CHAPTER 4

POLARIZATION EVOLUTION OF A SOLITON TYPE PULSE IN BIREFRINGENT OPTICAL FIBERS

4.1 INTRODUCTION

Apart from the chromatic dispersion, there is another dispersion called PMD that is applicable only to SMF. Even though the fiber is called ‘single mode’, it actually carries two modes under one name. These modes are linear-polarized waves that propagate within the fiber in two orthogonal planes. Ideally, each of the modes carries half of the total light power. If the fiber has ideal symmetric cross sectional properties, both the modes propagate at the same velocity and arrive at the fiber end simultaneously. Thus, signal travel along the fiber remains undisturbed and the presence of the polarized modes goes unnoticed. But there is some asymmetry in every fabricated fiber, but the most likely times for serious asymmetry to occur are during the fiber-cabling and splicing processes. Under this condition, both the modes do not travel with same velocity and hence come at the end of the fiber with different timings. In a nutshell, the pulse spreading caused by a change of fiber polarization properties is called PMD.

Hence, ordinary SMF do not normally preserve the polarization state of the propagating light. As a consequence, polarization fluctuations can be
observed at the output of fiber and hence enhance the bit error rate. In the worst case, polarization fluctuations may even extinguish the detected signal completely. Thus either polarization stabilization or an active control of State Of Polarization (SOP) is indispensable.

4.2 POLARIZATION ANALYSIS IN SINGLE MODE FIBERS

In an ideal circular single mode fiber, two orthogonal independent degenerate modes (i.e., the orthogonal polarizations) always do exist. These modes are degenerated since both are defined by the same propagation constant or by the same velocity of propagation. In general, the electric field in an optical fiber is always a linear superposition of these two eigen polarizations or eigenmodes. As eigenmodes are independent, they propagate without any mutual disturbance.

4.2.1 EIGENMODES

In a real SMF, various asymmetries can be observed. As a result, the degeneration of both orthogonal polarizations is removed and they propagate with different velocities instead of the same velocity. Thus these modes actually become two different modes called the eigenmodes.

Besides noncircular fiber core and asymmetrical lateral stress, various other internal and external imperfections can be observed along the fiber. Typical additional imperfections are bending, torsions and symmetrical
distributions of the refraction index. Moreover, variations in temperature may also influence the quality of light propagation through the optical fiber since different thermal coefficients of expansion with respect to fiber core and jacket yield again asymmetrical internal stresses. Independent of type, all these perturbations break the circular waveguide geometry and influence the velocity of propagation of both the eigenmodes.

In addition the change in the velocity of propagation, fiber distortion such as mentioned above also yield an undesired mode coupling. Because of this, a reciprocal exchange of energy takes place. As a result, both modes influence each other and they are no longer independent. They are not eigenmodes now since only independent modes are called eigenmodes. However, it can be shown that coupled modes can always be expressed in terms of two independent eigenmodes again, taking into account the perturbations of the fiber. The orthogonal polarizations of these new defined eigenmodes differ from the orthogonal polarizations of the eigenmodes, which were existing in the absence of fiber perturbations. The orthogonal directions of polarization are called the principal axes. If, for example, fiber exhibits an elliptical core, principal axes are the small and the large half-axes of the elliptical cross-section of the fiber core. In order to discuss the polarization effects, it is presumed that the principal axes of the eigenmodes are same as the x- and y- directions of a rectangular Cartesian coordinate
system. In this coordinate system, propagation constants are represented by \( \beta_x \) and \( \beta_y \). The axis of fiber that is the direction of propagation is the \( z \)-direction. We also presume that the fiber is perturbed only by an axial asymmetry, which does not change with the position variable \( z \).

An important quantity to assess how polarization is changed or maintained in a single mode fiber is given by the difference

\[ \Delta \beta = \beta_x - \beta_y. \] (4.1)

of both propagation constants \( \beta_x \) and \( \beta_y \). Normalizing this difference by \( 2\pi/\lambda \), where \( \lambda \) denotes the wavelength of the light, the dimensionless quantity \( D = \Delta \beta \lambda / 2\pi \) called the double refraction or birefringence is obtained. The birefringence in a fiber is a measure of the difference in the effective indices of the two orthogonally polarized modes. Thus, single mode fibers with different velocities of propagation of both their eigenmodes are normally called birefringent fibers. The SOP repeats itself after a distance equal to \( 2\pi/\Delta \beta \) and this length is referred to as the beat length of the fiber, \( L_b \). The beat length is a measure of the birefringence of the fiber; the smaller the beat length the larger is the corresponding birefringence.

4.3 POLARIZATION OF LIGHT

Light is demonstrated by its intensity, wavelength and polarization. The polarization plays a vital role in the pulse profile in fiber. Augustin Jean
Fresnel and et al., completely explained the major optical phenomena of interference, diffraction and polarization on the basis of wave theory. Further, Fresnel had successfully applied the wave theory to the problem of the propagation and polarization of light in anisotropic medium, that is, crystals. However Fresnel and others were unable to describe the unpolarized light on the basis of wave theory of light. Finally in the year 1852, Sir George Gabriel Stokes had been able to show that unpolarized light could be described within the limit of wave theory of light. He formulated the polarized light in terms of measured quantities (observables) not like his predecessors to describe unpolarized light in terms of amplitudes. Further, Stokes showed that his intensity formulation of the polarized could be used to describe not only unpolarized and partially but also completely polarized. Thus, his formulation was applicable to any state of polarized light. Stokes introduced four parameters that could characterize any state of polarized light, now known as the Stokes polarization parameters.

4.3.1 STOKES POLARIZATION PARAMETERS.

Analysis of the ellipse showed that for special cases it led to forms which can be interpreted as linearly polarized and circularly polarized light. This description of light in terms of the polarisation ellipse is very useful because it enables us to describe by means of a single equation various states of polarized light. However, this representation is inadequate for several
reasons. As the beam of light propagates through space, it is found that in a plane transverse to the direction of propagation, the light vector traces out an ellipse or some special form of an ellipse, such as a circle or a straight line in a time interval of the order of $10^{15}$ sec. This period of time is clearly too short to allow us to follow the tracing of ellipse. This fact, therefore, immediately prevents us from ever observing the polarization ellipse. Another limitation is that the polarization ellipse is only applicable to describing light which is completely polarized. It cannot be used to describe either unpolarized light or partially polarized light. This is a particularly serious limitation because in nature light is very often unpolarized or partially polarized. Thus, the polarization ellipse is an idealization of the true behavior of light; it is correct at any given instant of time. These limitations forced us to consider an alternative description of polarized light in which only observed or measured quantities enter. The situation is the same as when wave equation and its solution are dealt with, neither of which can be observed. The average value of the optical field has to be used which in the present case requires the represent action of polarized light in terms of observables.

Henri Poincarè, a famous nineteenth-century French mathematician and physicist, discovered around 1980 that the polarization ellipse could be represented on complex plane. Further, he discovered that this plane could be
represented onto a sphere. The sphere which Poincarè devised is extremely useful for dealing with polarized light problems and appropriately enough which is represented in Fig.(4.1).

![Poincarè sphere](image)

Figure 4.1
Poincarè sphere

In 1852, Sir George Gabriel Stokes discovered that the polarization behavior could be represented in terms of observables. He found that any state of polarized light could be completely described by four measurable quantities ($S_0, S_1, S_2, S_3$), known as the stoke polarization parameters. Poincarè sphere provides remarkable insights into the manner in which polarized light behaves in its interaction with polarizing elements. The most remarkable properties are that any point on sphere can be described in terms
of the three Stokes parameters $S_1$, $S_2$ and $S_3$ for elliptically polarized light, and the magnitude of interaction of a polarized beam with an optical polarizing element corresponds to a rotation of the sphere; the final point describes the new set of Stokes parameters.

The first parameter $S_0$ expresses the total intensity of the optical field. The remaining three parameters describe the polarization state. These stokes parameters are a logical sequence of the wave theory. Further it gives a complete description of any polarization of light. If Stokes parameters are formed in terms of column matrix, the so called Stokes vector leads to not only measurable but observable.

4.3.2 DERIVATION OF THE STOKES POLARIZATION PARAMETERS

Experiments of Fresnel and others led to the discovery that light consisted only of two transverse components. These two were perpendicular to each other and for convenience assumed to be propagating in the $z$ direction. When the propagator is eliminated between the transverse components, then the transverse components are represented by

$$E_x(t) = E_{ox}(t) \cos(\omega \tau + \delta_x(t)),$$
$$E_y(t) = E_{oy}(t) \cos(\omega \tau + \delta_y(t)).$$

(4.2)

where $E_{ox}(t)$, $E_{oy}(t)$ are the instantaneous amplitudes, $\omega$ is the instantaneous angular frequency, $\delta_x(t)$ and $\delta_y(t)$ are the instantaneous phase factors. As the
field propagates, $E_x(z, t)$ and $E_y(z, t)$ give rise to a resultant vector. This vector describes a locus of points in the space, and the curve generated by those points can be written as

$$\frac{E_x^2(t)}{E_{ox}^2} + \frac{E_y^2(t)}{E_{oy}^2} - 2 \frac{E_x(t) E_y(t)}{E_{ox} E_{oy}} \cos(\delta(t)) = \sin^2(\delta(t)), \quad (4.3)$$

where $\delta(t) = \delta_x(t) - \delta_y(t)$.

The above equation is recognized as the equation of ellipse and shows at any instant of time the locus of points described by the optical field as it propagates in an ellipse. This behavior is called as optical polarization. The above polarization ellipse is considered to be very important, because it gives the different states of polarization. For monochromatic radiation, the amplitudes and phases are constant for all time, and hence Eqn.(4.3) reduces to

$$\frac{E_x^2(t)}{E_{ox}^2} + \frac{E_y^2(t)}{E_{oy}^2} - 2 \frac{E_x(t) E_y(t)}{E_{ox} E_{oy}} \cos(\delta) = \sin^2(\delta), \quad (4.4)$$

where $E_{ox}$, $E_{oy}$ and $\delta$ are constants.

$E_x$ and $E_y$ continue to be implicitly dependent on time, as is evident from Eqn.(4.2). In order to represent Eqn.(4.4) in terms of the observables of the optical field, an average over the time observation is to be considered and the Eqn.(4.4) is

$$\langle \frac{E_x^2(t)}{E_{ox}^2} \rangle + \langle \frac{E_y^2(t)}{E_{oy}^2} \rangle - 2 \langle \frac{E_x(t) E_y(t)}{E_{ox} E_{oy}} \rangle \cos(\delta) = \sin^2(\delta), \quad (4.5)$$

where
\[ \langle E_i(t)E_j(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} E_i(t)E_j(t) \, dt, \quad \text{with } i, j = x, y. \]

After algebra, Eqn. (4.5) is written as
\[ \left( E_{ax}^2 + E_{ay}^2 \right) - \left( E_{ax}^2 - E_{ay}^2 \right) - \left( 2E_{ax}E_{ay} \cos(\delta) \right)^2 = \left( 2E_{ax}E_{ay} \sin(\delta) \right)^2. \quad (4.6) \]

The quantities inside the parenthesis are
\begin{align*}
S_0 &= E_{ax}^2 + E_{ay}^2, \\
S_1 &= E_{ax}^2 - E_{ay}^2, \\
S_2 &= 2E_{ax}E_{ay} \cos(\delta), \\
S_3 &= 2E_{ax}E_{ay} \sin(\delta).
\end{align*} \quad (4.7)

with \( S_0^2 = S_1^2 + S_2^2 + S_3^2. \) \quad (4.8)

The four equations given by Eqn.(4.7) are the Stokes parameters for a plane wave. The first Stokes parameter \( S_0 \) is the total intensity of light. The parameter \( S_1 \) describes the amount of linear horizontal or vertical polarization, the parameter \( S_2 \) describes the amount of linear \(+45^\circ\) or \(-45^\circ\) polarization and the parameter \( S_3 \) describes the amount of right or left circular polarization contained within the beam. Stokes vectors are expressed in terms of intensities and they are real quantities. Using Schwarz’s inequality, one can show that for any state of polarized light the Stokes parameters always satisfy the relation, \( S_0^2 \geq S_1^2 + S_2^2 + S_3^2. \) The equality sign applies when there is completely polarized light and the inequality sign when there is partially polarized light or unpolarized light. The orientation angle \( \psi \) of the polarization ellipse is given by
\[
\tan(2\psi) = \frac{2 E_{ox} E_{oy} \cos \delta}{E_{ox}^2 - E_{oy}^2}.
\]

Then the ellipticity angle \( \chi \) was given by

\[
\sin(2\chi) = \frac{2 E_{ox} E_{oy} \sin \delta}{E_{ox}^2 + E_{oy}^2}.
\]

The Stokes parameters enable us to describe the degree of polarization \( P \) for any state of polarization

\[
P = \frac{\sqrt{S_0^2 + S_1^2 + S_2^2}}{S_o} \quad 0 \leq P \leq 1.
\]

To obtain stokes parameters of an optical beam, one must always take average of the polarization ellipse. However, the time averaging process can be formally bypassed by representing the real optical amplitudes of Eqn.(4.2).

The Stokes parameters for a plane wave are now obtained from the formulas

\[
\begin{align*}
S_0 &= E_x E_x^* + E_y E_y^*, \\
S_1 &= E_x E_x^* - E_y E_y^*, \\
S_2 &= E_x E_y^* - E_y E_x^*, \\
S_o &= i (E_x E_y^* - E_y E_x^*)
\end{align*}
\]

4.3.3 POLARIZATION EVOLUTION OF A SOLITON PULSE

Single mode fibers are actually bimodal because of the presence of birefringence [1-4]. This birefringence creates two principal transmission axes within the fiber known as the fast and slow axes. Splitting of the pulse takes place when an optical pulse is launched into a single mode fiber. The two
parts of the pulse reach the fiber end at slightly different times, bringing in two changes with the injected pulse. First, the pulse gets broadened and second, the SOP of a soliton pulse gets modified. So the SOP is allowed to change and take different values. Meanwhile the SOP of a pulse can also be changed due to the relative phase of the pulse [5]. Hence polarization would occur in a single mode fiber only in the presence of birefringence. Pulse propagation in a single mode birefringent fiber is governed by the well known CNLS equations, which was systematically derived by Menyuk [4]. Therefore, it is clear that an OSFC system shows polarization dependent losses affecting the system performance. Thus there is great interest in analyzing, controlling, and manipulating the polarization state of a soliton pulse in a fiber. This has an increasingly vital role in OSFC.

As has been discussed in the introduction, in the case of a birefringent fiber, each pulse generally consists of two polarization components that trap each other through nonlinear Kerr effects. Such a pulse is referred to as a vector soliton in optics literature. It should be noted that a vector soliton is just a solitary wave solution, not a soliton in the strict mathematical sense. Thus, vector solitons consist of two field components that mutually self-trap in a nonlinear medium. These solitons were first suggested by Manakov [6] for the Kerr nonlinearity, which leads to two coupled NLS equation. A new phenomenon in the coupled NLS equations is that two vector solitons can
form a perfectly stationary bound state if they have the same phase in one component and a $\pi$-phase difference in the other component. Physically, stationary two vector-soliton states can be formed because the attracting force in the in-phase polarization balances the repelling force in the out-of-phase polarization [7]. The existence of such stationary bound states was first established through numerical means by Haelterman et al., [7]. The analytical construction of such bound states was first achieved by Yang by an asymptotic tail-matching method [8]. In that work, the spacing between vector solitons in a stationary configuration was obtained explicitly. Experimentally, stationary two-vector-soliton states have been observed in photorefractive materials [9].

In addition to the above studies, several analytical and numerical methods are available to study the evolution of the SOP of a soliton pulse propagating along a birefringent optical fiber. Several effective analytical methods have been developed to analyze this polarization evolution [5, 10-14]. For example, Hutchings [10] et al., applied an eigenvalue approach to study the polarization evolution of a pulse along the waveguide. For the first time, Stokes parameters were used by Daino, Gregory, and Wabnitz [11] to study the effect of polarization in nonlinear couplers. From the integro-differential equation, they have directly shown that the trajectories of the motion of the Stokes parameters can be plotted on the Poincarè sphere.
Evangelides, Mollenauer, Gordon, and Bergano [12] have applied the Stokes parameter formalism to the experimental study of the polarization of soliton pulse propagation in fibers.

Akhmediev and Soto-Crespo [5, 13] have studied the different SOP of a soliton pulse with Kerr nonlinearity. To describe them, they have constructed the integro-differential equation, found stationary solutions for two special cases of the fiber parameters, and discussed the oscillating and rotating phase. Combinations of these two motions have been considered for the general case. For the first time, Barad and Silberberg [14] have experimentally confirmed the polarization evolution and instability of a soliton pulse in a birefringent optical fiber. The main advantage of using the Stokes parameters is that they provide the simplest and straightforward analytical definition of the different states of polarization of light. Many researchers use Stokes parameters to describe the polarization changes of a wave propagating in an optical medium [5, 15]. The use of Stokes parameters has an advantage in providing a rather straightforward analytical definition of the different SOP of light. Höök and Karlsson [16] demonstrated the existence of a stationary soliton like solution in the NLSE.

In what follows, the SOP of a soliton like pulse is expressed in terms of the Stokes parameters by constructing the integro-differential equation.
Many researchers [16–18] have studied the influence of Fourth Order dispersion (FOD). In this chapter the effect of FOD is also analyzed using stokes parameter formalism, and it is found that higher values of FOD broaden the pulse width. Lower values of the FOD reduce the pulse width, which will be very useful in optical communication.

4.4 THEORETICAL MODEL

In the ever-growing communication world, an increase in the information carrying capacity of the fiber optic communication system is possible. To increase the capacity of the system, the shorter pulses have to be used, which can be achieved at larger light intensities. With the increase of the intensity, it is necessary to take into consideration the next order term in the expression of the full refractive index

\[ n = n_0 + n_2 I - n_4 I^3, \]

(4.13)

where \( I = |q|^2 \).

Under the above criterion, the general form of cubic-quintic nonlinear Schrödinger equation is given by

\[ iq_z = q_n + |q|^2 q + \gamma |q|^4 q, \]

(4.14)

where \( q \) is a complex-valued function of the variables (z, t). The above equation takes into account a quintic nonlinearity correction to the cubic (Kerr) nonlinearity, which should be taken into account for larger light intensities.
Further, when $\gamma = 0$, the above equation becomes the well-known cubic nonlinear Schrödinger equation, which is used to describe the propagation of the envelope of a light pulse in an optical fiber which has a Kerr-type nonlinear refractive index. For short pulses and high input peak power pulse the refractive index cannot be described by Kerr-type nonlinearity, as the index is then influenced by higher-order nonlinearities. In materials with high nonlinear coefficients, such as semiconductors, and semiconductor-doped glasses, the saturation of the nonlinear refractive-index change is no longer negligible at moderately high intensities and should be taken into account. The above Eqn.(4.14) is the correct model to describe the propagation of the envelope of a light pulse in higher-order refractive index.

Therefore, in this chapter, the pulse propagation in a birefringent optical fiber with quintic effect described in terms of CNLS type equations is considered. In a reference frame traveling along the $z$-axes with the common group velocity, the governing equations take the form,

\begin{align}
&iq_{1z} + q_{1rr} + 2|q_1|^2 + B|q_2|^2 q_1 + \gamma (|q_1|^2 + B|q_2|^2)^2 q_1 + \rho q_1 + \kappa q_2 = 0, \\
&iq_{2z} + q_{2rr} + 2|q_2|^2 + B|q_1|^2 q_2 + \gamma (|q_2|^2 + B|q_1|^2)^2 q_2 - \rho q_2 + \kappa q_1 = 0.
\end{align}

(4.15)

Here, $q_1$ and $q_2$ are the slowly varying envelopes of the two linearly polarized components of the field along the $x$ and $y$ directions, $z$ is the propagation direction, $\tau$ is the normalized retarded time, $B$ is the cross phase modulation, $\gamma$
is the quintic effect, $\rho$ and $\kappa$ are the coupling and cross-coupling coefficients respectively.

The presence of quintic effect in the above equation breaks the integrability of the system under consideration. And consequently, (symmetry breaking) instability occurs in the system, which is considered to be the essential requirement for the switching purpose in the birefringent fibers. In the following section, the (symmetry breaking) instability in the birefringent fiber is investigated.

For the sake of convenience the field components is represented in the form

$$q_1 = Q_1 e^{iq}, q_2 = Q_2 e^{iq}. \tag{4.16}$$

where $q$ is the soliton parameters, $2\pi/q$ is proportional to the soliton period, and $2\sqrt{2}q$ is proportional to the energy of a soliton. On substitution the Eqn.(4.15) becomes

$$iQ_1 - (q - \rho)Q_1 + Q_{1rr} + 2(|Q_1|^2 + B|Q_2|^2)Q_1 + \gamma(|Q_1|^2 + B|Q_2|^2)^2 Q_1 + \kappa Q_2 = 0,$$

$$iQ_2 - (q + \rho)Q_2 + Q_{2rr} + 2(|Q_2|^2 + B|Q_1|^2)Q_2 + \gamma(|Q_2|^2 + B|Q_1|^2)^2 Q_2 + \kappa Q_1 = 0. \tag{4.17}$$

The Stokes parameters are defined by [19, 20]
\( s_0 = |Q_1|^2 + |Q_2|^2, \)
\( s_1 = |Q_1|^2 - |Q_2|^2, \)
\( s_2 = Q_1^*Q_2 + Q_1Q_2^*, \)
\( s_3 = -i(Q_1^*Q_2 - Q_1Q_2^*). \)  

(4.18)

From the above equation, it is easy to see that \( s_0^2 = s_1^2 + s_2^2 + s_3^2, \) \( s = (s_1, s_2, s_3) \) can be viewed as the components of a Stokes vector, having modulus \( s_0. \) They are defined at certain points on the Poincarè sphere that identifies the polarization ellipse uniquely. The above equations are also called as differential Stokes parameters, and these four parameters are real functions of both \( z \) and \( \tau. \) Assuming that all the fields must decay to zero at infinity, using Eqn.(4.18), Eqn.(4.17) can be written as

\[
\frac{d}{dz} \int s_0 d\tau = 0, \\
\frac{d}{dz} \int s_1 d\tau = -2\kappa \int s_3 d\tau, \\
\frac{d}{dz} \int s_2 d\tau = 2\rho \int s_2 d\tau + 2(1 - B) \int s_1 s_2 d\tau + \gamma(1 - B^2) \int s_0 s_2 d\tau, \\
\frac{d}{dz} \int s_3 d\tau = 2\kappa \int s_1 d\tau - 2\rho \int s_2 d\tau - 2(1 - B) \int s_1 s_3 d\tau + \gamma(1 - B^2) \int s_0 s_3 d\tau.
\]  

(4.19)

The above equations are called as integro-differential equations of the general system. In the above equation, the first equation represents the conservation of total energy of the solitary pulse. And the remaining three integral equations give the information about the rotation of Stokes vector, which can be understood by writing the above equation in vector form.
In order to get the solution the method of separation of variables is used. In this connection, it is assumed that both the field components $Q_1$ and $Q_2$ have common average profile [5, 14]. Hence the solution for the field components $Q_1$ and $Q_2$ can be written as

$$Q_1 = X(z)f(r); \quad Q_2 = Y(z)f(r)$$

(4.20)

where $X(z)$ and $Y(z)$ are complex amplitudes and $f(r)$ is a real function defining the common profiles. With the above substitution of Eqn.(4.20) in (4.19), we obtain

$$\frac{dS_0}{dz} = 0,$$

$$\frac{dS_1}{dz} = -2\kappa S_3,$$

$$\frac{dS_2}{dz} = 2\rho S_1 + 2gS_1S_2 + \gamma hS_0S_1S_3,$$

$$\frac{dS_3}{dz} = 2\kappa S_1 - 2\rho S_2 - 2gS_2S_3 - \gamma hS_0S_1S_2.$$  

(4.21)

The above equation can be written in vectorial form

$$\frac{d\vec{S}}{dz} = 2\kappa [e_2 \times \vec{S}] + 2\rho [e_1 \times \vec{S}] + 2gS_1[e_1 \times \vec{S}] + \gamma hS_0S_1[e_1 \times \vec{S}].$$  

(4.22)

The above vector equation is used to describe the rotation of Stokes vector, which will be discussed in the following special case. Here

$\vec{S} = (S_1, S_2, S_3), \quad e_1$ and $e_2$ are unit vectors rotating along the axes x and y respectively and the integrated Stokes parameters divided by the integral

$$\int_{-\infty}^{\infty} f^2 d\tau$$

are given by,
\[ S_0 = (|X|^2 + |Y|^2), \]
\[ S_1 = (|X|^2 - |Y|^2), \]
\[ S_2 = (X^*Y + XY^*), \]
\[ S_3 = -i(X^*Y - XY^*). \] (4.23)

The above Stokes parameters are function of \( z \) alone, not on \( (z, t) \) as in the case of differential Stokes parameters. Here also, for completely polarized light, the Stokes parameters always satisfy the relation \( S_0^2 = S_1^2 + S_2^2 + S_3^2 \). In Eqn.(4.21), \( g \) and \( h \) are nonlinear birefringent coefficients, which are defined in the following manner

\[ g = \left(1 - B\right) \frac{\int f^4 dt}{\int f^2 dt} \]
\[ h = \left(1 - B^2\right) \frac{\int f^6 dt}{\int f^2 dt}. \] (4.24)

These nonlinear coefficients \( g \) and \( h \) are related to nonlinear beat length \( L_{nl} \) and is found to be proportional to these coefficients as \( L_{nl} \approx \pi g \) and \( L_{nl} \approx \pi h \).

According to Evangelides et al., [12], at any given point, the same state of polarization applies to the soliton pulse when the two field components have the same amplitude and phase profiles. Under this criteria, Eqn.(4.17) admits the following pulse envelope
\[
Q_1 = \left[ \frac{2[q-(\rho+k)]}{\sqrt[3]{4(q-\rho)^2 + 1}} \cosh \left[ 2\sqrt{(q-\rho) \tau} \right] + 1 \right]^{\gamma/2} \quad Q_2 = Q_1. \tag{4.25}
\]

If the two field components have the same amplitude and phase profiles, then the same SOP applies to the soliton pulse, at any given point throughout the fiber. However, when the two field components have the different amplitudes and phase profiles, surely different states of polarization will be present during the propagation of the soliton pulse. In the following section the polarization evolution of a soliton pulse is discussed. Because of the complicated mathematical structure of Eqn.(4.21) the following special case is considered.

4.4.1 SPECIAL CASE (B = 1 and k = 0).

To analyze the SOP in the special case, we consider B = 1 and k = 0 in Eqn.(4.21). In this case both the Stokes parameters S_0 and S_1 are conserved but the Stokes vector \( \vec{S} \) now rotates around the \( \hat{e}_1 \) axis with frequency \( \omega = 2\rho \) onto the Poincare sphere. When B = 1 and k = 0, the solution to Eqn.(4.21) is found to be of the form

\[
S_1 = S_0 \cos 2\chi,
\]
\[
S_2 = S_0 \sin 2\chi \sin (2\rho z + \phi),
\]
\[
S_3 = S_0 \sin 2\chi \cos (2\rho z + \phi). \tag{4.26}
\]
Here $2\chi$ is the ellipticity angle formed between $S$ and $e_1$. The constant of integration $\phi$ can be determined from the initial conditions. The ellipticity angle can vary from $-\pi/2$ to $+\pi/2$ including zero. Now we have the following Stokes parameters $S=(0, \pm S_0 \cos (2\rho z + \phi), \pm S_0 \sin (2\rho z + \phi))$ on the Poincaré sphere when the ellipticity angle $2\chi = \pm \pi/2$. In order to find the envelopes $Q_1$ and $Q_2$, first $X$ and $Y$ are found first and then $f$ in the following manner. Using Eqn.(4.23), $X$ and $Y$ are found to be

$$X(z) = \sqrt{\frac{S_0}{2}} \exp[i(\rho z + \phi)],$$

$$Y(z) = \sqrt{\frac{S_0}{2}} \exp[-i(\rho z + \phi)].$$

(4.27)

for $2\chi = \pm \pi/2$, where $\phi$ is the initial phase of the rotation. Similarly, equation for $f(\tau)$ is governed by

$$\left(\frac{df}{d\tau}\right)^2 = q f^2 - S_0 f^4 - \frac{\gamma S_0^2 f^6}{3}. \quad (4.28)$$

Using the standard integral functions, the solution for $f$ is found to be [21]

$$f(\tau) = \left[ \frac{2q}{\left(\frac{4}{3}(q)\gamma + 1\right) \cosh[2\sqrt(q)\tau] + 1} \right]^{\gamma/2}. \quad (4.29)$$

Therefore, the solutions in terms of $Q_1$ and $Q_2$ can be written as
\[ Q_1 = \left[ \frac{q}{\left( \frac{4}{3}(q)^\gamma + 1 \right) \cosh[2(\sqrt{q})r] + 1} \right]^{1/2} e^{i(\rho z + \phi)}, \]

\[ Q_2 = \left[ \frac{q}{\left( \frac{4}{3}(q)^\gamma + 1 \right) \cosh[2(\sqrt{q})r] + 1} \right]^{1/2} e^{-i(\rho z + \phi)}. \]  

(4.30)

4.4.2 EVOLUTION PLOTS FOR CIRCULAR CASE:

To study the polarization evolution of a soliton pulse, the above field envelopes \( Q_1 \) and \( Q_2 \) are substituted in the Stokes parameter. To analyze this evolution, the parameter values are chosen as \( q = 0.6, \rho = 0.5 \) and \( \gamma = 0.3 \). The Stokes parameter \( s_0 \) is observed as one soliton plot and the parameter \( s_1 \) is zero. The variations of one-soliton plots are observed in the remaining \( s_2 \) and \( s_3 \) parameters. These evolutions are shown in the following Figs.(4.2) and (4.3). Here it should be noted that the variations observed in the two parameters are mainly because of the presence of \( \rho \) in the exponential term. It is also observed that when the parameter \( \rho \) is equal to zero, all the Stokes parameters are found to be one soliton plot. It means that when the coefficient \( \rho \) is assumed to be absent, all the Stokes parameters remain constant for any input polarization.
Figure 4.2. Polarization evolution plots for $s_0$, $s_2$ and $s_3$
Figure 4.3. Contour plots for $s_0$, $s_2$ and $s_3$
4.4.3 LINEAR POLARIZATION

In the case of linear polarization, $2\chi = 0$, and hence $S_1 = \pm S_0$, $S_2 = 0$ and $S_3 = 0$. By using these parameters, get the relations for $X$ and $Y$ cannot be obtained. Therefore, in order to find the relations for $X$ and $Y$, Eqn. (4.26) has to be considered as it is i.e., another set of the Stokes parameters $S = [\pm S_0 \cos \theta, \pm S_0 \sin \theta \cos (2pz + \phi), \pm S_0 \sin \theta \sin (2pz + \phi)]$ on the Poincare sphere is obtained. In this case, $X$ and $Y$ are found to be,

$$X(z) = \sqrt{\frac{S_0}{2}} \cos \left(\frac{2X}{2}\right) \exp \left[i \left(\rho z + \phi\right)\right],$$

$$Y(z) = \sqrt{\frac{S_0}{2}} \sin \left(\frac{2X}{2}\right) \exp \left[-i \left(\rho z + \phi\right)\right],$$

and the solutions $Q_1$ and $Q_2$ are constructed in the form,

$$Q_1 = \sqrt{\frac{q}{\left[\sqrt{4/3(q)} + 1\right] \cosh[2(\sqrt{q})r] + 1}} \cos \left(\frac{\theta}{2}\right) e^{i(\rho z + \phi)},$$

$$Q_2 = \sqrt{\frac{q}{\left[\sqrt{4/3(q)} + 1\right] \cosh[2(\sqrt{q})r] + 1}} \sin \left(\frac{\theta}{2}\right) e^{-i(\rho z + \phi)},$$

where $\theta = 2\chi$ is the ellipticity angle which defines the relative values of the two components. The phase difference between them is denoted by $\phi$. From the above equation, it is clearly concluded that both the fields are linearly polarized.
4.4.4 EVOLUTION PLOTS FOR LINEAR POLARIZATION:

Similarly, by substituting the above field envelopes in Eqn.(4.18), the evolution is analysed by fixing the ellipticity angle arbitrarily. In this case, $s_1$, $s_2$, and $s_3$ reveal interesting results. For $\theta/2 = 0, \pi/6, \pi/3, \pi/2$ and $\pi$ values, the Stokes parameter plot is analysed. For $\theta/2 = 0, \pi$ values, parameter $s_1$ becomes both positive and negative, whose plot is observed as bright and dark soliton respectively. Further, no oscillations are observed in both $s_2$ and $s_3$ plots, i.e., all the four parameters are found to be constant during the propagation. For $\theta/2 = \pi/6, \pi/3$ values, parameter $s_1$ becomes both positive and negative, whose plot is observed as bright and dark soliton respectively. In this case, in addition to $s_0$, $s_1$ is also found to be constant during the propagation. For $\theta/2 = \pi/4$, parameter $s_1$ becomes zero and hence plot is not observed, but oscillations are observed in both $s_2$ and $s_3$ for the above mentioned value.

4.4.5 HORIZONTAL/VERTICAL POLARIZATION:

It is known that the Eqn.(4.17) admits two stationary solutions. In Eqn.(4.17), to analyze the linear polarization, it is assumed that $Q_2$ term is zero. In other words, it is evident that the $Q_1$ field components are possible only along the slow axis. Now the solution for $Q_1$ is found to be
\[ Q_1 = \left( \frac{2(q - \rho)}{\left( \frac{4}{3}(q - \rho)^2 + 1 \right) \cosh \left[ 2 \sqrt{(q - \rho)} \tau \right] + 1} \right)^{1/2}, \quad Q_2 = 0. \quad (4.33) \]

From the above equation, it is concluded the linearly polarized solitary waves are possible along the slow axis. Similarly, by making \( Q_1 \) term is zero in Eqn.(4.17), it is obtained that

\[ Q_2 = \left( \frac{2(q + \rho)}{\left( \frac{4}{3}(q + \rho)^2 + 1 \right) \cosh \left[ 2 \sqrt{(q + \rho)} \tau \right] + 1} \right)^{1/2}, \quad Q_1 = 0. \quad (4.34) \]

The above solitary wave solution is linearly polarized along the fast axis. From the above solitary wave solutions, one can also get some idea about horizontal (or) and vertical polarization of a solitary wave. It should be noted that the above said two polarizations along the slow and fast axes are possible only in the presence of birefringence. If the birefringence is absent, then the solitary wave solutions in Eqns(4.33) and (4.34), degenerate into a soliton of single NLS equation of the form

\[ \sqrt{Q_1^2 + Q_2^2} = \left[ \frac{2q}{\left( \frac{4}{3} q^2 + 1 \right) \cosh \left[ 2 \sqrt{q} \tau \right] + 1} \right]^{1/2}. \quad (4.35) \]

The above solitary wave solution can be linearly polarized along any direction in the \( Q_1 \) and \( Q_2 \) plane. The above equation also gives information about the
energy of the solitary wave. So from this equation, one can very easily understand that the energy of the solitary wave depends not only on the soliton parameter $q$, but also on quintic term $\gamma$.

4.5 SYMMETRY BREAKING INSTABILITY

Symmetry breaking instability is found to occur in the present dynamical system since they have failed to satisfy integrability properties of the soliton as has been discussed in the introduction. Kockaert et al [22] have reported the symmetry-breaking instability of the soliton bound state. Therefore in the following, the phenomenon of symmetry instability existing in the dynamical system is discussed and it is considered to be the essential requirement for the switching devices [22, 23]. At present all optical switching phenomenon is based on the spatial symmetry-breaking instability whose advantage with respect to existing switches is to operate with arbitrarily low power threshold [22].

To study the symmetry breaking instability in our dynamical system, system of Eqn.(4.17) is integrated, not by direct integration method. The following two steps are adopted. In the first step, the system of Eqn.(4.17) is reduced to the well-known anharmonic oscillator type equation without any change in the physical parameters of the defining equation. That is, the reduced anharmonic oscillator equation contains all the information of the
system under consideration. Depending on the physical parameters of the
system, the stability, instability and symmetry breaking instability in the
birefringent fiber are discussed. For this purpose, the anharmonic oscillator
equations are derived from the evolution of the Stokes parameters. This
analysis shows how the evolution of the Stokes parameter is governed by the
anharmonic oscillator equation. To derive anharmonic equation from the
evolution of the Stokes parameters, we find the invariants which are admitted
by the system of Eqns.(4.18) and (4.21) are found to be

\[
S_0^2 = S_1^2 + S_2^2 + S_3^2 = R, \\
\rho S_1 + \frac{g}{2} S_1^2 + \frac{\gamma h}{4} S_0 S_1^3 + \kappa S_2 = \Gamma. \tag{4.36}
\]

The above set of invariants determines a set of closed curves, which are
trajectories of the motion. This can be easily seen from the phase plane
diagram. By using Eqns.(4.21) and (4.36), the following nonlinear differential
type equation in terms of \( S_1 \) are obtained.

\[
\frac{d^2 S_1}{dz^2} + \alpha S_1 + \beta S_1^2 + \gamma S_1^3 = 4 \rho \Gamma, \tag{4.37}
\]

where, \( \alpha = 4(\kappa^2 + \rho^2) + 4 \Gamma g + 2 \Gamma \gamma h S_0, \)

\( \beta = 6 \rho g + 3 \gamma \rho h S_0, \)

and \( \gamma = 2g^2 + \gamma h S_0(1 + g) + \frac{\gamma^2 h^2}{2} S_0^2. \)
When the physical parameter $\rho = 0$, the exact non-driving Duffing oscillator equation is obtained. Therefore, the above nonlinear differential type equation reduces to the Duffing oscillator type equation.

$$\frac{d^2 S_1}{dz^2} + \alpha S_1 + \gamma S_1^3 = 0. \quad (4.38)$$

The above equation is the well known unforced and undamped Duffing oscillator equation and it can be solved in terms of standard Jacobian elliptic function as follows,

$$S_1(z) = B cn\{z[\gamma (\alpha_1^2 + B^2)/2]^{1/2}, B^2/(\alpha_1^2 + B^2)^{1/2}\}, \quad (4.39)$$

where $cn(u, k)$ is the Jacobian elliptic function and $B = B(R, \Gamma)$ is a constant which depends upon the initial conditions. Initially it is assumed that $S_1(0) = B$ and $dS_1/dz = 0$.

As has been discussed in the aforementioned section, both the anharmonic and unforced & undamped oscillator equations are integrated to study the stability, instability and symmetry-breaking instability of a soliton pulse in the birefringent fiber. The symmetry-breaking instability is investigated from Eqn.(4.37) and both stability and instability of a soliton in a birefringent fiber are investigated from Eqn.(4.38).
4.5.1 PHASE-PLANE DIAGRAM FOR STABILITY:

The phase-plane diagrams in Figs. (4.4a) and (4.4b) give the information about the stability of the system under consideration for the above choices of the physical parameters.
4.5.2 PHASE-PLANE DIAGRAM FOR INSTABILITY

Figures (4.5a) and (4.5b) show the instability of the dynamical system with symmetry. It means that the energy exchanges symmetrically from one mode to the other i.e. energy exchanges equally from one mode to the other.

4.5.3 PHASE-PLANE DIAGRAM FOR SYMMETRY BREAKING INSTABILITY

Contrary to the above Figs.(4.5a) and (4.5b) it is seen that, the symmetry property breaks in Figs.(4.6a) and (4.6b) mainly because of the presence of the nonlinear birefringence in the anharmonic oscillator type equations. From the above figures, it is obvious that the energy can be exchanged asymmetrically from one mode to the other in the birefringent fiber. This is the reason why symmetry breaking occurs in the dynamical system considered. Once there is
energy exchange property from one mode to other, then phenomenon of switching in the birefringent fiber can be very easily explained, which is beyond the scope of this work. Thus the possibility of symmetry breaking instability i.e. asymmetrical energy exchange between the modes in the birefringent fiber is investigated.

4.6 POLARIZATION EVOLUTION IN THE PRESENCE OF FOURTH ORDER DISPERSION

In this section the effect of FOD is analyzed using Stokes parameter formalism [24] and it is found that higher values of FOD broaden the pulse width. Lower values of the FOD reduce the pulse width, which will be very useful in optical communication. The propagation of soliton type pulse in birefringent optical fibers is modeled by CNLS equations which describe the interaction of chromatic dispersion, cubic Kerr nonlinearity, and fiber birefringence. The polarization dynamics of a soliton pulse propagating along a birefringent optical fiber with cubic nonlinear effect and fourth order dispersion can be represented by the following CNLSE equations [4, 16].

\[
\begin{align*}
\frac{i}{\beta} \frac{\partial U}{\partial z} + & \frac{1}{2} \frac{\partial^2 U}{\partial \tau^2} + \beta U + \left( |U|^2 + B |V|^2 \right) U - g \frac{\partial^4 U}{\partial \tau^4} = 0, \\
\frac{i}{\beta} \frac{\partial V}{\partial z} + & \frac{1}{2} \frac{\partial^2 V}{\partial \tau^2} - \beta V + \left( |V|^2 + B |U|^2 \right) V - g \frac{\partial^4 V}{\partial \tau^4} = 0.
\end{align*}
\]

(4.40)

Here U and V are the slowly varying envelopes of the two linearly polarized components of the field along the x and y direction, the propagation direction z is divided by the dispersion length Z₀. The parameter \( \tau \) is the normalized
retarded time represented by $T_o/1.76$, where $T_o$ is the full width at half maximum (FWHM) of a hyperbolic secant pulse. The coefficient ‘B’ refers to the ratio of cross phase modulation term to the self phase modulation term. Here $\beta$ represents the self coupling term, which represents half the difference between the propagation constant. In physical units, $\beta$ is the physical birefringence strength (in m$^{-1}$) multiplied by the dispersion length $Z_o$ (in meters). For $\beta > 0$, the anisotropy of the dispersion relations corresponds to the usual associations of the U and V polarizations with the fast and slow axes of the fiber respectively. This distinction between the fast and slow axes along with the birefringence parameter $\beta$, plays a vital role in the amplitude dynamics of the pulse. For our further investigation, the solution of Eqn.(4.40) is represented in the following form

$$U(z, \tau, q) = u(z, \tau) e^{iqz},$$

$$V(z, \tau, q) = v(z, \tau) e^{iqz}. \tag{4.41}$$

where $q$ is the soliton parameter, $2\pi/q$ is proportional to the soliton period and $2\sqrt{2}q$ is proportional to the energy of soliton. The value of cross phase modulation term $B$ in relation to the self phase modulation term is represented as

$$B = \frac{2 + 2 \sin^2(\alpha)}{2 + \cos^2(\alpha)}. \tag{4.42}$$

where $\alpha$ is the ellipticity angle which can take the values from $-\pi$ to $\pi$. When $\alpha$ is equal to $0^\circ$, it represents the linear birefringent fiber where the cross coupling parameter takes the value $B=2/3$. If $\alpha = 90^\circ$, it describes the circular
birefringent case, the parameter $B$ takes the value of 2. In this work, by applying Stokes parameter formalism to the CNLS equations, the polarization evolution of a soliton pulse for linear birefringent case is analysed.

4.6.1 STATIONARY POINTS WHEN $2 \alpha = \pm \frac{\pi}{2}$

It is a well known fact that ellipticity angle takes values from $-\pi$ to $\pi$ including zero on the Poincaré sphere. Now the stationary point for the same values of the ellipticity angle is to be calculated. When $2\alpha = 0$ and $\pm \frac{\pi}{2}$, the corresponding stationary points are found to be of the form $S = (\pm1,0,0)$ and $(0, \pm\sin(2\beta z + \Psi), \pm\cos(2\beta z + \Psi))$ respectively. Using these stationary points, it is possible to calculate $X$ and $Y$ thereby characterizing the polarization evolution on the Poincare sphere. Now the values of $X$ and $Y$ for the arbitrary values of the ellipticity angles are calculated.

$$X = \cos(\alpha) \exp(i\beta + \frac{S_1 g_1}{6} z + \Psi),$$

$$Y = i\sin(\alpha) \exp(-i(\beta + \frac{S_1 g_1}{6} z + \Psi)).$$

(4.43)

As the first set of stationary points does not allow the calculation of the values of $X$ and $Y$, the same for the latter case are calculated. In this case, the values of $X$ and $Y$ are found to be
\[ X = \frac{\exp[i(\beta z + \Psi)]}{\sqrt{2}}, \quad Y = i\left(\frac{\exp[-i(\beta z + \Psi)]}{\sqrt{2}}\right). \]  

(4.44)

\[ u(z, \tau) = X(z) f(\tau), \quad v(z, \tau) = Y(z) f(\tau). \]  

(4.45)

With the help of above Eqns.(4.44) and (4.45), the following equation is obtained from Eqn.(4.40),

\[ -q f + \frac{f_{\tau\tau}}{2} + \frac{5}{6} f^3 - g f_{\tau\tau\tau} = 0. \]  

(4.47)

On solving the above equation \( f(\tau) \) is obtained [16] to be

\[ f(\tau) = \frac{9}{\sqrt{100g}} \text{sech}^2 \left( \frac{\tau}{\sqrt{40g}} \right), \]  

(4.48)

when \( q = \frac{1}{25g} \).

---

**Figure 4.7**

\( f(\tau) \) versus the time for various values of the fourth order dispersion (FOD) coefficient \( g \)
Figure 4.7 portrays $f(t)$ versus the time for various values of the fourth order dispersion (FOD) coefficient $g$. From the plot, it is clear that as the value of fourth order dispersion increases, the effect due to the same is not negligible and it has much influence on the system. That is, for higher values of the FOD, pulse width gets broadened, which is clearly depicted in Fig.(7). But for low value of the FOD, the pulse width has been reduced considerably and during this process amplification of pulse also takes place. Thus the FOD plays an indispensable role on the system. Usually lower values of the FOD are always preferred to get the shorter pulse width, which is important in optical fiber communication system these days to increase the transmission bit rate. Recently, Pitois and Millot [25] experimentally investigated the influence of the fourth order dispersion on the onset of scalar spontaneous modulational instability in a single mode fiber. Now, the solution for Eqn.(4.46), using Eqn.(4.45) is

$$u(z,\tau) = \sqrt{\frac{g}{200}} \sec h^2 \left[ \frac{\tau}{\sqrt{40g}} \right] e^{i(\beta z + \psi)},$$

$$v(z,\tau) = i \sqrt{\frac{g}{200}} \sec h^2 \left[ \frac{\tau}{\sqrt{40g}} \right] e^{-i(\beta z + \psi)}. \quad (4.49)$$

Using Eqn.(4.41), the Eqn.(4.49) is rewritten as

$$U(z,\tau) = \sqrt{\frac{g}{200}} \sec h^2 \left[ \frac{\tau}{\sqrt{40g}} \right] e^{i((\nu+\beta)z + \psi)},$$
\[ U(z, \tau) = i \sqrt{\frac{9}{200g}} \sec h^3 \left( \frac{\tau}{\sqrt{40g}} \right) e^{i(\sigma-\beta)z-\eta}. \]  \hspace{1cm} (4.50)

when \( q = \frac{1}{25g} \).

The above equations represent the solution of Eqn.(4.40) and it is for a particular case of \( 2\alpha = \pm \frac{\pi}{2} \), which reveals the right and left circular polarizations.

### 4.7 RESULTS AND DISCUSSION

In this chapter, the pulse propagation in birefringent fiber with quintic self and cross phase modulation effects is investigated. Using the Stokes parameter formalism, the integro-differential Stokes parameters are derived to describe polarization evolution of a soliton pulse. From them, the solitary wave solutions are constructed. Also, the polarization evolution plots for the four Stokes parameters are discussed. After constructing the conserved quantities, anharmonic and unforced & undamped Duffing oscillator equation have been derived. From these oscillator equations, the stability, instability and symmetry-breaking instability of a soliton pulse in a system under consideration are investigated. Thus in this chapter, to the best of our knowledge, anharmonic and unforced & undamped Duffing oscillator equations are derived for the first time from the birefringent fiber point of view. In addition, the nonlinear pulse propagation through a linear...
birefringent optical fiber in the presence of fourth order dispersion has also been considered. From the results, it has been found that higher values of FOD, broaden the pulse width and lower values of the same reduce the pulse width which, in turn, will be useful in optical communication.
4.8 REFERENCES


