CHAPTER 2

BRIGHT AND DARK SOLITARY WAVE SOLUTIONS FOR HIGHER ORDER NLS EQUATION WITH CUBIC AND QUINTIC EFFECTS

2.1 INTRODUCTION

Optical solitons have a promising potential to become principal carriers in telecommunications due to their capability of propagating long distances without attenuation and all optical switching [1-5] etc. Therefore, considerable attention is being paid theoretically and experimentally to analyze the dynamics of optical solitons in optical waveguides (for example, silica fibers under different contexts) [1-6]. Such investigations are helpful for realizing optical soliton applications, particularly in soliton based optical communication systems [7, 8] and nonlinear optical switches [9-11]. The waveguides used in such optical systems are usually of the Kerr type. Consequently, the dynamics of light pulse is described by the family of Nonlinear Schrödinger (NLS) equation with cubic nonlinear terms [1-3]. However, as the intensity of the incident pulse becomes stronger, non-Kerr nonlinearity effect comes into play and due to this additional effect, the physical features and the stability of NLS solitons can change. The way through which non-Kerr nonlinearity influences NLS soliton propagation is described by the NLS family of equations with higher-degree nonlinear terms.
Therefore, investigations on these evolution equations become important from a theoretical point of view. Particularly this importance has received a boost after the experimental observation of the multistability of solitons in non-Kerr fibers [1-3]. In general, the non-Kerr effects are not completely integrable and cannot be solved exactly by the inverse scattering transform method. In such nonintegrable systems, therefore, the details of soliton interaction during collision cannot be described exactly and hence are still open to debate. However, numerical stimulations [1-3] show that even the slightest change from Kerr nonlinearity results in the two solitons annihilating each other, merging or creating many new solitons, depending on the initial inclination of the two solitons and their shapes. But besides the important problem of computer time, the numerical approach is not very appealing in the sense that it is not a simple task to get physical insight from purely numerical experiments. The idea, therefore, is to use approximate analytical methods such as the perturbation technique, variational method, etc., in order to compensate for the lack of exact results. By treating the quintic nonlinear terms due to the non-Kerr effect as perturbations of the cubic NLS equations, i.e., restricting the effects of quintic nonlinearity to be less predominant than the cubic terms, the NLS equations are studied both analytically and numerically [1-3]. The basic evolution equation with cubic-quintic nonlinearity, which describes the soliton evolution in non-Kerr media with parabolic nonlinearity, is discussed in the following section.
2.2 BASIC EVOLUTION EQUATIONS

Before embarking into the discussion of evolution equation, the origin of the quintic nonlinearity in the medium is discussed. The interest for considering cubic-quintic (CQ) nonlinearity in our model stems from a nonlinear correction to the medium's refractive index in the form \( \delta n = n_2 I - n_4 I^2 \), \( I \) being the light intensity and the coefficients \( n_2, n_4 > 0 \) determine the nonlinear response of the media. Although, formally it may be obtained by an expansion of the saturable nonlinearity \( \delta n = n_2 I \left[ 1 + \left( \frac{n_4}{n_2} \right) I \right]^{-1} \), it has the handicap of being always self-focusing \( \frac{d(\delta n)}{dl} > 0 \). However, the CQ model changes the sign of focusing at a critical intensity \( I_c = \frac{n_2}{2n_4} \). An experimental measurement of the nonlinear dielectric response in the Para-toluene sulfonate (PTS) optical crystal aptly models the above mentioned insights near 1600nm [1].

The CQ nonlinearity can be achieved by doping a fiber with semiconductor materials. One should have positive sign (\( n_2^{(1)} > 0 \)) and large saturation intensity (\( I_{sat}^{(1)} \)). The other should have a negative sign (\( n_2^{(2)} < 0 \)) with nearly same magnitude and low saturation intensity i.e., (\( I_{sat}^{(2)} << I_{sat}^{(1)} \)). Thus, this model needs special investigation to explore the
ultrashort pulse propagation. Of course, the value of the quintic nonlinearity is low, when compared to cubic nonlinearity, which is unavoidable when the ultrashort pulse propagation is considered. Thus, the cubic-quintic model needs special investigation to explore the ultrashort pulse propagation in the nonlinear media.

Generally when high optical intensities (or materials with high nonlinear coefficients even at moderate optical intensities, for example, semiconductor doped glasses, organic polymers, thin liquid-filled capillaries, etc.) are considered, it is necessary to take into account higher power nonlinearities arising from an expansion of the refractive index in powers of intensity $I$ of the light pulse: $n = n_0 + n_2 I + n_4 I^2 + \ldots$, where $n_0$ is the linear refractive index coefficient and $n_2, n_4, \ldots$, are nonlinear refractive index coefficients $[3]$.

In the case of $n = n_0 + n_2 I + n_4 I^2$, the wave equation for high-intensity light pulse propagation in an isotropic single mode optical fiber with a circular cross section and fiber axis $z$ can be written as

$$\nabla^2 E - \frac{1}{C^2} \frac{\partial^2 D_L}{\partial t^2} = \frac{1}{C^2} \frac{\partial^2 D_{NL}}{\partial t^2}.$$  \hspace{1cm} (2.1)

where $C$ is the speed of light, the linear part $D_L$ and the nonlinear part $D_{NL}$ of
the electric-field displacements are related to the electric field \( E(\mathbf{r}, t) \) by the relation

\[
D_L = \int \varepsilon(t') E(t-t') \, dt',
\]

(2.2)

and

\[
D_{NL} = \varepsilon_2 |E|^2 + \varepsilon_4 |E|^2.
\]

(2.3)

in which \( \varepsilon = n_0^2 \), \( \varepsilon_2 = 2 n_2 n_0 \) and \( \varepsilon_4 = 2 n_4 n_0 \).

A solution of Eqn.(2.1) is sought in the form

\[
E = e \, R(\mathbf{r}) \, A(z, t) \, e^{i(\beta z - \omega t)}.
\]

(2.4)

where \( e \) is a unit vector in the direction of wave polarization, \( R(\mathbf{r}) \) describes the transverse field modes, in which \( \mathbf{r} \) is a two-dimensional vector in the \( x-y \) plane and \( A(z, t) \) is a slowly varying amplitude. Here we assume that \( R(\mathbf{r}) \), which is mainly defined by the linear effects, corresponds to the modal distribution of the fundamental fiber mode \( HE_{11} \), for simplicity. Then, from Eqn.(2.1) to (2.4), assuming the temporal dispersion of the dielectric permittivity to be small, using the slowly varying envelope approximation, the following nonlinear partial differential equation for \( A(z, t) \) can be obtained [12].

\[
i \left[ \frac{A_z}{v_g} - \frac{1}{2} k_{\text{slow}} A_r - \frac{i}{6} k_{\text{slow}} A_m + \frac{k n_2}{n_0} \alpha_0 |A|^2 A \right.
+ \frac{k n_4}{n_0} \beta_0 |A|^4 A + \frac{i n_2}{v_g n_0} \left( |A|^2 A \right)_z + \frac{i n_2}{v_g n_0} \left( |A|^4 A \right)_r = 0.
\]

(2.5)

where the subscript \( \omega \) of the wave number \( k \) (i.e., \( k_\omega \)) indicates
differentiation of \( k \) with respect to \( \omega \), the subscripts \( z \) and \( t \) of \( A \) indicate differentiation of \( A \) with respect to the coordinates \( z \) and \( t \), respectively, and the numerical values of the parameters \( \alpha_0 \) and \( \beta_0 \) depend on the form of the function \( R(r) \).

It is convenient now to transform the Eqn.(2.5) to a reference frame moving with group velocity \( v_g \), and to introduce dimensionless variables

\[
q = \frac{A}{|A_0|},
\]

\[
\gamma = \frac{2 n_4 \beta_0 |A_0|^2}{n_2 \alpha_0},
\]

\[
\gamma_1 = \frac{-k_0 \omega_0}{3(-k_0 \omega_0)} \sqrt{\frac{1}{z_{NL}(-k_0 \omega_0)}},
\]

\[
\alpha_2 = \frac{2}{v_g} \sqrt{n_2 \alpha_0 |A_0|^2} \sqrt{\frac{k n_0}{-k_0 \omega_0}},
\]

\[
\alpha_3 = \frac{2 n_4 \beta_0 |A_0|^2}{v_g} \sqrt{\frac{|A_0|^2}{k n_0 n_2 \alpha_0(-k_0 \omega_0)}},
\]

\[
z_{NL}^{-1} = \frac{k n_2 \alpha_0 |A_0|^2}{n_0},
\]

\[
t \rightarrow \sqrt{\frac{1}{z_{NL}(-k_0 \omega_0)} \left( t - \frac{z}{v_g} \right)},
\]
in which $z_{NL}$ characterizes the nonlinear properties of the fiber and $|A_0|$ is a measure of the maximum amplitude of the input pulse. Now Eqn.(2.5) takes the form

$$i q_z + q_{tt} + 2|q|^2 q + \alpha |q|^4 q + i \alpha_1 q_{tt} + i \alpha_2 \left(q^2 q\right) + i \alpha_3 \left(q^4 q\right) = 0.$$  \hspace{1cm} (2.6)

The above equation describes the effects of quintic nonlinear terms proportional to the real parameters $\alpha$ and $\alpha_3$ on the dynamics of the pulse envelope allowing SPM and higher-order linear and nonlinear dispersions. For pulse widths greater than 100 fs, one can neglect the last three terms of Eqn.(2.6) and the resulting equation is a well-studied [13,14] simple normalized NLS equation with cubic-quintic nonlinear terms.

2.3 THEORETICAL MODEL

In this chapter, having realized the importance of cubic-quintic nonlinearity, recently, Radakrishnan, Lakshmanan and Kundu [12] proposed the Higher order NLS (HNLS) equation with the CQ nonlinear terms arising in non-Kerr media. They investigated the integrable systems of coupled HNLS equations with some simplifications in the model coefficients and found the Lax-pair, conserved quantities and exact soliton solutions. More recently, Hong [14] considered the similar cubic-quintic model with the
higher terms and found the analytical bright and dark solitary wave solutions by applying the complex envelope functions ansatz.

It well known that to increase the information carrying capacity of the OFC system, one has to reduce the pulse width. It also means that the intensity of incident pulse has been increased. Under this condition, the dynamics of the pulse is then described by a family of NLS equation with higher order linear and nonlinear effects such as third order dispersion and quintic self-steepening and quintic stimulated Raman Effect [12, 14]. For the first time, Du et al., [13] introduced a novel technique called coupled amplitude phase to solve the NLS equation. Following their work, Palacios et al., [15] applied the same technique to the HNLS equation and obtained the dark soliton solution.

In order to investigate the bright and dark solitary wave solutions, the following higher order cubic and quintic nonlinear effects are considered in the non-Kerr medium where the pulse dynamics can be described by [12, 14],

\[
i \frac{\partial U}{\partial z} + \frac{\partial^2 U}{\partial t^2} + |U|^2 U + |U|^4 U + i \frac{\partial^3 U}{\partial t^3} + i \alpha_1 \frac{\partial (|U|^2 U)}{\partial t} + i \alpha_2 U \frac{\partial (|U|^2)}{\partial t} + i \alpha_3 U \frac{\partial (|U|^4)}{\partial t} = 0.
\]

The above equation reduces to NLSE, only when all the higher order linear and nonlinear terms are zero, for which soliton solutions are well
known [4, 5, 16]. Now, the generation of bright and dark solitons with the help of coupled phase amplitude method is discussed. The coupled phase amplitude ansatz

\[ U(\xi, \tau) = P(\tau + \beta \xi) \exp(i(\kappa \xi - \omega \tau)). \]

is rewritten as

\[ U(\xi, \tau) = P(\chi)\exp(i(\kappa \xi - \omega \tau)), \tag{2.8} \]

where \( \chi = \tau + \beta \xi \) and \( P \) is real.

Using Eqn.(2.8) the Eqn.(2.7) becomes,

\[ \kappa P - iP_x \beta = P_x \left( -2i \omega - 3i \omega^2 + 3i \alpha_1 P^2 + 2i \alpha_2 P^2 + 5i \alpha_3 P^4 + 4i \alpha_4 P^4 \right) \]

\[ + P_{xx} \left( 1 + 3 \omega \right) + iP_{xxx} + P \left( -\omega^2 - \omega^3 \right) + P^3 \left( 1 + \omega \alpha_1 \right) + P^3 \left( 1 + \omega \alpha_3 \right). \]

Separation of real and imaginary parts in the above equation leads to

\[ -P_x \beta = P_x \left( -2 \omega - 3 \omega^2 + 3 \alpha_1 P^2 + 2 \alpha_2 P^2 + 5 \alpha_3 P^4 + 4 \alpha_4 P^4 \right) + P_{xxx}, \tag{2.9} \]

\[ \kappa P = P_{xx} \left( 1 + 3 \omega \right) + P \left( -\omega^2 - \omega^3 \right) + P^3 \left( 1 + \omega \alpha_1 \right) + P^3 \left( 1 + \omega \alpha_3 \right). \tag{2.10} \]

Equation (2.9) is rewritten as,

\[ P_{xxx} = P_x \left( -\beta + 2 \omega + 3 \omega^2 - 3 \alpha_1 P^2 - 2 \alpha_2 P^2 - 5 \alpha_3 P^4 - 4 \alpha_4 P^4 \right). \]

Integrating the above equation leads to

\[ P_{xx} = \left( -\beta + 2 \omega + 3 \omega^2 \right) P - \left( \frac{3 \alpha_1 + 2 \alpha_2}{3} \right) P^3 - \left( \frac{5 \alpha_3 + 4 \alpha_4}{5} \right) P^5. \tag{2.11} \]

Rewriting the Eqn.(2.10)

\[ P_{xx} = \left( \frac{\kappa + \omega^2 + \omega^3}{1 + 3 \omega} \right) P - \left( \frac{1 + \omega \alpha_1}{1 + 3 \omega} \right) P^3 - \left( \frac{1 + \omega \alpha_3}{1 + 3 \omega} \right) P^5. \tag{2.12} \]
Equations (2.11) and (2.12) are equivalent only if the following conditions are satisfied

\[
(-\beta + 2\omega + 3\omega^2) = \left(\frac{\kappa + \omega^2 + \omega^5}{1+3\omega}\right),
\]

(2.13)

\[
\frac{3\alpha_1 + 2\alpha_2}{3} = \left(\frac{1 + \omega^2}{1+3\omega}\right),
\]

(2.14)

\[
\frac{5\alpha_3 + 4\alpha_4}{5} = \left(\frac{1 + \omega^2}{1+3\omega}\right).
\]

(2.15)

Equation (2.13) gives \(\kappa\) as

\[
\kappa = (-\beta - 3\omega\beta + 2\omega + 8\omega^2 + 8\omega^3).
\]

(2.16)

From Eqns. (2.14) and (2.15), \(\omega\) becomes

\[
\omega = \frac{30 - 15\alpha_1 - 10\alpha_2 - 15\alpha_3 - 12\alpha_4}{6(5\alpha_1 + 5\alpha_2 + 5\alpha_3 + 6\alpha_4)}.
\]

(2.17)

Having calculated the physical parameters such as propagation constant and frequency in terms of the physical parameters of the nonlinear medium, theoretically the generation of bright and dark solitons is discussed.

### 2.3.1 BRIGHT SOLITON SOLUTION

In the context of nonlinear optics the envelope soliton is called a bright soliton because it corresponds to a pulse of light. The broadening effects of light pulse due to anomalous dispersion is, exactly balanced by the narrowing effects of focusing nonlinearity and the pulse never changes (in the limit of
zero loss) its shape when propagating along the fiber. Eqn.(2.11) is modified as

\[
P_{xx} P_{x} = (-\beta + 2\omega + 3\omega^2) P P_x - \left(\frac{3\alpha_1 + 2\alpha_2}{3}\right) P^3 P_x - \left(\frac{5\alpha_3 + 4\alpha_4}{5}\right) P^5 P_x.
\]

Integrating the above equation, the sextic anharmonic oscillator equation is

\[
\frac{1}{2} P_x^2 - (-\beta + 2\omega + 3\omega^2) \frac{P_x^2}{2} + \left(\frac{3\alpha_1 + 2\alpha_2}{12}\right) P^4 + \left(\frac{5\alpha_3 + 4\alpha_4}{30}\right) P^6 + C = 0. \tag{2.18}
\]

Rearranging the above equation

\[
P_x^2 = (-\beta + 2\omega + 3\omega^2) P^2 - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right) P^4 - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right) P^6 + C, \tag{2.19}
\]

where C is an arbitrary constant of integration. From the above equation, it is possible to get the different analytical solutions for different values of the constant of integration C. Among these solutions, our prime aim to investigate only the bright soliton solution. The Eqn.(2.19) describes the motion of a classical particle traveling in a sextic potential well having the following potential form

\[
U(P) = (-\beta + 2\omega + 3\omega^2) \frac{P^2}{2} - \left(\frac{3\alpha_1 + 2\alpha_2}{12}\right) P^4 - \left(\frac{5\alpha_3 + 4\alpha_4}{30}\right) P^6.
\]

Equation (2.19) becomes

\[
\left(\frac{dP}{d\chi}\right)^2 = (-\beta + 2\omega + 3\omega^2) P^2 - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right) P^4 - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right) P^6.
\]

Assuming the constant of integration to be equal to zero the above equation becomes
\[ \frac{dP}{d\chi} = \sqrt{(-\beta + 2\omega + 3\omega^2)P^2 - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right)P^4 - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right)P^6}, \]

\[ d\chi = \frac{dP}{\sqrt{(-\beta + 2\omega + 3\omega^2)P^2 - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right)P^4 - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right)P^6}}, \]

\[ \int \frac{dP}{\sqrt{(-\beta + 2\omega + 3\omega^2)P^2 - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right)P^4 - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right)P^6}} = \int d\chi. \]

On integration

\[ \chi = \int \frac{dP}{P \sqrt{(-\beta + 2\omega + 3\omega^2)P^2 - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right)P^4 - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right)P^6}}. \]

By giving transformation \( P^2 = q \) the above equation is modified as,

\[ \chi = \int \frac{dq}{2q \sqrt{(-\beta + 2\omega + 3\omega^2) - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right)q - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right)q^2}}. \]  \hspace{1cm} (2.20)

Making a transformation, \( q = \frac{1}{t^2} \) and

\[ \chi = \int \frac{-\left(\frac{1}{t^2}\right)dt}{2 \left(\frac{1}{t}\right) \sqrt{(-\beta + 2\omega + 3\omega^2) - \left(\frac{3\alpha_1 + 2\alpha_2}{6}\right)t - \left(\frac{5\alpha_3 + 4\alpha_4}{15}\right)t^2}}. \]

the equation is
\[
\chi = \int \frac{-dt}{2 \sqrt{X_1} \sqrt{t^2 - \left( \frac{X_2}{X_1} \right) t - \left( \frac{X_3}{X_1} \right)}}
\]

where \( X_1 = (-\beta + 2\omega + 3\omega^2) \),

\[
X_2 = \frac{3\alpha_1 + 2\alpha_2}{6}
\]

and \( X_3 = \frac{5\alpha_3 + 4\alpha_4}{15} \).

Solving the above equation

\[
t = \left( \frac{X_2}{2X_1} \right) + \sqrt{\left( \frac{X_2^2}{4X_1^2} \right) \left( \frac{X_3}{X_1} \right) \cosh(2\sqrt{X_1} \chi)}.
\]

Using \( P^2 = q = t^{-1} \), then above equation is modified as,

\[
P(\chi) = \left( \frac{X_2}{2X_1} \right) + \sqrt{\left( \frac{X_2^2}{4X_1^2} \right) \left( \frac{X_3}{X_1} \right) \cosh(2\sqrt{X_1} \chi)} \right)^{-\frac{1}{2}}. \tag{2.21}
\]

The effect of positive nonlinearity and the anomalous dispersion fight out to give birth to solitons in the optical fibers. These resulting solitons are referred to as bright solitons. By using Eqns.(2.8) and (2.21) the bright soliton solution is

\[
U(\xi, \tau) = \left( \frac{3\alpha_1 + 2\alpha_2}{6} \right) \left[ \frac{1}{2(-\beta + 2\omega + 3\omega^2)} \right]^{-\frac{1}{2}} \cosh \left( 2\sqrt{-\beta + 2\omega + 3\omega^2} \chi \right) \exp[i(\kappa\xi - \omega\tau)]]
\]

\tag{2.22}
The above solution represents the bright solitary wave solution for the HNLS equation with quintic effects for which Soliton plot is as shown in Figs.(2.1) and (2.2). The bright soliton is depicted in Fig.(2.1) for different values of the physical parameters.

\[
\begin{align*}
\mathcal{R} &= \sqrt{\left(\frac{3\alpha_1 + 2\alpha_2}{6}\right)^2 \left(-\beta + 2\omega + 3\omega^2\right) + 4\left(-\beta + 2\omega + 3\omega^2\right)^2 \left(\frac{5\alpha_4 + 4\alpha_4}{15}\right)}
\end{align*}
\]

Figure 2.1
Bright solitary wave solution for different values of the physical parameters \(\beta = 1, \omega = -0.085, \alpha_1 = -0.02, \alpha_2 = 0.01, \alpha_3 = 0.018, \alpha_4 = 0-0.001\) and \(\kappa = 0.01\)
Further, to have an idea about the variation of pulse width and amplitude with respect to input power, the 2D plot for various values of the input power is depicted in Fig.(2.2). From the plot, it is clear that the pulse width is reduced and hence amplitude increases as the value of input power is increased.

2.3.2 DARK SOLITON SOLUTION

The generation of dark solitons was first predicted by Hasegawa and Tappert [5] and Zakharov and Shabat [16] and experimentally demonstrated by Emplit et al., [17]. The bright soliton is a pulse on a zero-intensity background, while a dark soliton appears as an intensity dip in an infinitely extended constant background. Apart from the inverse intensity profile, an additional unique feature of a dark soliton is its specific phase profile. The
dark soliton phase chirp is a monotonic and odd function of the spatial coordinate. Recently, increased interest in dark spatial solitons has become connected with their possible application in optical logic devices [18] and waveguide optics as dynamic switches and junctions [19]. They are also considered for signal processing and communication applications because of their inherent stability [20]. In fact, the influence of noise and fiber loss on dark solitons is much lesser than that of bright solitons [21]. The advantages of dark solitons over bright solitons are that, they are considered for signal processing and communication applications because of their inherent stability [20]. In fact, the influence of noise and fiber loss on dark solitons is much lesser than that of bright solitons [21]. Dark solitons can be created without a threshold value in the input pulse power, which is not possible in the bright soliton. The force between the dark solitons is always repulsive whereas it is either attractive or repulsive in the case of bright solitons depending upon their relative phase [4].

To derive dark soliton solution analytically for higher order NLS with cubic and quintic term, Eqn.(2.18) is rewritten as,

\[
\left( \frac{dP}{d\chi} \right)^2 = -\Gamma P^2 - \frac{K}{2} P^4 - \frac{\psi}{3} P^6 + C,
\]

where \( \Gamma = (\beta - 2\omega - 3\omega^2) \),

\[
K = \left( \frac{3\alpha_1 + 2\alpha_2}{3} \right),
\]
and $\Psi = \left( \frac{5\alpha_3 + 4\alpha_4}{5} \right)$.

The transformation $P = \frac{1}{\sqrt{R}}$ to the above equation leads to

$$\left( \frac{dR}{d\chi} \right)^2 = -4C \left( \frac{\Psi}{3C} + \frac{K}{2C} R + \frac{\Gamma}{C} R^3 - R^2 \right).$$  \hspace{1cm} (2.23)

The Eqn.(2.23) is rewritten as

$$\left( \frac{dR}{d\chi} \right)^2 = -4C (R - e_1)(R - e_2)(R - e_3),$$  \hspace{1cm} (2.24)

where

$$e_1, e_2, e_3 = \left( \frac{5\alpha_3 + 4\alpha_4}{15C} \right);$$

$$-e_1e_2 - e_1e_3 - e_2e_3 = \left( \frac{3\alpha_1 + 2\alpha_2}{6C} \right),$$

$$e_1 + e_2 + e_3 = \left( \frac{\beta - 2\omega - 3\omega^2}{C} \right).$$

On rearranging the Eqn.(2.24), we get

$$\frac{dR}{(R - e_1)(R - e_2)(R - e_3)} = d\chi \sqrt{-4C}.$$  \hspace{1cm} (2.25)

On integration this becomes

$$\int_{e_2}^{e_3} \frac{dR}{(R - e_1)(R - e_2)(R - e_3)} = \int_{e_2}^{e_3} \sqrt{-4C} d\chi,$$  \hspace{1cm} (2.26)

the solution oscillates between $e_2$ and $e_3$ where $e_1 < e_2 < e_3$ are the real roots of $e$. In Jacobi elliptical function, the solution is
\[ R = e_3 + (e_2 - e_3) \text{sn}^2\left(\sqrt{C(e_1 - e_3)} \chi, m\right). \]  

(2.27)

where

\[ e_1 = \frac{2 - k^2}{3}, \quad e_2 = \frac{2k^2 - 1}{3}, \quad e_3 = \frac{k^2 + 1}{3}; \quad m = k^2 \frac{e_3 - e_2}{e_3 - e_1} \text{ and } \chi = (z - vt). \]

where \( m \) is the modulation parameter and \( v \) is the velocity of the soliton pulse.

The above solution represents the dark solitary wave solution for the HNLS equation with quintic effects for which the 3D and 2D dark soliton plots are shown in Figs.(2.3) and (2.4) respectively for various values of the physical parameters of the system considered. From the plot 2D, it is clear that the pulse width reduces and hence amplitude increases as the value of input power is increased as in the case of bright soliton.

Figure 2.3
Dark solitary wave solution for different values of the physical parameters
\[ e_1 = -0.1, \quad e_2 = 0.7, \quad e_3 = 0.3, \quad v = 0. \]
2.4 RESULTS AND DISCUSSION

By means of coupled amplitude phase technique, an analytical bright solitary wave solution for the HNLS equation, which describes the femtosecond pulse propagation in the non-Kerr media, is found. In this case both quintic self-steepening and delayed Raman response terms have been included because of the additional physics requirement in the femto second pulse propagation range. It is interesting to mention that the dark soliton solution [22] discussed in this chapter is in good agreement with earlier results [23, 24].
2.5 REFERENCES


