 (2002)” (See Appendix).
§ 2.1 BASIC OBSERVATIONS

Definition 2.1.1 A subset $S$ of $V$ of a non-trivial graph $G$ is said to be a complementary connected domination set, if $S$ is a dominating set and $G[V-S]$ is connected. The minimum cardinality of such $S$ is called complementary connected domination number and is denoted by $\gamma_{cc}(G)$.

Remark 2.1.2 The following results can be easily obtained from the definitions of the respective parameters.

(i) If $m \geq 1$, then $\gamma_{cc}(G)$ exists.

(ii) If $G = P_n$ ($n \geq 3$), then $\gamma_{cc}(G) = n - 2 = \gamma_c(G)$. Also for $G = P_2$, $\gamma_{cc}(G) = 1$.

(iii) If $G = C_n$, then $\gamma_{cc}(G) = n - 2 = \gamma_c(G)$.

(iv) If $G$ is the corona $H \circ K_1$ for some connected graph $H$, then $\gamma_{cc}(G) = n / 2 = \gamma_c(G)$.

(v) If $G$ is a star graph, then $\gamma_{cc}(G) = n - 1$.

(vi) For any graph $G$, $\gamma(G) \leq \gamma_{cc}(G)$.

(vii) If $P_k$ ($k > 3$) is a subgraph of the graph $G$, then $\gamma_{cc}(G) \leq n - 2$.

Remark 2.1.3 Let $G_1, G_2, ..., G_m$ be the components of $G$. 
Then \( \gamma_{cc}(G) = \min \left\{ \sum_{1 \leq i \leq m} |V(G_k)| + \gamma_{cc}(G_i) \right\} \).

Hereafter in this chapter, by a graph \( G \), we mean a simple connected graph with the number of vertices \( n > 1 \).

**Theorem 2.1.4** Let \( G \) be a graph with \( n > 2 \). Then there exists a \( \gamma_{cc}(G) \)-set \( S \) which contains all the pendant vertices of \( G \).

**Proof** Let \( S \) be any \( \gamma_{cc}(G) \)-set. Let \( u \) be a pendant vertex not in \( S \). Then \( u \) is adjacent to a non–pendant vertex \( v \in S \). If \( |V - S| = 1 \), then \( \gamma_{cc}(G) = n - 1 \). Now \( S_1 = S \cup \{u\} - \{v\} \) is clearly a \( \gamma_{cc}(G) \)-set with all pendant vertices in \( G \). So let us assume that \( |V - S| > 1 \). Note that \( v \) is the only vertex adjacent to \( u \) and so \( u \) is isolated in \( G[V - S] \), it is a contradiction to \( S \) is a \( \gamma_{cc}(G) \)-set. Hence \( S \) contains all the pendant vertices of \( G \).

\( \square \)

**Remark 2.1.5** It is clear that any \( \gamma_{cc}(G) \)-set \( S \) of \( G \) with \( |S| \leq n - 2 \) contains all pendant vertices of \( G \).

**Lemma 2.1.6** Let \( v \) be a cut vertex of a graph \( G \) and \( S \) be a \( \gamma_{cc}(G) \)-set of \( G \). If \( v \in S \), then all vertices except one component of \( G - v \) belong to \( S \).
**Proof** Suppose there exist vertices \( u \) and \( w \) in two different components of \( G - v \) and not in \( S \). Since \( v \) is on every \( u - w \) path of \( G \) and \( v \in S \), the vertices \( u \) and \( w \) are not connected in \( G[V-S] \), a contradiction.

\[ \square \]

**Theorem 2.1.7** Let \( T \) be a tree. Then \( \gamma_{cc}(T) = n/2 \) if and only if \( T \) is of the form \( HoK_1 \) for some tree \( H \).

**Proof** If \( T = HoK_1 \), then \( T \) has even number of vertices and \( \gamma_{cc}(G) = n/2 \).

Conversely, suppose \( n = 2 \), then the result is trivial. Assume that \( G \) is a graph with \( \gamma_{cc}(G) = n/2 \) and \( n > 2 \). Let \( S \) be a \( \gamma_{cc}(G) \)-set containing all the pendant vertices (Theorem 2.1.4). Any vertex in \( S \) cannot dominate two or more vertices in \( V - S \). Otherwise \( T \) contains a cycle. Hence a vertex \( u \in S \) cannot have more than one neighbor in \( V - S \). Also each vertex in \( V - S \) has at least one neighbor in \( S \). If \( k \) (\( k > 0 \)) vertices in \( S \) have less than \( k \) neighbors in \( V - S \), then since \( \gamma_{cc}(G) = n/2 \), there exists \( p \) (\( p > 1 \)) vertices in \( V - S \) adjacent to less than \( p \) vertices in \( S \). This is impossible since \( V - S \) is connected. Therefore, each vertex in \( V - S \) is adjacent to only one vertex in \( S \). Since \( |S| = n/2 \), it is enough to prove that \( S \) contains only pendant vertices of \( T \).

Now supposing \( S \) contains a non-pendant vertex \( u \), then at least one vertex \( u_1 \neq u \) not adjacent to the vertices in \( V - S \), by Lemma 2.1.6. We
see that, for the two vertices $u_1$ and $u$, there will be exactly one neighbor in $V - S$, which is impossible by the above argument.

**Remark 2.1.8** There do exist graphs $G$ (figures given below) for which $\gamma_{cc}(G) = n / 2$. Further, we could not obtain a necessary and sufficient condition for $\gamma_{cc}(G) = n / 2$.

(i) 

(ii) 

(iii) 

(iv) 

(v)
**Theorem 2.1.9** For a tree $T$, $\gamma_{cc}(T) \geq \text{diam} (T) - 1$.

**Proof** Let $u$ and $v$ be the two vertices in $T$ such that $\text{dist}(u, v) = \text{diam}(T)$. Clearly $u$ and $v$ are pendant vertices. By Theorem 2.1.4, one can find a $\gamma_{cc}(G)$-set $S$, with $v$ and $v$ in $S$. Let $u_1$ be the first vertex, which is not in $S$ from $u$ and $v_1$ be the first vertex, which is not in $S$ from $v$ along the path
of the diameter. If \( u_1 \) and \( v_1 \) are adjacent, then \( \gamma_{cc}(T) \leq \text{diam}(T) - 1 \).

Suppose \( u_1 \) and \( v_1 \) are not adjacent, then by the definition of the \( \gamma_{cc}(G) \)-set, \( u_1 \) and \( v_1 \) are connected in \( V - S \). Hence each vertex in the \( u_1-v_1 \) path is adjacent to at least one vertex in \( S \). Therefore \( \gamma_{cc}(T) \leq \text{diam}(T) - 1 \). \( \square \)

**Remark 2.1.10** If \( T = P_n \), then \( \gamma_{cc}(T) = \text{diam}(T) - 1 \).

**Theorem 2.1.11** For any graph \( G \), \( \gamma_{cc}(G) \leq n - \delta \), where \( \delta \) is the minimum degree of the graph \( G \).

**Proof** If the minimum degree \( \delta = 1 \), the result is true. Let \( v \) be a vertex in \( V(G) \) such that \( \text{deg}(v) = \delta > 1 \). Then \( G \) must have a cycle and so \( \gamma_{cc}(G) \leq n - 2 \). If \( G[N(v)] \) is connected, then \( \gamma_{cc}(G) \leq |N(v)| = n - \delta \). On the other hand, let \( u_1, u_2, \ldots, u_k \) \((k < \delta - 1)\) be the vertices in \( N(v) \) and not adjacent to a vertex \( u \in N(v) \). Since each \( u_i \) is of degree at least \( \delta \), each \( u_i \) is adjacent to a vertex not in \( N[v] \). If we choose \( S_1 = V - N[v] \cup \{u\} \), then it dominates the vertices of \( G \). Since \( G[V-S_1] \) is connected, \( \gamma_{cc}(G) \leq |S_1| = n - \delta \). \( \square \)

**Theorem 2.1.12** Let \( G \) be a graph. Then \( \gamma_{cc}(G) = n - 1 \) if and only if \( G \) is a star.
Proof If $G$ is a star, then $\gamma_{cc}(G) = n - 1$. Conversely, assume that $\gamma_{cc}(G) = n - 1$. Suppose $G$ contains a cycle, then $\gamma_{cc}(G) \leq n - 2$. Therefore $G$ must be a tree. If $n = 2$, then the result is obvious. If $G$ is not a star and $n \geq 3$, then $\text{diam}(G) \geq 3$. In this case $P_4$ is a sub graph of $G$ and hence $\gamma_{cc}(G) \leq n - 2$, a contradiction to $\gamma_{cc}(G) = n - 1$. \qed

Remark 2.1.13 In view of the above Theorem 2.1.12, if $G$ is not a star, then $\gamma_{cc}(G) \leq n - 2$ ($n \geq 3$). However the following are a few graphs for which $\gamma_{cc}(G) = n - 2$.

(i) 

(ii) $P_n$      (iii) $C_n$      (iv) $K_2 \circ nK_1$

Figure 2.2
§ 2.2.0 RELATIONS WITH OTHER PARAMETERS

In this section, we obtain results connecting Complementary Connected Domination Number and other domination parameters.

Theorem 2.2.1 Let S be any $\gamma_{cc}(G)$–set of the graph G. If $\gamma(G) = \gamma_{cc}(G)$, then S cannot have cut vertices.

Proof Suppose $|S| = n - 1$. Then G must be a star. Since $\gamma(G) = 1$ for a star, $n = 2$. That is $G = K_2$ and hence the theorem is obvious. Assume that $|S| \leq n - 2$. Then S contains all the pendant vertices (Remark 2.1.5).

Now it is enough to prove that, if S is a $\gamma_{cc}(G)$–set having a cut vertex v, then $\gamma(G) \neq \gamma_{cc}(G)$. Since $v \in S$, one of the components of G – {v} must be in S (Lemma 2.1.6). Clearly v is of degree greater than one.

Case 1: Suppose G – {v} has isolated vertices, then all these isolated vertices would be in S. The vertex v is enough to dominate the isolated vertices and hence $\gamma(G) < \gamma_{cc}(G)$.

Case 2: Suppose G – {v} has no isolated vertices. Note that each vertex in one of the components of G – {v} is in S. Now, for selection of the $\gamma$–set, one need to take all vertices of the connected component of G – {v} and so $\gamma(G) < \gamma_{cc}(G)$.
Remark 2.2.2 Any $\gamma_{cc}$-set of $C_5$ does not have cut vertices. However $\gamma_{cc}(C_5) = 3$, $\gamma(C_5) = 2$. Hence the converse of the theorem is not true.

Proposition 2.2.3 If $G$ is a tree, then $\gamma_{cc} \geq \beta_o$.

Proof Let $S$ be a $\gamma_{cc}$-set containing all the pendant vertices of $G$ (Theorem 2.1.4). Suppose $S$ has no non-pendant vertices, then $S$ is independent and so $\gamma_{cc} = \beta_o$. Here we see that $G = G[N[V - S]]$. On the other hand, assume that $S$ contains a non-pendant vertex $u$, that is $G \neq G[N[V - S]]$. Note that all the vertices in $G - N[V - S]$ must be in $S$ where as all these need not be in a $\beta_o$-set of $G$. Suppose $\beta_o > \gamma_{cc}$, then $\beta_o$ of $N[V - S]$ must be greater than $\gamma_{cc}$ of $G[N[V - S]]$ which is impossible. Thus $\gamma_{cc} \geq \beta_o$.

Corollary 2.2.4 For any tree $T$, $\gamma_{cc}(T) \geq n / 2$.

Proof Since $T$ is a tree, $\beta_o(T) \geq n / 2$ [20], by the Proposition 2.2.3, $\gamma_{cc}(T) \geq n / 2$.
Corollary 2.2.5  For any tree $T$, $\gamma_{cc}(T) = \gamma(T)$ if and only if $T = HoK_1$ for some tree $H$.

Proof  We have $\gamma(T) \leq n / 2$ (Theorem 1.24 [20]) and $\gamma_{cc}(T) \geq n / 2$ (Corollary 2.2.4). Therefore $\gamma_{cc}(T) = \gamma(T)$ if and only if $\gamma(T) = \gamma_{cc}(T) = n / 2$. That is if and only if $T = HoK_1$ (Theorem 1.29, [10] and Theorem 2.1.7).

Proposition 2.2.6  Let $G$ be a graph. If $\gamma_{cc}(G)$–set $S$ is independent, then $V - S$ is a dominating set and $\gamma + \gamma_{cc} \leq n$.

Proof  Since $S$ is independent, every vertex in $S$ is adjacent to at least one vertex and so $V - S$ and so $V - S$ is a dominating set. Hence $\gamma \leq |V - S|$. This gives that $n = |S| + |V - S| \geq \gamma_{cc} + \gamma$.

Theorem 2.2.7  Let $T$ be a tree with $n \geq 3$ and $S$ be a $\gamma_{cc}(T)$–set. Then $V - S$ is a dominating set if and only if every vertex $v \in T$ of degree greater than one is a support.

Proof  Assume that every vertex $v \in T$ with $\deg(v) \geq 2$ is a support. The set of all pendant vertices will form a $\gamma_{cc}$–set and therefore $V - S$ is a dominating set.
Conversely, assume that $V - S$ is a dominating set. If $|S| = n - 1$, then $T$ is a star and so nothing to prove. Otherwise $|S| < n - 1$, then $S$ contains all the pendant vertices (Remark 2.1.5). If $S$ contains only pendant vertices, then each vertex of degree greater than one is a support. On the other hand $S$ contains a non-pendant vertex. Let $v$ be a vertex, which is not a support.

**Case 1:** If $v \in S$, then there exists a pendant vertex $w \in S$ such that entire $v - w$ path is in $S$. Note that $w$ is not covered by $V - S$, a contradiction to $V - S$ is a dominating set.

**Case 2:** If $v \not\in S$, then there exists a $x \in S$ such that $v$ is dominated by $x$ in $S$. By the assumption $x$ is not a pendant vertex. Hence there exists a pendant vertex $w$ in $S$ such that $x - w$ path is fully in $S$. Also $w$ is not covered by $V - S$, a contradiction to the assumption.

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**Proposition 2.2.8** Let $S$ be a $\gamma_{cc}$-set of a graph $G$ such that $V - S$ is a dominating set. Then $S$ has no cut vertices.

**Proof** Suppose $S$ has a cut vertex say $x$, then by Lemma 2.1.6, all the vertices of one of the components of $G - \{x\}$ are in $S$. Hence $V - S$ cannot dominate all the vertices of $S$. 

\[\]
Remark 2.2.9 The converse of the Theorem 2.2.8 need not be true. For example, $C_5$ has no cut vertices. We see that any $\gamma_{cc}(G)$-set $S$ of $C_5$ contains 3 adjacent vertices. Note that $S$ contains no cut vertices where as $V - S$ is not a dominating set.

Proposition 2.2.10 If cut $T$ is the number of cut vertices of a tree $T$ ($n > 2$), then $\gamma_{cc}(T) \geq \text{cut } T$.

Proof Let $S$ be a $\gamma_{cc}(G)$-set containing all the pendant vertices of $T$ (Theorem 2.1.4). The vertices of $V - S$ are cut vertices of $T$. Let $v$ be a vertex in $V - S$. By Lemma 2.1.6, corresponding to every vertex $v$ in $V - S$, there must be at least one pendant vertex in $S$. Hence $\gamma_{cc}(T) \geq \text{cut } T$. □

Corollary 2.2.11 For any tree $T$, if $\text{cut } T = \gamma_{cc}(T)$, then $\gamma_{cc}(T) = \gamma_e(T)$.

Proof It is known that the set of all non-pendant vertices form a $\gamma_e$-set. Hence $\gamma_e(T) = \text{cut } T = \gamma_{cc}(T)$. □

Corollary 2.2.12 For any tree $T$, $\gamma_{cc}(T) \geq \gamma_e(T)$.

Proof The corollary is immediate from $\gamma_e(T) = \text{cut } T$ (Corollary 1.40, [19]). □
Proposition 2.2.13 Let $T$ be a tree: If $\gamma_{cc}(T) = l(T)$, then $\beta_o(T) = \gamma_{cc}(T)$, where $l(T)$ is the number of leaves in the tree $T$.

Proof One can observe that there exists a $\beta_o$-set containing all the pendant vertices of the tree $T$ (Theorem 5.1.4). Since all the pendant vertices dominate $T$, $\beta_o(T) = l(T)$. Hence $\beta_o(T) = \gamma_{cc}(T)$. 

Corollary 2.2.14 If $\gamma_{cc}(T) = l(T)$, then $\gamma(T) + \gamma_{cc}(T) = n$ and $\gamma(T) + \beta_o(T) = n$.

Proof If $\gamma_{cc}(T) = l(T)$, then $T$ contains $n - l(T)$ supports. Since there is a $\gamma$-set (Theorem 5.3.1) containing all its supports, $\gamma(T) = n - l(T)$. That is $\gamma(T) + \gamma_{cc}(T) = n$. The other part of the Corollary follows from the Proposition 2.2.13.

Theorem 2.2.15 Let $T$ be a tree and $s_t$ be the number of supports in $T$. Then $\gamma_{cc} \geq n - s_t$.

Proof If $\gamma_{cc} = n - 1$, then $T$ is a star and hence in this case $s_t = 1$. Thus $\gamma_{cc} = n - s_t$. Now assume that $\gamma_{cc} \leq n - 2$. Let $S$ be a $\gamma_{cc}$-set of $T$. Clearly $S$ contains all the pendant vertices (Remark 2.1.5). If every $u$ in $V - S$ is adjacent to at least one pendant vertex, then every vertex of $V - S$ is a
support. Otherwise there exists a \( u \in V - S \), which is not a support. By the
Lemma 2.1.6, all vertices of one of the components of \( G - \{ u \} \) are in \( S \).
Hence \( u \) corresponds to a support in \( S \). This implies that in either case
every vertex in \( V - S \) corresponds to a support and this correspondence is
also one-to-one. Thus, in this case also, \( s_t \geq |V - S| \). From this the result
follows.

\[ S_t \leq |V - S| \]

From this the result follows.

**Corollary 2.2.16** For a tree \( T \), if \( \text{diam}(T) \leq 3 \), then \( \gamma_{cc} + s_t = n \).

**Proof** If \( \text{diam}(T) \leq 3 \), then by the assumption, \( P_5 \) is not a sub graph of \( T \)
and so every vertex of degree greater than or equal to 2 is adjacent to at
least one pendant vertex. Hence every \( \gamma_{cc} \)-set \( S \) of \( T \) contains only
pendant vertices of \( T \) which gives that \( \gamma_{cc} + s_t = n \).

**Corollary 2.2.17** If \( \gamma_{cc}(T) + l(T) = n \), then \( \gamma_{cc}(T) = \gamma_e(T) \), where \( l(T) \) is
the number of leaves of a tree \( T \).

**Proof** The result is the consequence of the fact that \( \gamma_e(T) = n - l(T) \),
where \( l(T) \) is the number of leaves of the tree \( T \) (Corollary 1.40, [19]).
**Theorem 2.2.18** Let $v$ be the vertex of the graph $G$ with deg($v$) = $k$. If every $u \in N(v)$ is adjacent to at least one vertex not in $N[v]$, then $\gamma_{cc} \leq n - k$.

**Proof** Let $v \in V(G)$ and deg($v$) = $k$. Since every vertex in $N[v]$ is adjacent to at least one vertex not in $N[v]$, it is easy to see that $S = V(G) - N[v] \cup \{u\}$, for $u \in N(v)$, is a dominating set of $G$. Now $|S| = |V(G) - N[v] \cup \{u\}| = n - (k + 1) + 1 = n - k$. Also note that $G[V - S]$ is connected. Hence $\gamma_{cc}(G) \leq |S| = n - k$. 

**Corollary 2.2.19** Let $v$ be a vertex of the graph $G$ with maximum degree $\Delta$. If $u \in N(v)$ has at least one adjacent vertex not in $N[v]$, then $\gamma_{cc} \leq n - \Delta$.

**Theorem 2.2.20** For any connected graph $G$ with $n$ vertices and $m$ edges, $\left\lceil \frac{n}{\Delta+1} \right\rceil \leq \gamma_{cc} \leq 2m - n + 1$. Also $\gamma_{cc} = 2m - n + 1$ if and only if $G$ is a star.
Proof Since \( \lceil n / (A + 1) \rceil \leq \gamma \) (Theorem 1.26, [21]) and \( \gamma \leq \gamma_{cc} \), the lower bound of the equality is true. For the upper bound, it is known that \( \gamma_{cc} \leq n - 1 \). That is \( \gamma_{cc} \leq 2(n - 1) - n + 1 \). Since \( G \) is connected, \( m \geq n - 1 \) and so \( \gamma_{cc} \leq 2m - n + 1 \).

Suppose \( \gamma_{cc} = 2m - n + 1 \), then \( 2m - n + 1 \leq n - 1 \). i.e., \( 2m \leq 2n - 2 \) which implies that \( m \leq n - 1 \). This means that \( G \) must be a tree with \( m = n - 1 \) and \( \gamma_{cc} = n - 1 \). Clearly \( G \) is a star. The other part is trivial. \( \Box \)

**Theorem 2.2.21** If \( \overline{G} \) is a connected complement of a connected bipartite graph \( G \), then \( \gamma_{cc}(\overline{G}) = \gamma(\overline{G}) = 2 \).

**Proof** Let \( G \) be a connected bipartite graph. Then \( G \) cannot be a complete bipartite graph. Otherwise \( \overline{G} \) is disconnected. Let \( (X, Y) \) be a partition of \( V(G) \). Note that \( G[X] \) and \( G[Y] \) are complete in \( \overline{G} \). Since \( G \) is not complete bipartite, there exists a vertex \( v \) in \( X \) not adjacent to a vertex \( u \in Y \). Clearly \( u \) and \( v \) are adjacent in \( \overline{G} \). Choose \( u_1 \in N(u) \) and \( v_1 \in N(v) \). Let \( S = \{u_1, v_1\} \). Then \( S \) is a \( \gamma_{cc} \)-set in \( \overline{G} \). Since \( \gamma(\overline{G}) \) cannot be 1, we observe that \( \gamma_{cc}(\overline{G}) = \gamma(\overline{G}) = 2 \). \( \Box \)
Let G be a connected graph. If H is a connected spanning subgraph of G, then one can see that $\gamma_{cc}(G) \leq \gamma_{cc}(H)$. Let $e_t = \min \{k / k$ is the number of pendant edges in a connected spanning tree of G}, where the minimum runs over all connected spanning trees of G.

**Theorem 2.2.22** If G is a connected graph with $n > 2$, then $\gamma_{cc} \geq e_t$.

**Proof** Let S be a $\gamma_{cc}$-set. Since G[V – S] is connected, G[V – S] has a spanning tree T (say). Consider the spanning tree of G formed by adding the vertices of S to T such that each vertex in S is adjacent to only one vertex in T. Thus T has at least $\gamma_{cc}$ pendant edges. Hence $\gamma_{cc} \geq e_t$. \qed