CHAPTER–I

INTRODUCTION

1.1 GENERAL

The evolution of many physical systems is described by nonlinear ordinary or partial differential and difference equations depending on whether a system is continuous or discrete. During the past one decade or so, remarkable progress has been made in understanding the integrability [1-5] and nonintegrability [6-15] of nonlinear dynamical systems. Integrable systems are rather limited and in general expected to show regular behaviour whereas nonintegrable systems are capable of showing regular as well as complicated irregular motion called chaotic [6-15].

The knowledge of integrability of a differential equation or a system is very important in physics because we are interested in finding a solution to a physical problem. Integrability can be considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative information to understand the dynamics of the system globally [1-5].

Even though a unique definition of integrability is not available in mathematical literature the term integrability is often attached to dynamical systems which can be integrated sufficient number of times to obtain an exact analytic solution. A classical Hamiltonian (constant energy) system with N-degrees of freedom is said to be integrable if there exist N-independent integrals of motion whose mutual Poisson brackets vanish [4]. Equations solvable by inverse scattering transform [2,16] and satisfying Painlevé property [3-5,17-20] are also
called integrable.

An ordinary differential equation of the form

\[ \dot{x}_i = F_i(x_1, \ldots, x_n; t), \quad i = 1, 2, \ldots, n \] (1.1)

where \( F_i \)'s are rational in \( x_1, x_2, \ldots, x_n \) and analytic in \( t \) is said to have the Painlevé property if there is a Laurent series expansion with \( n - 1 \) arbitrary expansion coefficients apart from the pole position being arbitrary [18]. Weiss et al. [20] have shown how the Painlevé property could also be defined for partial differential equations. Algorithms have been developed in order to determine whether a given ordinary differential equation [19] or partial differential equation [20] has the Painlevé property. In recent times the method has been applied to a variety of dynamical systems, both Hamiltonian and non-Hamiltonian, and a considerable number of new integrable systems have been identified [3-5,20-39].

In order to identify the integrable parameters of a given dynamical system, two more techniques have widely been used, namely, Lie symmetry analysis and the direct method of finding involutive integrals of motion. Lie advocated almost a century ago the study of differential equations through invariance analysis under one-parameter continuous transformation groups associated with symmetry properties as a means of analysing them. This method has been further developed by Ovsjannikov [40], Bluman and Cole [41] and others [42,43]. The study of Lie symmetries plays a prominent role because once again it not only gives the integrable parameters of the problem but also gives the associated integrals of motion in a straightforward fashion. Integrable properties of some of the nonlinear systems have been studied through Lie symmetry
Certain nonlinear dynamical systems exhibit complicated irregular motion termed as chaotic in phase space depending on the nature of the system parameters, energy, strength of external forcing and so on. Chaos is the phenomenon of occurrence of random type evolution in completely deterministic nonlinear dynamical systems with sensitive dependence on initial states. The occurrence of chaos has been studied in many theoretical model equations, experimental systems and electronic circuits [6-15].

On many occasions chaos is a beneficial feature as it explains transition from laminar to turbulent fluid flow and multi-photon infra-red absorption. It also enhances mixing and chemical reactions and provides a vigorous mechanism for transporting heat and so on [60-62]. However there are practical situations where one wishes to avoid or control chaos so as to improve the performance of the dynamical system. For example, increasing drag in flow systems, erratic fibrillations of heart beating, complicated circuit oscillations are some situations where chaos is harmful. This is why controlling of chaos has become one of the most interesting and challenging programs in the field of nonlinear dynamics. The ability to control chaos, that is to convert chaotic oscillations into a desired regular orbit would be beneficial in working with a particular system. The possibility of purposeful selection and stabilization of specific orbits in a chaotic system using minimal predetermined efforts provides a unique opportunity to maximize the output of a dynamical system. On the other hand, the simultaneous presence of periodic orbits in a chaotic system is of great use in bringing the system from chaos to order. Interestingly, migration from a chaotic orbit to a coexisting periodic motion is possible by
migration control algorithms. It is thus of great practical importance to develop suitable control and migration methods and analyse their efficacy. In recent years, much interest has been focused on these types of problems [15,63-79].

As mentioned earlier, a characteristic property of chaotic dynamics is the extreme sensitive dependence on initial conditions. Essentially, trajectories with close initial conditions quickly become uncorrelated. A consequence of this is that two perfectly identical chaotic systems never synchronize. However, a few years ago Pecora and Carroll [80,81] showed that synchronization can be achieved for a class of chaotic systems. After their seminal work great interest has been focused on synchronization of chaotic systems and several methods of synchronization have been proposed and their applicability has been studied in many theoretical model equations and electrical systems [12,77]. A method developed for chaos synchronization is the method of one way coupling between two identical chaotic systems [82-89]. In this configuration, the response chaotic system variables follow identically the drive chaotic system variables for the appropriate one-way coupling strength. Very recently, synchronization of coupled oscillators and higher dimensional systems received considerable interest [90-100]. Synchronization has been studied in coupled lasers [90], coupled sine maps [91], pulse coupled relaxation oscillators [92], hyperchaotic systems [94,95,97], the five-dimensional Lorenz system [96] and the spatio-temporal chaotic system [98]. The new effect of phase synchronization of weakly coupled self-sustained chaotic oscillators has been found [99]. Leung [100] investigated slowing down of synchronization of limit cycle near the boundaries of synchronization domains in two van der Pol oscillators with various forms of coupling. Both controlling and synchronization of chaotic systems
have changed our outlook on chaos, paving ways for new and exciting technological applications: spread spectrum, secure communications of analog and digital signals [15,77].

In the nonlinear dynamics literature, the case of one nonlinear oscillator perturbed by a sinusoidal forcing is highly documented. Recently, the case of coupled oscillators has received growing interest [101-131]. This is due to the fact that coupled oscillators provide fundamental models for the dynamics of various physical, electromechanical, chemical and biological systems. Ordered and chaotic behaviours in a two coupled van der Pol oscillators or self-sustained oscillators [101,105,106,109-112,115-117,119,123], application of Shilnikov theorem to a van der Pol oscillator coupled to a Duffing oscillator [101], synchronization of coupled chaotic oscillators [100,120,124] and activated rate process in a double well Duffing oscillator coupled to a slow harmonic mode [125] have been studied. Denardo et al. [126] observed parametric instability in a linearly coupled unforced Duffing oscillators. The dynamics of two coupled maps has also been reported [103,104,114]. A chain of coupled Duffing oscillators was investigated by Dressler and Lauterborn [108]. Umbberger et al. [107] observed domain-like spatial structures in a chain of driven Duffing oscillators. Thus, there is a great deal of interest in the study of coupled oscillators.

Motivated by the above, we wish to investigate integrability and chaotic dynamics in two coupled, damped and periodically driven Duffing oscillators governed by the equations
\begin{align}
\ddot{x} &= -d\ddot{x} - 2A_1 x - 4\alpha_1 x^3 - 2\delta x y^2 + f_1 \cos \omega t, \quad (1.2a) \\
\ddot{y} &= -d\ddot{y} - 2A_2 y - 4\alpha_2 y^3 - 2\delta x^2 y + f_2 \cos \omega t, \quad (1.2b)
\end{align}

where $2A_i$, $\alpha_i$, $d$ and $\delta$ are natural frequency, Duffing term, damping coefficient and coupling strength respectively. The two coupled Duffing oscillators has been used to model Soret driven Bénard convection [113], vibrations of a stretched string [102], motions of nonlinear circular plates [122] and so forth [8,118,121]. The choice $d = 0$ and $f_1 = f_2 = 0$ in eqs.(1.2) reduces to the undamped and unforced two coupled anharmonic oscillators. The integrable choices of it has been studied earlier [5,27,36,127,128]. When $\delta = 0$, the system (1.2) decouples into two Duffing oscillators. The integrability of a single damped and unforced Duffing oscillator has been studied by various authors [22,37,129,130]. Elliott [102] studied the resonance behaviour and Naber-goj et al. [121] investigated the stability of nonoscillating solution in (1.2) with $f_2 = 0$. Applying the method of multiple scales Nayfeh and Vakakis [122] analysed subharmonic frequency response curves. The interaction between high and low frequency modes is analysed by Nayfeh and Nayfeh [118]. Stagliano et al. [131] found doubling bifurcations of destroyed tori.

1.2 PRESENT WORK

In the present thesis we consider the two coupled Duffing oscillators eqs.(1.2) and carry out investigation on

(i) integrability,

(ii) occurrence of chaotic dynamics,
(iii) migration from a chaotic attractor to a coexisting periodic attractor,  
(iv) synchronization of chaotic orbit and  
(v) effect of various noises.  
The thesis is organised as follows.  

1.2.1 Painlevé Analysis  
In Chapter II, we perform Painlevé analysis to the unforced \( f_1 = f_2 = 0 \) two coupled Duffing oscillators eqs.(1.2). We obtain the parametric restrictions for which the system becomes integrable. The system is found to be integrable only for the parametric restrictions \( \alpha_1 = \alpha_2, \delta = 6\alpha_1, A_1 = A_2, \) and \( d = \pm 3\sqrt{A_1} \). For the integrable cases under the transformation \( u = x + y, v = x - y \) the system (1.2) decouples into two single oscillators. Exact analytical solution involving elliptic function with exponential amplitude and argument is constructed for the decoupled system.  

1.2.2 Generalised Lie Symmetries of Two Coupled Duffing Oscillators  
In Chapter III, we present the invariance analysis for a set of coupled first order ordinary differential equations under a one-parameter Lie group of transformations. Then we investigate the invariance properties of force-free, two coupled Duffing oscillators. We treat this system as first order coupled ordinary differential equations and find out the Lie symmetries associated with the system for certain parametric choices. From Noether's theorem we identify the integrals of motion for the corresponding parametric choices. Two independent integrals of motion are found for the choices.
(i) $\alpha_1 = \alpha_2$, $\delta = 6\alpha_1$, $A_1 = A_2$, $d = \pm 3\sqrt{A_1}$.

(ii) $\alpha_1 = 16\alpha_2$, $\delta = 3\alpha_1/4$, $A_1 = A_2$, $d = \pm 3\sqrt{A_1}$ and

(iii) $16\alpha_1 = \alpha_2$, $\delta = 3\alpha_2/4$, $A_1 = A_2$, $d = \pm 3\sqrt{A_1}$.

We note that choice (i) is identified by P-analysis while the remaining two choices are not obtained from Painlevé analysis.

1.2.3 Chaotic Behaviour of a Two Coupled Duffing Oscillators

We study the occurrence of chaotic motion of the periodically driven two coupled Duffing oscillators eqs.(1.2) in Chapter IV. We consider the three physically interesting potential wells, namely,

(i) single well with infinite height potential,

(ii) potential with a hump at the centre and

(iii) single well with finite height hump potential.

We numerically study the dynamics of the system by varying the amplitude of the external forces for fixed values of the other parameters. We show the occurrence of multiple periodic attractors and their period doubling cascades. Then we study the onset of horseshoe chaos using Melnikov-analytical technique. We analyse the nature of flow on the perturbed manifold by an averaging procedure. We obtain the analytical threshold condition for horseshoe chaos and compare it with the numerical result on the onset of chaos.

1.2.4 Migration Control in a Periodically Driven Two Coupled Duffing Oscillators

Having studied bifurcation and chaotic dynamics in the system (1.2) we next consider the problem of migration from one coexisting attractor to another coexisting attractor with a special emphasis on chaos to periodic. Starting
from a chaotic dynamics we show how it is possible to bring the system to a regular motion by means of the control algorithms such as open-plus-closed-loop control method, adaptive control algorithm, Chen and Dong method and Singer-Wang-Bau feedback method. We study the applicability and efficacy of the methods by calculating the region of stable migration and the variation of the required perturbation and the variation of recovery time on the stiffness of the control signal.

1.2.5 Synchronization of Subsystems in the Two Coupled Duffing Oscillators

Chapter VI is concerned with the synchronization of subsystems of the two coupled Duffing oscillators. For a specific choice of the parameter value the system has the coexistence of six chaotic attractors. We briefly point out their characteristic features. Numerically, we construct the basin of attraction of the coexisting attractors. For two attractors it is found to be simple disconnected straight lines while for the other attractors it appears very complex. We show that phase portraits of (two)subsystems are distinct for two attractors and identical for four attractors. Of these four attractors, state variables of subsystems are perfectly synchronized for two attractors and asynchronized for the other two attractors. We investigate the possibility of synchronization of subsystems by employing a continuous feedback method. We study the efficiency of the method using synchronization time.

1.2.6 Noise Induced Jumps in the Two Coupled Duffing Oscillators

Finally, Chapter VII is devoted to the study of the influence of various external noises such as Gaussian, additive dichotomous, amplitude dependent,
uniform and Lévy. For noise level less than a critical value four chaotic attractors, one in each of the four potential wells, coexist. For a range of noise level intermittent switching between the coexisting attractors is observed. We characterize the intermittent dynamics using power spectrum, average and relative residence times on each precoupled attractor and probability distribution of time interval between successive switchings. Also, we study the variation of probability distribution of state variables in the Poincaré map as a function of noise strength. Then we compare the effect of various noises. Further, we show the occurrence of crisis-induced and type-I intermittencies in the absence of external noise.