CHAPTER II

PAINLEVÉ ANALYSIS

2.1 INTRODUCTION

The prime aim of the present thesis is to study various important nonlinear phenomena in the two coupled Duffing oscillators eqs.(1.2). The first step towards the understanding of the dynamics of a given nonlinear dynamical system is to investigate its integrability. Previously, integrability of the unforced and single Duffing oscillator eqs.(1.2) ($\delta = 0, f = 0$) [37,38,129,130] and undamped and unforced two coupled Duffing oscillators eqs.(1.2) ($f = 0, d = 0$) [5,36,127,128] were reported.

In this Chapter we investigate the integrability of the unforced and damped two coupled Duffing oscillators eqs.(1.2)($f = 0$). Applying Painlevé (P) analysis we obtain integrable choices. The system is found to be integrable for two parametric choices for $d \neq 0$. To substantiate the result of the P-test an exact analytical solution is constructed for the integrable choices. The system is found to decoupled into two single damped Duffing oscillators under a change of variables. The general solution is obtained in terms of exponential and Jacobian elliptic functions.

The plan of the Chapter is as follows. To be self contained, in section 2.2 we briefly outline the salient features of P-analysis. In section 2.3 we perform the P-test to the unforced two coupled Duffing oscillators eqs.(1.2). The integrable limits are identified [132]. In section 2.4 we construct the explicit
analytical solution for the integrable cases. Section 2.5 contains conclusion.

2.2 PAINLEVÉ ANALYSIS - BASIC IDEA[4]

As mentioned in Chapter-I, an ordinary differential equation in the complex domain is said to be of P-type if all the movable singularities of the solutions are poles. A necessary condition for an $n$th order ordinary differential eq. (1.1) to have the P-property is that there is a Laurent series expansion with $(n - 1)$ arbitrary expansion coefficients. (For historical reviews on the development of P-analysis, for example, see refs.[1,3-5]). Ablowitz-Ramani-Segur (ARS) [18,19] proposed an algorithm to check whether a given system of ordinary differential equations does or does not possess the P-property.

The main assumption on which the ARS algorithm rests is that the dominant behaviour of the solutions in the neighbourhood of a movable singularity $t^*$ is of the form

$$x_i \approx a_{i0}(t - t^*)^{p_i}, \quad t \to t^*. \tag{2.1}$$

The ARS algorithm consists of the following three steps:

(i) the dominant behaviours

(ii) the resonances and

(iii) the constants of integration.

The solution of an ordinary differential equation in the neighbourhood of a singularity can be written in the form of the Laurent series as

$$x_i(t) = (t - t^*)^{p_i} \sum_{k=0}^{\infty} a_{ik} (t - t^*)^k, \tag{2.2}$$
where \( t^* \) is a pole position and the leading order \( p_i \)'s are negative integers or rational functions.

(i) DOMINANT BEHAVIOURS

The first step is to determine the leading order behaviours of \( x_i \) in the neighbourhood of a movable singularity \( t^* \) in the form (2.1), as \( t \to t^* \), \( a_{i0} = \text{constant} \). Substituting eq.(2.1) in (1.1) one can find all possible \( p_i \)'s for which two or more terms in each equation balance each other while the others can be ignored as \( t \to t^* \). For each choice of \( p_i \) the terms that can balance are called dominant terms (leading terms). If all the allowed \( p_i \)'s are negative integers then for each of them the dominant term can be viewed as the first term of the Laurent series around a movable pole. The solution may correspond to the strong \( P - property \). If any of the \( p_i \)'s is a rational fraction the algorithm is again applicable and may be related to the so called weak \( P - property \). In both the cases, the solution takes the form of the Laurent series (2.2).

(ii) RESONANCES

The leading order analysis only tells us about the behaviour of a solution at a singularity. In order to determine the behaviour in the neighbourhood of the singularity, a local expansion must be constructed. If the singularity is indeed a movable pole then this expansion will be a simple Laurent series (2.2). In this case the position \( t^* \) of the singular value of \( t \) corresponds to one of the \( n \) integration constants. For an \( n \)th order system there are still \( n - 1 \) such arbitrary constants to be sought. The powers of \( t \) at which these arbitrary
constants enter are called resonances. To find resonances, we substitute

\[ x_i \approx a_{i0} \tau^{p_i} + \Omega_i \tau^{p_i+r}, \quad r > 0, \quad i = 1, 2, ..., n \]  \hspace{1cm} (2.3)

where \( \tau = (t - t^*) \). We then retain the leading order terms in \( \Omega_i \). The reduced equation will be of the form

\[ Q(\tau) \Omega = 0, \quad \Omega = (\Omega_1, ..., \Omega_n), \]  \hspace{1cm} (2.4)

where \( Q(\tau) \) is an \( n \times n \) matrix with \( r \) appearing only in its diagonal elements. Then the resonance values are determined from the roots of the equation

\[ \det Q(\tau) = 0. \]  \hspace{1cm} (2.5)

(iii) DETERMINATION OF INTEGRATION CONSTANTS

The final step in the P-analysis is the determination of integration constants. For this purpose we substitute the truncated expansion

\[ x_i = a_{i0} \tau^{p_i} + \sum_{k=0}^{r_\tau} a_{ik} \tau^{p_i+k}, \]  \hspace{1cm} (2.6)

where \( r_\tau \) is the largest positive root of eq.(2.5). At the resonances, one usually finds some condition termed 'compatibility condition' that has to be satisfied in order to secure arbitrariness of the coefficient.
2.3 PAINLEVÉ ANALYSIS OF THE TWO COUPLED DUFFING OSCILLATORS[132]

In this section we apply the P-analysis to the unforced and damped two coupled Duffing oscillators equations. The two coupled Duffing oscillators eqs.(1.2) with \( f = 0 \) is written as

\[
\begin{align*}
\ddot{x} &= -d \dot{x} - 2A_1 x - 4\alpha_1 x^3 - 2\delta x y^2, \\
\ddot{y} &= -d \dot{y} - 2A_2 y - 4\alpha_2 y^3 - 2\delta x^2 y.
\end{align*}
\tag{2.7a/7b}
\]

2.3.1 Leading Order Behaviours

To start with, we assume the leading orders to be

\[
x \approx a_0 \tau^p, \quad y \approx b_0 \tau^q, \quad \tau = (t - t^\ast) \to 0
\tag{2.8}
\]

where \( t^\ast \) is an arbitrary singularity. To determine \( p, q, a_0 \) and \( b_0 \) we use (2.8) in (2.7) and obtain pairs of leading order equations

\[
\begin{align*}
a_0 p(p-1) \tau^{p-2} &= -da_0 p \tau^{p-1} - 2A_1 a_0 \tau^p - 4\alpha_1 a_0^3 \tau^{3p} - 2\delta a_0 b_0^2 \tau^{p+2q}, \\
b_0 q(q-1) \tau^{q-2} &= -db_0 q \tau^{q-1} - 2A_2 b_0 \tau^q - 4\alpha_2 b_0^3 \tau^{3q} - 2\delta b_0 a_0^2 \tau^{2p+q}.
\end{align*}
\tag{2.9/10}
\]

These equations immediately reveal that three different types of leading orders are possible. They are given in table (2.1).

The three different solution branches, eqs.(2.11 – 2.13) must be tested for the P-property. The next step is to perform the resonance analysis.
Table 2.1 Leading orders and the corresponding values of $a_0$ and $b_0$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Leading orders</th>
<th>Values of $a_0$ and $b_0$</th>
<th>Equation Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$p = -1$, $q = -1$</td>
<td>$a_0^p = (2\alpha_2 - \delta)/(\delta^2 - 4\alpha_1\alpha_2)$, $b_0^q = (2\alpha_1 - \delta)/(\delta^2 - 4\alpha_1\alpha_2)$</td>
<td>(2.11)</td>
</tr>
<tr>
<td>2.</td>
<td>$p = -1$, $q = \frac{1}{2}[1 + (1 + \frac{4\delta}{\alpha_1})^{\frac{1}{2}}] \geq \frac{1}{2}$</td>
<td>$a_0^p = -1/(2\alpha_1)$, $b_0^q = \text{arbitrary}$</td>
<td>(2.12)</td>
</tr>
<tr>
<td>3.</td>
<td>$p = -1$, $q = \frac{1}{2}[1 - (1 + \frac{4\delta}{\alpha_1})^{\frac{1}{2}}] &gt; -1$</td>
<td>$a_0^p = -1/(2\alpha_1)$, $b_0^q = \text{arbitrary}$</td>
<td>(2.13)</td>
</tr>
</tbody>
</table>

2.3.2 Resonances

To find the resonances, that is, the values of $r$ at which arbitrary constants will enter in the expansions of the solutions near the singularity at $t = t^*$, we write

$$x \approx a_0 r^p + \Omega_1 r^{p+r}, \quad y \approx b_0 r^q + \Omega_2 r^{q+r}. \quad (2.14)$$

We substitute (2.14) in (2.7) to obtain resonances. Retaining the leading order terms, we obtain a system of linear algebraic equation

$$M_2(r) \Omega = 0, \quad \Omega = (\Omega_1, \Omega_2), \quad (2.15)$$

where $M_2(r)$ is a $2 \times 2$ matrix dependent on $r$. In order to have a nontrivial set of solutions $(\Omega_1, \Omega_2)$ we require the determinant of $M_2(r)$ equal to zero.

**Case 1: $p = -1, q = -1$ (eq.(2.11))**

When $p = -1$ and $q = -1$ the form of $M_2(r)$ is given by

$$M_2(r) = \begin{pmatrix} (r - 1)(r - 2) + 8\alpha_1 a_0^2 - 2 & 4\delta a_0 b_0 \\ 4\delta a_0 b_0 & (r - 1)(r - 2) + 8\alpha_2 b_0^2 - 2 \end{pmatrix}. \quad (2.16)$$
Then the \( \text{det} \ M_2(r) = 0 \) leads to the equation

\[
(r^2 - 3r - 4)(r^2 - 3r - \mu) = 0,
\]

(2.17a)

where

\[
\mu = 4\left[1 + 2(\alpha_1^2 + \alpha_2^2)\right].
\]

(2.17b)

Thus the resonances, the roots of the eq. (2.17a) are

\[
r = -1, 4, \frac{1}{2}[3 \pm (9 - 4\mu)^{1/2}] .
\]

(2.18)

The root \(-1\) corresponds to the arbitrariness of \(t^*\) in (2.8). All the other resonances must have positive integer values as a necessary condition for (2.14) to be a Laurent series and (2.7) to possess the P-property. This happens for special values of \(\mu\). Equation (2.18) along with (2.11) then leads to the following two possibilities listed in the table (2.2):

<table>
<thead>
<tr>
<th>Case</th>
<th>(\mu)</th>
<th>(\alpha_1 a_0^2 + \alpha_2 b_0^2)</th>
<th>(r)</th>
<th>(\delta)</th>
<th>Equation Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(i)</td>
<td>2</td>
<td>(-1/4)</td>
<td>(-1, 1, 2, 4)</td>
<td>(2[(\alpha_1 + \alpha_2) \pm (\alpha_1^2 + \alpha_2^2 - \alpha_1 \alpha_2)^{1/2}])</td>
<td>(2.19a)</td>
</tr>
<tr>
<td>1(ii)</td>
<td>0</td>
<td>(-1/2)</td>
<td>(-1, 0, 3, 4)</td>
<td>(\sqrt{4\alpha_1 \alpha_2})</td>
<td>(2.19b)</td>
</tr>
</tbody>
</table>

**Case 2:** \(p = -1, q = \frac{1}{2}[1 + \left(1 + \frac{4\mu}{\alpha_1}\right)^{1/2}] \geq \frac{1}{2} \) (eq.(2.12))

For the leading order given by eq.(2.12), the expression for \(M_2(r)\) becomes

\[
M_2(r) = \begin{pmatrix} r^2 - 3r + 8\alpha_1 a_0^2 & 0 \\ 4\delta a_0 b_0 & r^2 + 2rq - r \end{pmatrix}.
\]

(2.20)
From \( \text{det } M_2(r) = 0 \), the resonance values are obtained as

\[
r = -1, 0, (1 - 2q), 4. \tag{2.21}
\]

In eq.(2.21), for \((1 - 2q) \geq 0\) we must have \(q \leq 1/2\). But this is in general contradictory to the leading order singularity nature, namely, \(q \geq 1/2\) (eq.(2.12)). Thus, the associated P-branch can have less number of arbitrary constants.

**Case 3:** \(p = -1, q = \frac{1}{2}[1 - (1 + (\delta/\alpha_1))^{1/2}] > -1\) (eq.(2.13))

For the leading order, eq.(2.13), \(M_2(r)\) is again given by the eq.(2.20). Now we have two distinct cases.

**Case 3(i):**

\(q = 0\) and so \(\delta = 0\), the uncoupled case.

**Case 3(ii):**

\[
q = -1/2, \quad 3\alpha_1 = 4\delta, \quad r = -1, 0, 2, 4. \tag{2.22}
\]

Thus, for the two coupled Duffing oscillators eqs.(2.7), we identify three sets of full resonances, namely (2.19a), (2.19b) and (2.22). The resonance analysis only tells us which coefficients should be arbitrary, and this has to be verified by checking the full recursion relations.

2.3.3 Identifying the Arbitrary Constants of Integration

To verify the existence of a sufficient number of arbitrary constants we
introduce the series expansion

\[ x \approx a_0 \tau^p + \sum_{k=1}^{4} a_k \tau^{p+k}, \quad y \approx b_0 \tau^q + \sum_{k=1}^{4} b_k \tau^{q+k}, \] (2.23)

in (2.7). Equating the coefficients of the powers of \((\tau^{p+k-2}, \tau^{q+k-2})\) to zero, we obtain a system of linear algebraic equations for \(a_k\) and \(b_k\). For a system containing parameters, say, \(\alpha_1, \alpha_2, \ldots\) one will often find that arbitrariness is only obtained for special values of the parameters. We will now deal with each one of the cases 1(i), 1(ii) and 3(ii) separately. For the case 1(i) we give the analysis in detail while for the remaining two cases we present main results only.

**Case 1(i): Equation (2.19a)**

The resonance values are \(r = -1, 1, 2, 4\). They imply that in addition to \(\tau^r\), three more arbitrary constants exist for the associated series expansion (2.23) of (2.7). Consequently, to satisfy P-property \(a_1(\text{or } b_1), a_2(\text{or } b_2)\) and \(a_4(\text{or } b_4)\) must be arbitrary which we verify now.

From the coefficients of \((\tau^{-2}, \tau^{-2})\) we obtain

\[ da_0b_0 - 2b_0a_1(6\alpha_1a_0^2 + \delta b_0^2) - 4\delta a_0b_0^2b_1 = 0, \] (2.24a)

\[ da_0b_0 - 2a_0b_1(6\alpha_2b_0^2 + \delta a_0^2) - 4\delta a_0^2b_0a_1 = 0. \] (2.24b)

Subtracting eq.(2.24b) from eq.(2.24a) gives

\[ a_1a_0(\delta - 6\alpha_1) + b_1b_0(\delta - 6\alpha_2) = 0. \] (2.25)

Thus, for \(a_1(\text{or } b_1)\) to be arbitrary we require \(\delta = 6\alpha_1\) (or \(6\alpha_2\)). For this choice of
δ, from (2.19a) we further find that α₁ = α₂ and hence \( a₀ = b₀ = -1/(8α₁) \). Next, from the coefficients of \((τ⁻¹, τ⁻¹)\) we obtain

\[
Aₙ a₀ - \frac{3}{2} a₂ - \frac{3}{2} b₂ + 6α₁ a₀(a₁^2 + b₁^2) + 12α₁ a₀ a₁ b₁ = 0, \tag{2.26a}
\]

\[
A₂ a₀ - \frac{3}{2} a₂ - \frac{3}{2} b₂ + 6α₁ a₀(a₁^2 + b₁^2) + 12α₁ a₀ a₁ b₁ = 0, \tag{2.26b}
\]

where we have used the choices \( a₀ = b₀, α₁ = α₂ \) and \( δ = 6α₁ \), obtained above.

Subtracting (2.26b) from (2.26a) we get

\[
(0).a₂ + (0).b₂ + a₀(A₁ - A₂) = 0. \tag{2.27}
\]

Here \( a₂ \) (or \( b₂ \)) can be arbitrary and \( A₁ = A₂ \). Further we have

\[
a₂ + b₂ = \frac{2}{3} a₀ \left( A₁ - \frac{d₂}{12} \right). \tag{2.28}
\]

In a similar manner, equating terms of order \((τ⁰, τ⁰)\) we uniquely determine the coefficients \( a₃ \) and \( b₃ \) and obtain

\[
a₃ + b₃ = \frac{1}{27} da₀ \left( 9A₁ - d² \right). \tag{2.29}
\]

Finally, from the coefficients of \((τ¹, τ¹)\) the compatibility condition for \( a₄ \) (or \( b₄ \)) to be arbitrary is

\[
2d(a₃ + b₃) = -(a₂ + b₂) \left[ A₁ + 6α₁(a₁ + b₁)^2 + 12α₁ a₀(a₂ + b₂) \right],
\]

\[
= \frac{1}{a₀} \left( a₂ + b₂ \right) \left[ \frac{d²a₀}{12} - \left( \frac{3}{2} \right)(a₂ + b₂) - A₁ a₀ \right]. \tag{2.30}
\]
Using (2.28) the right hand side of (2.30) is found to be zero. Then substituting for \((a_3 + b_3)\) from (2.29) in (2.30) we get

\[
a_0 d^2 (9A_1 - d^2) = 0. \tag{2.31}
\]

This implies that \(a_4\) (or \(b_4\)) is arbitrary for

\((i)d = 0,
(ii)d = \pm 3\sqrt{A_1}.

Thus, the Laurent series for the eqs.(2.7) with \(p = -1, q = -1\) is seen to have three arbitrary parameters besides \(t^*\). We note that the general solution of eqs.(2.7) can be characterized by four arbitrary parameters (integration constants). From the above analysis we can say that for case 1(i) the system (2.7) possesses P-property for

\[
\alpha_1 = \alpha_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = 0, \tag{2.32}
\]

\[
\alpha_1 = \alpha_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = \pm 3\sqrt{A_1}. \tag{2.33}
\]

The choice \(d = 0\) (eq.(2.32)) corresponds to the undamped anharmonic oscillator.

**Case 1(ii): Equation (2.19b)**

In this case the resonance values are \(r = -1, 0, 3, 4\) with the parametric condition \(\delta = 2\sqrt{\alpha_1\alpha_2}\). From the coefficients of \((r^{-3}, r^{-3})\) in eqs.(2.7) we obtain

\[
2\alpha_1 a_0^2 + \delta b_0^2 = -1, \tag{2.34a}
\]

\[
2\alpha_2 b_0^2 + \delta a_0^2 = -1. \tag{2.34b}
\]
For nontrivial solutions of $a_0$ and $b_0$ we must have

\[
\begin{vmatrix}
2\alpha_1 & \delta \\
2\alpha_2 & \delta \\
\end{vmatrix} = 0
\]

which sets $\alpha_1 = \alpha_2$. Subtracting eq.(2.34b) from (2.34a) we get

\[
(2\alpha_1 - \delta)a_0^2 - (2\alpha_1 - \delta)b_0^2 = 0. 
\tag{2.35}
\]

From the above equation the condition for $a_0$ (or $b_0$) to be arbitrary is $\delta = 2\alpha_1$.

Next, from the coefficients of $(r^{-2}, r^{-2})$ and $(r^{-1}, r^{-1})$ in eqs.(2.7) we determine $a_1, b_1, a_2$ and $b_2$ as

\[
a_1 = -\frac{a_0}{6}, \quad b_1 = -\frac{b_0}{6}, 
\tag{2.36}
\]

\[
a_2 = \frac{a_0}{3} \left[ \left( A_2 - \frac{d^2}{12} \right) + (A_1 - A_2)(3 + 4\alpha a_0^2) \right], 
\tag{2.37a}
\]

\[
b_2 = \frac{b_0}{3} \left[ \left( A_2 - \frac{d^2}{12} \right) + (A_1 - A_2)4\alpha a_0^2 \right]. 
\tag{2.37b}
\]

Comparing the coefficients of $(r^0, r^0)$ in (2.7) we obtain the following set of equations

\[
a_0a_3 + b_0b_3 = -\frac{d^3}{864\alpha_1}, 
\tag{2.38a}
\]

\[
(0).a_3 + (0).b_3 = d \left[ (A_1 - A_2) \left\{ 1 + \left( \frac{8\alpha^2a_0^2}{3} \right)(a_0^2 - b_0^2) - 2\alpha_1a_0^2 \right\} + \left( \frac{2\alpha_1}{3} \right)(a_0^2 - b_0^2)(A_2 - \frac{d^2}{12}) \right]. 
\tag{2.38b}
\]
Equation (2.38) implies that $a_3$ (or $b_3$) is arbitrary if

$$A_1 = A_2, \quad d^2 = 12A_2.$$  \hfill (2.39)

From the coefficients of $(r', r)$ we obtain

$$(0).a_4 + (0).b_4 = da_3(1 - \alpha_1a_0^2 - \alpha_1a_0b_0) + db_3(1 - \alpha_1b_0^2 - \alpha_1a_0b_0).$$ \hfill (2.40)

Now using (2.38) and (2.40) we can determine both the coefficients $a_3$ and $b_3$. But from the resonance condition (2.19b) we require either $a_3$ or $b_3$ as arbitrary. Because the coefficients $a_3$ and $b_3$ are fixed we conclude that the case 1(ii) is of non-P-type for $d \neq 0$. However, for $d = 0$, the right hand side of eq.(2.40) becomes zero and hence $a_4$ (or $b_4$) is arbitrary without any further restrictions on the parameters. Thus, for the case 1(ii) the system possesses P-property for

$$\alpha_1 = \alpha_2, \quad \delta = 2\alpha_1, \quad d = 0, \quad A_1 \text{ and } A_2 \text{ arbitrary.}$$ \hfill (2.41)

**Case 3(ii): Equation (2.22)**

Proceeding as before, from the coefficients of $(\tau^{-1}, \tau^{-1/2})$ in (2.7) we find that $b_2$ is arbitrary only if

$$3\alpha_1 = 4\delta, \quad d^2 + 3A_1 - 12A_2 = 0, \quad \delta^2 - 18\delta\alpha_2 + 72\alpha_2^2 = 0.$$ \hfill (2.42)

From the last condition in (2.42) we obtain $\delta = 6\alpha_2$ or $12\alpha_2$.

From the coefficients of $\tau^{-1}$ in (2.7) we obtain an equation containing terms $b_2$ and powers of $b_0$ and constant terms only. Since $b_0$ and $b_2$ are arbitrary
we equate the coefficients of $b_2$ and various powers of $b_0$ to zero separately which leads to the condition $d = 0$. That is, the case $3(ii)$ is non-P-type for $d \neq 0$.

Therefore, the case $3(ii)$ passes P-test only if

\[
3\alpha_1 = 4\delta, \quad \alpha_1 = 8\alpha_2, \quad A_1 = 4A_2, \quad d = 0.
\]  

(2.43)

\[
3\alpha_1 = 4\delta, \quad \alpha_1 = 16\alpha_2, \quad A_1 = 4A_2, \quad d = 0.
\]  

(2.44)

Thus, for $d = 0$ system (2.7) possesses P-property for four sets of parametric restrictions given by eqs.(2.32), (2.41), (2.43) and (2.44) and when $d \neq 0$, it passes the P-test only for the parameters set given by eq.(2.33).

2.4 ANALYTICAL SOLUTION FOR THE INTEGRABLE CASES

For $d = 0$, Lakshmanan and Sahadevan [27, 36, 127] explicitly constructed second integrals of motion, the first being the Hamiltonian, in order to substantiate the complete integrability. Now we construct the analytical solution of the eqs.(2.7) for the integrable choices given by eq.(2.33). For $d \neq 0$, with the choice (2.33), the system (2.7) under the transformation $u = x + y$, $v = x - y$ decouples into the following two single oscillators:

\[
\ddot{u} + 3\sqrt{A_1}\dot{u} + 2A_1u + 4\alpha_1u^3 = 0, \quad (2.45)
\]

\[
\ddot{v} + 3\sqrt{A_1}\dot{v} + 2A_1v + 4\alpha_1v^3 = 0. \quad (2.46)
\]

The solution of the above damped Duffing oscillator equation can be eas-
ily determined [38]. For positive damping and \( \alpha_1 > 0 \), under the transformation

\[
W = \sqrt{\frac{2\alpha_1}{A_1}}u \exp\left(\sqrt{A_1} t\right), \quad Z = -\sqrt{2} \exp\left(-\sqrt{A_1} t\right),
\]  

(eq. (2.45) can be reduced to

\[
\frac{d^2 W}{dZ^2} + W^3 = 0.
\]  

Equation (2.48) has a Jacobian elliptic function solution [133]

\[
W = W_0 \text{cn}(W_0 z; k), \quad z = Z - Z_0, \quad k^2 = \frac{1}{2},
\]  

where \( W_0 \) and \( Z_0 \) are integration constants. From (2.48) and (2.49) the solution of (2.45) is written as

\[
u(t) = \sqrt{\frac{A_1}{2\alpha_1}} W_0 \exp\left(\sqrt{A_1} t\right) \text{cn}(W_0 z; k),
\]  

where

\[
z = -\sqrt{2} \exp\left(-\sqrt{A_1} t\right) - Z_0.
\]  

For \( \alpha_1 < 0 \), using the transformation (2.47) with \( \alpha_1 = |\alpha_1| \), (2.45) becomes

\[
\frac{d^2 W}{dZ^2} - W^3 = 0
\]  

which has the solution [133]

\[
W = W_0/\text{cn}(W_0 z; k'),
\]  

25
where
\[ z = Z - Z_0, \quad k'^2 = 1 - k^2 = \frac{1}{2} \]  
and \( W_0, Z_0 \) are the integration constants. The solution of (2.45) can now be written as
\[ u(t) = \sqrt{\frac{A_1}{2|\alpha_1|}} W_0 \exp(\sqrt{A_1} t) [\text{cn}(W_0 z; k')]^{-1}, \]  
(2.53a)
where
\[ z = -\sqrt{2} \exp(-\sqrt{A_1} t) - Z_0. \]  
(2.53b)

The solution for the negative damping \( d = -3\sqrt{A_1} \) can be easily obtained from (2.50) and (2.53) by replacing \( t \) by \(-t\) as
\[ x(t) = \sqrt{\frac{A_1}{2|\alpha_1|}} W_0 \exp(\sqrt{A_1} t) [\text{cn}(W_0 z; k')]^{-1}, \]  
(2.54a)
where
\[ z = -\sqrt{2} \exp(\sqrt{A_1} t) - Z_0. \]  
(2.54b)

2.5 CONCLUSION

In this Chapter we have applied the P-analysis to the nonlinearly coupled, force-free Duffing oscillators. Specific sets of parameters for which the system becomes integrable are obtained. For \( d \neq 0 \) the system is found to be integrable for parametric restrictions given by eq.(2.33). For the integrable cases explicit analytical solution is constructed. For the system (2.7) with \( d = 0 \) four integrable choices were identified [27,36,127], namely,

(i) \( \alpha_1 = \alpha_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \)

(ii) \( \alpha_1 = \alpha_2, \quad \delta = 2\alpha_1, \quad A_1 \text{ and } A_2 \text{ arbitrary}, \)
(iii) $\alpha_1 = 16\alpha_2$, $\delta = 6\alpha_2$, $A_1 = 4A_2$,

(iv) $\alpha_1 = 8\alpha_2$, $\delta = 6\alpha_2$, $A_1 = 4A_2$.

However, when the damping term is added the system is found to be integrable only for $\alpha_1 = \alpha_2$, $\delta = 6\alpha_1$, $A_1 = A_2$ and $d = \pm 3\sqrt{A_1}$. In the next Chapter the invariance properties of the two coupled Duffing oscillators are studied by the method of Lie symmetries. For the uncoupled, damped and forced Duffing oscillator (eq.(2.7a) with $\delta = 0$) Parthasarathy and Lakshmanan [37] applied Painlevé analysis and found that the system is integrable for $d = \pm 3\sqrt{A_1}$, $A_1 > 0$, $\alpha_1$ arbitrary and $f = 0$. That is, the system is nonintegrable in the presence of the periodic external force. For the two coupled Duffing oscillators also one can perform P-analysis in the presence of the external periodic force $f\cos wt$. Since such analysis involves tedious calculations we have not carried out this.