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The Painlevé property, integrability and chaotic behaviour of a two-coupled Duffing oscillators

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Abstract. Integrability and chaotic behaviour in a two-coupled Duffing oscillators are studied. The coupling is nonlinear. Painlevé test is performed to identify integrable cases of damped- and force-free system. Exact analytical solutions are given for the integrable cases. Effect of external periodic forces for (i) single well with infinite height potential, (ii) potential with a hump at the centre and (iii) single well with finite height hump potential are numerically investigated. Occurrence of multiple attractors and period doubling cascades of coexisting attractors is presented.

Keywords. Coupled Duffing oscillators; Painlevé analysis; chaos.

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1. Introduction

The evolution of many phenomena in nature is described by nonlinear ordinary or partial differential and difference equations depending on whether the concerned system is continuous or discrete. During the last decade or so, remarkable progress has been made in understanding the integrability and nonintegrability of nonlinear dynamical systems [1–4]. Integrable systems are rather limited and are in general expected to show regular behaviour whereas nonintegrable systems are capable of showing regular as well as complicated irregular motions in phase space. The knowledge of integrability of a differential equation or a system is very important in physics because we are interested to find a solution to the physical problem. Integrable limits of several physical systems were obtained by Painlevé analysis [4–10].

An ordinary differential equation is said to have the Painlevé property if all movable singularities of the solutions are poles. The term 'strong Painlevé' is used when the solution in the neighbourhood of an arbitrary singularity \( t* \) can be expressed as \( \tau = (t - t*)^{-p} \), where \( p \) is an integer determined from the leading order, so that the movable algebraic or logarithmic branch points as well as essential singularities are excluded. Ramani, Dorizzi and Grammaticos [7] have introduced the so-called weak Painlevé property. By weak Painlevé property it is meant that the solution in the neighbourhood of the movable singularity \( t* \) can be expressed as an expansion in powers of \( \tau = (t - t*)^{-1/q} \), where \( q \) must be a natural number that depends purely on the leading order behaviour of the singularity and the nature of the potentials. Equations having the Painlevé property might be easier to integrate or solve analytically.
Recently, a series of papers [11–15] devoted to the identification of integrability of damped anharmonic oscillator

\[ \ddot{x} + f_1(t)\dot{x} + f_2(t)x + f_3(t)x^3 = 0 \]  

using the Painlevé test was published. The integrable choices of the undamped two coupled anharmonic oscillators

\[ \ddot{x} = -2A_1x - 4\alpha_1x^3 - 2\delta xy^2, \]
\[ \ddot{y} = -2A_2y - 4\alpha_2y^3 - 2\delta x^2y, \]

has also been studied by various authors [9, 16–18]. System (2) with linear damping is written as

\[ \ddot{x} = -d_1\dot{x} - 2A_1x - 4\alpha_1x^3 - 2\delta xy^2, \]
\[ \ddot{y} = -d_2\dot{y} - 2A_2y - 4\alpha_2y^3 - 2\delta x^2y, \]

where \( 2A_i = \Omega_{oi}, \alpha_i, d \) and \( \delta \) are natural frequency, Duffing term, damping coefficient and coupling strength respectively. Equation (3) models two-coupled Duffing oscillators and has been used to model Soret driven Bénard convection [19], vibrations of a stretched string [20], motions of nonlinear circular plates [21] and so forth [2, 22, 23]. Thus the study of integrable and nonintegrable properties of (3) is not only of theoretical but also of practical interest. The choice \( d = 0 \) reduces to the undamped anharmonic oscillator system. When \( \delta = 0 \) we have decoupled Duffing oscillators.

In this paper, first we wish to apply Painlevé analysis to study the integrability of (3). It is important to investigate the dynamics of the system including chaotic behaviour in the nonintegrable limit. It is well-known that for chaotic behaviour to occur, a fixed point of a dissipative continuous dynamical system must undergo a Hopf bifurcation thereby developing a limit cycle motion. As the control parameter is varied this limit cycle may bifurcate further leading to chaotic motion. Using linear stability analysis we found that the fixed points of (3) do not undergo Hopf bifurcation for any nonzero value of the parameters. Thus system (3) cannot show chaotic behaviour. However, when the system is subjected to external periodic forces a variety of interesting behaviours such as period doubling phenomenon, coexistence of multiple attractors, chaotic motion and merging of attractors occur.

The paper is organized as follows. To be self-contained, in § 2 we briefly outline the salient features of Painlevé analysis. In § 3 we perform the Painlevé test to the coupled Duffing oscillators. The integrable limits are identified. Then we obtain the explicit analytical solution for the integrable cases in § 4. Section 5 is devoted to the study of the influence of external periodic forces. The dynamics is numerically investigated by varying the amplitude of the forces for three physically interesting potentials. We show the occurrence of multiple periodic attractors and period doubling of coexisting attractors culminating in chaos. Section 6 contains conclusions.

2. Painlevé analysis

A necessary condition for an nth order ordinary differential equation of the form

\[ \dot{x}_i = F_i(x_1, \ldots, x_n, t), \quad i = 1, 2, \ldots, n, \]

where \( x_1, \ldots, x_n \) are dependent variables and \( F_i \) are functions of the variables and time.
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where $F_i$ are rational in $x_1, \ldots, x_n$ and analytic in $t$ to have the Painlevé ($P$-) property is that there is a Laurent series expansion with $(n - 1)$ arbitrary expansion coefficients. The $P$-analysis essentially consists of three steps, dealing with the dominant behaviours, the resonances and the constants of integrations, respectively [5–9].

i) Dominant behaviours: The first step is to determine the leading order behaviours of $x$ in the neighbourhood of a movable singularity $t^*$ in the form $x_i \approx a_{i0}(t - t^*)^{p_i}$, as $t \to t^*$, $a_{i0} = \text{constant}$. If all the allowed $p_i$'s are negative integers, the solution may correspond to the strong $P$-property and if any of the $p_i$'s is a rational fraction, the solution may be associated with the weak $P$-property. In either case, the solution takes the form of a Laurent series,

$$x_i(t) = (t - t^*)^{p_i} \sum_{k=0}^{\infty} a_{ik}(t - t^*)^k. \quad (5)$$

ii) Resonances: The second step is to identify the powers of (5) at which the arbitrary parameters can enter, called resonances. Apart from $t^*$, we have $(n - 1)$ other arbitrary constants for (4). To find the resonances, we substitute

$$x_i \approx a_{i0}(t - t^*)^{p_i} + \Omega_i(t - t^*)^{p_{i0}}, \quad r > 0, \quad i = 1, \ldots, n \quad (6)$$

in (4) and retain the leading order terms in $\Omega_i$. The reduced equation will be of the form

$$Q(r) \cdot \Omega = 0, \quad \Omega = (\Omega_1, \ldots, \Omega_n), \quad (7)$$

where $Q(r)$ is an $n \times n$ matrix with $r$ appearing only in its diagonal elements. Then the resonance values are determined from the roots of the equation $\det Q(r) = 0$.

iii) The constants of integration: The final step verifies that in the Laurent series (5) at the resonance values, sufficient number of arbitrary constants exist without the introduction of logarithmic branch points. To do this, we substitute the truncated expansion

$$x_i = a_{i0}(t - t^*)^{p_i} + \sum_{k=1}^{r_i} a_{ik}(t - t^*)^{p_{i0} + k}, \quad (8)$$

where $r_i$ is the largest resonance value in (4) and determines the integration constants. At the resonances, one usually finds some condition termed 'compatibility condition' that has to be satisfied in order to secure arbitrariness of the coefficient.

3. The Painlevé property of nonlinearly coupled Duffing oscillators

3.1 Leading order behaviours

Let us apply $P$-analysis to (3). To start with, we assume the leading orders be

$$x \approx a_0 \tau^p, \quad y \approx b_0 \tau^q, \quad \tau = (t - t^*) \to 0. \quad (9)$$

To determine $p, q, a_0$ and $b_0$, we use (9) in (3) and obtain pairs of leading order equations

$$a_0 p(p - 1) \tau^{p-2} = -da_0 p \tau^{p-1} - 2A_1 a_0 \tau^p - 4a_1 \tau^3 - 2A_2 a_0 a_0^2 \tau^{p+4}, \quad (10a)$$

$$b_0 q(q - 1) \tau^{q-2} = -db_0 q \tau^{q-1} - 2A_2 b_0 \tau^q - 4a_2 b_0 \tau^3 - 2A_4 b_0^2 \tau^{q+4}. \quad (10b)$$

These equations immediately reveal that three different types of leading orders are possible. These are

\[ p = -1, \quad q = -1, \quad a_0^2 = (2a_2 - \delta)/(\delta^2 - 4a_1a_2), \]
\[ b_0^2 = (2a_1 - \delta)/(\delta^2 - 4a_1a_2). \]  
\[ (11) \]

\[ p = -1, \quad q = \frac{1}{2}[1 + (1 + (4\delta/a_1))^{1/2}] \geq \frac{1}{2}, \]
\[ a_0^2 = -1/(2a_1), \quad b_0^2 = \text{arbitrary.} \]  
\[ (12) \]

\[ p = -1, \quad q = \frac{1}{2}[1 - (1 + (4\delta/a_1))^{1/2}] > -1, \]
\[ a_0^2 = -1/(2a_1), \quad b_0^2 = \text{arbitrary.} \]  
\[ (13) \]

The three different solution branches, eqs (11–13) must be tested for the P-property. The next step is to carry out a resonance analysis.

3.2 Resonances

To find the resonances, that is, the values of the order \( r \) at which arbitrary constants will enter in the expansions of the solutions near the singularity at \( t = t^* \), we write

\[ x \approx a_0 t^r + \Omega_1 t^{r+\nu}, \quad y \approx b_0 t^r + \Omega_2 t^{r+\nu}. \]  
\[ (14) \]

We substitute (14) in (3) to obtain resonances. Retaining leading order terms, we obtain a system of linear algebraic equation

\[ M_2(r)\Omega = 0, \quad \Omega = (\Omega_1, \Omega_2), \]  
\[ (15) \]

where \( M_2(r) \) is a 2 x 2 matrix dependent on \( r \). In order to have nontrivial set of solutions \((\Omega_1, \Omega_2)\) we require the determinant of \( M_2(r) \) equal to 0.

Case 1: For equation (11), the form of \( M_2(r) \) is

\[ M_2(r) = \begin{pmatrix} (r-1)(r-2) + 8a_1a_0^2 - 2 & 4\delta a_0 b_0 \\ 4\delta a_0 b_0 & (r-1)(r-2) + 8a_2b_0^2 - 2 \end{pmatrix} \]  
\[ (16) \]

which then leads to the equation

\[ (r^2 - 3r - 4)(r^2 - 3r - \mu) = 0, \quad \mu = 4[1 + 2(\alpha_1a_0^2 + \alpha_2b_0^2)]. \]  
\[ (17) \]

Thus for (11) the resonances occur at

\[ r = -1, 4, \frac{3 \pm (9 - 4\mu)^{1/2}}{2}. \]  
\[ (18) \]

The root \(-1\) corresponds to the arbitrariness of \( t^* \) in (9). All the other resonances must have positive integer values as a necessary condition for (14) to be a Laurent series and (3) to possess the P-property. This happens for special values of \( \mu \). Equation (18) along with (11) then leads to the following two possibilities:

Case 1(i): \( \mu = 2, \quad \alpha_1a_0^2 + \alpha_2b_0^2 = -1/4, \quad r = -1, 1, 2, 4, \)  
\[ (19a) \]
\[ \delta = 2[(\alpha_1 + \alpha_2) + (\alpha_1^2 + \alpha_2^2 - \alpha_1\alpha_2)^{1/2}]. \]  
\[ (19b) \]

Case 1(ii): \( \mu = 0, \quad \alpha_1a_0^2 + \alpha_2b_0^2 = -1/2, \quad r = -1, 0, 3, 4, \)  
\[ (20a) \]
\[ \delta^2 = 4\alpha_1\alpha_2. \]  
\[ (20b) \]
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The resonance analysis is already yielding valuable information. If we are interested in the \( P \)-property, this immediately restricts \( \delta \) to those values given by (19b) and (20b) for the cases 1(i) and 1(ii) respectively.

Case 2: For eqs (12) and (13), the expression for \( M_2(r) \) degenerates to

\[
M_2(r) = \begin{pmatrix} r^2 - 3r + 8\alpha_1 a_0^2 & 0 \\ 4\delta a_0 b_0 & r^2 + 2rq - r \end{pmatrix}
\]

so that from \( \det M_2(r) = 0 \), the resonance values are

\[
r = -1, 0, (1 - 2q), 4.
\]

In (22), for \((1 - 2q) \geq 0\) we must have \( q \leq \frac{1}{3} \). But this is in general contradictory to the leading order singularity nature, \( q \geq \frac{1}{2} \), eq. (12). The only consistent case \( q = \frac{1}{2} \) requires both \( a_0 \) and \( b_0 \) to be arbitrary, which is not true as seen from (12). Thus, the associated \( P \)-branch can have less number of arbitrary constants.

Case 3: Using (13) in (21), we infer two possibilities:

Case 3(i): \( q = 0 \), and so \( \delta = 0 \), the uncoupled case.

Case 3(ii): \( q = -\frac{1}{3} \), \( 3\alpha_1 = 4\delta \), \( r = -1, 0, 2, 4 \). (23)

Thus, for the coupled Duffing equation (3), we identify three sets of full resonances, namely, (19), (20) and (23). The resonance analysis only tells us which coefficients should be arbitrary, and this has to be verified by checking the full recursion relations.

3.3 Identifying the arbitrary constants of integration

To verify the existence of a sufficient number of arbitrary constants we introduce series expansion

\[ x \approx a_0 t^\alpha + \sum_{k=1}^{4} a_k t^{\alpha+k}, \quad y \approx b_0 t^\beta + \sum_{k=1}^{4} b_k t^{\beta+k}, \]  

in (3) and equating the coefficients of the powers of \((t^{\alpha+k}, t^{\beta+k})\) to zero, we obtain a system of linear algebraic equations for \( a_k \) and \( b_k \). For a system containing parameters, say, \( a, b, c, \ldots \) one will often find that arbitrariness is only obtained for special values of the system parameters. We will now deal with each one of the cases 1(i), 1(ii) and 3(ii) separately. For the case 1(i) we give the analysis in detail while for the remaining two cases we present main results only.

Case 1(i): The resonance values \( r = 1, 2, 4 \) imply that in addition to \( t^\star \), three arbitrary constants exists. Thus, for system (3) to satisfy \( P \)-property \( a_1 \) (or \( b_1 \)), \( a_2 \) (or \( b_2 \)) and \( a_4 \) (or \( b_4 \)) must be arbitrary which we verify here. From the coefficients of \((t^{-2}, t^{-2})\) we obtain

\[
da_0 b_0 - 2b_0 a_1(6\alpha_1 a_0^2 + db_0^2) - 4\delta a_0 b_0 b_1 = 0,
\]

\[
da_0 b_0 - 2a_0 b_1(6\alpha_1 b_0^2 + \delta a_0^2) - 4\delta a_0 b_0 a_1 = 0.
\]

Then, from (25), for \( a_1 \) (or \( b_1 \)) to be arbitrary we require \( \delta = 6\alpha_1 \) (or \( 6\alpha_2 \)). For this choice of \( \delta \) values from (19b) we further find that \( \alpha_1 = \alpha_2 \) and hence \( a_0^2 = b_0^2 = -1/(8\alpha_1) \). Proceeding further, for \( a_2 \) (or \( b_2 \)) to be arbitrary, from the coefficients of \((t^{-1}, t^{-1})\) we
obtain $A_1 = A_2$ and hence we have

$$a_2 + b_2 = (2/3)a_0(A_1 - d^2/12).$$

(26)

In a similar manner, equating terms of order $(\tau^0, \tau^0)$ we uniquely determine the coefficients $a_3$ and $b_3$ and easily find

$$a_3 + b_3 = (1/27)da_0(9A_1 - d^2).$$

(27)

Finally, from the coefficients of $(\tau^1, \tau^1)$ the compatibility condition for $a_4$ (or $b_4$) to be arbitrary is

$$2d(a_3 + b_3) = -(a_2 + b_2)[A_1 + 6\alpha_1(a_1 + b_1)^2 + 12\alpha_1a_0(a_2 + b_2)],$$

$$= (1/a_0)(a_2 + b_2)[d^2a_0/12 - (3/2)(a_2 + b_2) - A_1a_0].$$

(28)

Using (26) the right hand side of (28) is found to be 0. Then, substituting $(a_3 + b_3)$ from (27) in (28) we obtain

$$a_0d^2(9A_1 - d^2) = 0.$$  

(29)

This means that $a_4$ (or $b_4$) is arbitrary only for $d = 0$ and $d = \pm 3\sqrt{A_1}$. The choice $d = 0$ corresponds to the undamped anharmonic oscillator. Thus, the Laurent series (5) for the Duffing equation (3) with $p = -1$, $q = -1$ is seen to have three arbitrary parameters besides $r^*$. We know that since (3) can be rewritten as four coupled first order ordinary differential equations, its general solution is characterized by four arbitrary parameters. The above analysis shows that these are manifested in the Laurent expansion (5) by the arbitrariness of $r^*$, $a_1$ (or $b_1$), $a_2$ (or $b_2$) and $a_4$ (or $b_4$). Thus, for case 1(i) the system possesses $P$-property for

$$a_1 = a_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = 0,$$

(30)

$$a_1 = a_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = \pm 3\sqrt{A_1}.$$  

(31)

Case 1(ii): The resonance values are $r = -1, 0, 3, 4$ with the parametric condition (20b). From the leading order analysis the conditions for $a_0$ (or $b_0$) to be arbitrary are

$$a_1 = a_2, \quad \delta = 2\alpha_1.$$  

(32)

Comparing the coefficients of $(\tau^0, \tau^0)$ in (3) we obtain the following set of equations

$$a_0a_3 + b_0b_3 = -d^3/(864\alpha_1),$$

(33)

$$0a_3 + 0b_3 = d[(A_1 - A_2)(1 + (8/3)\alpha_1^2a_0^2 - 2\alpha_1a_0^2)$$

$$+ (2/3)\alpha_1(a_0^2 - b_0^2)(A_2 - d^2/12)].$$

(34)

Equation (33) implies that $a_3$ (or $b_3$) is arbitrary while (34) gives the compatibility condition in terms of the parameters of the system. For (34) to be satisfied, we require

$$A_1 = A_2, \quad d^2 = 12A_2.$$  

(35)

From the coefficients of $(\tau^1, \tau^1)$ we obtain

$$0a_4 + 0b_4 = da_3(1 - \alpha_1a_0^2 - \alpha_1a_0b_0) + db_3(1 - \alpha_1b_0^2 - \alpha_1a_0b_0).$$  

(36)
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Now, using (34) and (36) we can determine both the coefficients $a_3$ and $b_3$. But from the resonance condition (20) we require either $a_3$ or $b_3$ as arbitrary. Since the coefficients $a_3$ and $b_3$ are fixed we conclude that case 1(ii) is of non-$P$-type for $d \neq 0$. However, for $d = 0$, the right hand side of (36) become zero and so $a_4$ (or $b_4$) is arbitrary without any further restrictions on the parameters. Thus, for the case 1(ii) the system possesses $P$-property for

$$a_1 = x_2, \quad \delta = 2x_1, \quad d = 0, \quad A_1 \text{ and } A_2 \text{ arbitrary.}$$

(37)

Case 3(ii): Proceeding as before, from the coefficients of $(\tau^{-1}, \tau^{-1/2})$ in (3) we find that $b_2$ is arbitrary only if

$$3x_1 = 4\delta, \quad d^2 + 3A_1 - 12A_2 = 0, \quad \delta^2 - 18\delta x_2 + 72x_2^2 = 0.$$  

(38)

From the last condition in (38) we obtain $\delta = 6x_2$ or $12x_2$. From the coefficients of $\tau^{-1}$ in (3) we obtain an equation containing terms $b_2$, and powers of $b_0$ and constant terms only. Since $b_0$ and $b_2$ are arbitrary we equate the coefficients of $b_2$ and various powers of $b_0$ to zero separately which leads to the condition $d = 0$. That is, case 3(ii) is non-$P$-type for $d \neq 0$. Thus, case 3(ii) passes $P$-test only if

$$3x_1 = 4\delta, \quad A_1 = 4A_2, \quad \alpha_1 = 8x_2, \quad d = 0,$$

$$3x_1 = 4\delta, \quad A_1 = 4A_2, \quad \alpha_1 = 16x_2, \quad d = 0.$$  

(39)  

(40)

Thus, for $d = 0$ system (3) possesses $P$-property for four sets of parametric restrictions given by (30), (37), (39) and (40) and when $d \neq 0$, (3) passes the $P$-test only for the parameters set given by (31).

4. Analytical solution for the integrable cases

For $d = 0$, Lakshmanan and Sahadevan [9, 17] explicitly constructed second integrals of motion, the first being the Hamiltonian, in order to substantiate the complete integrability. In this section we find the explicit analytical solution for the integrable cases with $d \neq 0$, namely,

$$\alpha_1 = \alpha_2, \quad \delta = 6\alpha_1, \quad A_1 = A_2, \quad d = \pm 3\sqrt{A_1}.$$  

(41)

For the above parametric choices, system (3) under the transformation $u = x + y$, $v = x - y$ decouples into two single oscillator

$$\dot{u} + 3\sqrt{A_1}u + 2A_1u + 4\alpha_1u^3 = 0,$$

$$\dot{v} + 3\sqrt{A_1}v + 2A_1v + 4\alpha_1v^3 = 0.$$  

(42)  

(43)

For positive damping and $\alpha_1 > 0$, under the transformation

$$W = \sqrt{2x_1/A_1}u \exp(\sqrt{A_1}t), \quad Z = -\sqrt{2}\exp(-\sqrt{A_1}t),$$

eq. (42) can be reduced to

$$d^2W/dZ^2 + W^3 = 0.$$  

(45)

Equation (45) has the Jacobian elliptic function solution [24]

$$W = W_0 cn(W_0 z; k), \quad z = Z - Z_0, \quad k^2 = 1/2$$  

(46)
where \( W_0 \) and \( Z_0 \) are arbitrary integration constants. From (44) and (46) the solution of (42) is written as

\[
u(t) = \sqrt{\frac{A_1}{2x_1}} W_0 \exp(-\sqrt{A_1} t) \text{cn}(W_0 z; k),
\]

\[
z = -\sqrt{2} \exp(-\sqrt{A_1} t) - Z_0.
\]

(47)

For \( \alpha_1 < 0 \), using the transformation (44) with \( x_1 = |\alpha_1| \), (42) becomes

\[
d^2W/dZ^2 - W^3 = 0
\]

(48)

which has the solution [24]

\[
W = W_0 / \text{cn}(W_0 z; k'),
\]

(49a)

where

\[
z = Z - Z_0, \quad k'^2 = 1 - k^2 = 1/2
\]

(49b)

and \( W_0, Z_0 \) are the arbitrary constants. The solution of (42) can now be written as

\[
u(t) = \sqrt{\frac{A_1}{2x_1}} W_0 \exp\left(-\sqrt{A_1} t\right) \left[\text{cn}(W_0 z; k')\right]^{-1},
\]

\[
z = -\sqrt{2} \exp(-\sqrt{A_1} t) - Z_0.
\]

(50)

The solution for the negative damping \( d = -3\sqrt{A_1} \) can be easily obtained from (47) and (50) by replacing \( t \) by \(-t\).

5. Regular and chaotic dynamics in the coupled Duffing oscillators

In the previous two sections we were concerned with the integrability and exact solutions of the coupled Duffing oscillators in the absence of external periodic forces. In order to study the occurrence of chaotic behaviour we consider the coupled Duffing oscillators driven by external periodic forces [2], namely,

\[
\dot{x} = -d\dot{x} - 2A_1 x - 4\alpha_1 x^3 - 2\delta xy^2 + f_1 \cos \Omega_1 t,
\]

\[
\dot{y} = -d\dot{y} - 2A_2 y - 4\alpha_2 y^3 - 2\delta yx^2 + f_2 \cos \Omega_2 t.
\]

(51)

Equation (51) models a variety of physical systems [2, 20, 22, 23]. Elliott [20] studied the resonance behaviour and Nabergoj et al [23] investigated the stability of nonoscillating solution in (51) with \( f_2 = 0 \). Applying the method of multiple scales Nayfeh and Vakakis [21] analyzed subharmonic frequency response curves. The interaction between high and low frequency modes is analyzed by Nayfeh and Nayfeh [22]. Recently, using Melnikov analytical method we have studied the occurrence of homoclinic bifurcations [25].

Equation (51) can be written as

\[
\dot{x} = -d\dot{x} - \partial V(x, y)/\partial x + f_1 \cos \Omega_1 t,
\]

\[
\dot{y} = -d\dot{y} - \partial V(x, y)/\partial y + f_2 \cos \Omega_2 t,
\]

where the potential function \( V \) is given by

\[
V(x, y) = A_1 x^2 + \alpha_1 x^4 + A_2 y^2 + \alpha_2 y^4 + \delta x^2 y^2.
\]

(52)
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Throughout our analysis we assume $A_2$ and $x_2$ to have the same signs as $A_1$ and $x_1$ respectively. Further, we assume that the coupling is weak. The shape of the potential varies with the signs of $A_1, A_2, x_1, x_2$. We consider the following cases.

**Case 1:** $A_1, A_2, x_1, x_2 > 0$ — single well with infinite height potential (figure 1a).

**Case 2:** $A_1, A_2 < 0, x_1, x_2 > 0$ — potential with a hump at the centre (figure 1b).

**Case 3:** $A_1, A_2 > 0, x_1, x_2 < 0$ — single well with finite height hump potential (figure 1c).

**Case 4:** $A_1, A_2, x_1, x_2 < 0$ — inverted single well potential (figure 1d).

The nature of the solutions of the system (51) depends on the shape of the potential. In cases (1) and (2) there exists globally bounded solutions since $V(x, y) \to \infty$ as $|x|$ and $|y| \to \infty$. However, for case (3) we may have unbounded solutions of exploding amplitudes for the choices of sufficiently large initial values since $V(x, y) \to -\infty$ as $|x|$ and $|y| \to \infty$. Finally, the potential with $A_1, A_2, x_1, x_2 < 0$ is physically uninteresting since the system has an exploding amplitude $V(x, y) \to -\infty$ as $|x|$ and $|y| \to \infty$. In the following we numerically study the occurrence of regular and chaotic dynamics in (51) for the first three potentials (cases 1–3).

5.1 Single well with infinite height potential

We fix the parameters at $A_1 = 0.005, A_2 = 0.01, x_1 = 10, x_2 = 10, \delta = 0.05, d = 3(A_1)^{1/2}$ and $\Omega_1 = \Omega_2 = 1$. We choose $f_1 = f_2 = f$. The forcing amplitude $f$ is varied from a small value. From our numerical studies we find the following. Figure 2a shows the occurrence of period doubling bifurcations, chaotic motion and window regions. For characterizing the regular and chaotic motion we have calculated the maximal Lyapunov exponent ($\lambda$). Variation of $\lambda$ against $f$ is plotted in figure 2b. Initially for

![Figure 1. (a) and (b)](image-url)
small values of $f$ the system exhibits symmetrical orbit. A typical orbit is shown in figure 3a for $f = 0.2$. As the parameter $f$ is varied the symmetrical orbit loses its stability and experiences a symmetry breaking bifurcation. The resulting asymmetric orbit is shown in figure 3b for $f = 0.8$. As $f$ is further increased the asymmetrical attractor undergoes period doubling cascade to chaos. Period-1, 2, 4 and 8 orbits are found in the interval $(0-0.896), (0.896-1.016), (1.016-1.04)$ and $(1.04-1.052)$ respectively. Onset of chaos is found at $f = 1.064$. A feature of chaotic regime is the presence of windows of periodic solutions interspersed throughout the range of their existence. Period-3 window occurs for $f \in (1.364-1.652)$ in which there is no chaotic behaviour. The developed chaos disappears at $f \approx 2.072$ by a period-1 limit cycle. In the chaotic regime as the parameter $f$ is increased the size of the attractor increases gradually as shown in the bifurcation diagram (2a). In figure 4 the Poincaré map of the chaotic attractor in $x - \dot{x}$ plane is plotted for $f = 1.7$.

5.2 Potential with a hump at the centre

We now fix $A_1 = -0.5$, $A_2 = -0.055$, $x_1 = 0.25$, $x_2 = 0.025$, $d = 0.4$, $\delta = 0.025$, $\Omega_1 = \Omega_2 = 1$. For small values of $f$ coexistence of two limit cycle orbits occur. As the parameter $f$ is increased both the orbits exhibit a cascade of period doubling leading to chaotic motion. Each coexisting attractor possesses its own basin of attraction, defined as the set of initial conditions from which the system evolves to a particular orbit.
Two-coupled Duffing oscillators

Figure 2. (a) Bifurcation diagram illustrating period doubling route to chaos. (b) Maximal Lyapunov exponent against the parameter \( f \) corresponding to the figure (2a).

Figure 3. Phase portrait of (a) symmetrical orbit for \( f = 0.2 \) and (b) asymmetrical orbit for \( f = 0.8 \).

Figures 5a and 5b shows the successive bifurcations of two coexisting attractors. Interestingly, both the attractors underwent bifurcations at the same \( f \) values. For example, both the period-1 orbits bifurcate to period-2 orbits at \( f \approx 0.255 \) and period-4 at \( f \approx 0.26475 \). Chaotic motion is first observed at \( f = 0.267 \). The chaotic attractors disappeared at \( f = 0.281 \). Due to crisis two period-1 limit cycles are found. Moreover, these orbits underwent period doubling which is clearly seen in figure 5. Further, we note a sudden expansion in the size of the chaotic attractors at \( f \approx 0.2715 \) and \( 0.3088 \).
Figure 4. Poincaré map of the chaotic attractor for $f = 1.7$.

Figure 5. Figures showing successive bifurcations of two coexisting period-1 attractors.
Two-coupled Duffing oscillators

For $f > 0.3165$ cross-well chaos is observed. Here, the two chaotic attractors merge into a single attractor. This is shown in figure 6 for $f = 0.6$.

5.3 Single well with finite height hump potential

Next we consider the case $A_1, A_2 > 0$ and $\alpha_1, \alpha_2 < 0$. The dynamics of the system has been investigated for the following fixed parameters $A_1 = 0.5, A_2 = 0.55, \alpha_1 = -1, \alpha_2 = -0.975, \delta = 0.025, d = 0.4$ and $\Omega_1 = \Omega_2 = 0.526$ thereby varying $f$ as done in the other two potential well cases. In this potential well also we have found coexistence of more than one stable periodic orbit and period doubling bifurcations. In contrast to the case 2 potential where both the period-1 orbits underwent period doubling bifurcations at same $f$ value, here the attractors are found to undergo period doubling bifurcations at different $f$ values. Figure 7 shows the phase portrait of three coexisting period-1 orbits for $f = 0.1143$. When $f$ is increased the limit cycle labelled as c in figure 7 alone persists while the other two become unstable at certain $f$ values and undergo cascades of period doubling bifurcations. Bifurcations of the period-1 orbit $a$ (b) to period-2 occur at $f = 0.1143375(0.11448)$; to period-4 at $f = 0.114481(0.1146225)$; to period-8 at $f = 0.1145184(0.114645)$.

Figure 8 shows successive period doubling process leading to chaotic motion of coexisting period-1 attractors. The Poincaré map of the two coexisting chaotic attractors at $f = 0.11466$ is given in figure 9. For comparison, same scales in $x$ and $\dot{x}$ coordinates are used. For clarity, in figure 9c we show the magnification of the chaotic attractor shown in figure 9b. As the parameter $f$ is increased beyond a certain critical
Figure 7. Phase portrait of three different coexisting period-1 attractors for $f = 0.1143$. The initial conditions used are $(x, y, z) = (0, 0.35, 0, 0)$ (a), $(-0.06, 0.35, 0, 0)$ (b) and $(0.1, 0, 0, 0)$ (c).

Figure 8. Period doubling bifurcations culminating in chaos of two coexisting attractors.

value of $f$, both the chaotic attractors merge together and form a single large chaotic attractor. This is caused by a crisis in which both the attractors fuse together and form a large attractor. Figure 9d shows the Poincaré map of such a chaotic attractor. By comparing figures (9a), (9b) and (9d) we note that the large attractor is indeed a mixture.
Two-coupled Duffing oscillators

Figure 9. (a, b) Poincaré map of the two coexisting chaotic attractors for $f = 0.11466$. (c) Enlargement of the attractor shown in (b). (d) Poincaré map of the chaotic attractor for $f = 0.11482$.

of the two coexisting attractors found at lower $f$ values. The coexistence of limit cycle c along with the chaotic attractors provides a mode of physical regulation as it allows to switch to a periodic regime upon suitable perturbation.

6. Conclusions

In this paper we have applied the singularity structure analysis (Painlevé analysis) to the nonlinearly coupled Duffing oscillators. Specific sets of parameters for which the system becomes integrable are obtained. For the integrable cases explicit analytical solution is constructed. For the system (3) with $d = 0$ four integrable choices were identified (cf. eqs (30), (37), (39) and (40)), namely,

i) $\alpha_1 = x_2$, $\delta = 6x_1$, $A_1 = A_2$,
ii) $\alpha_1 = x_2$, $\delta = 2x_1$, $A_1$ and $A_2$ arbitrary,
iii) $\alpha_1 = 6x_2$, $\delta = 6x_2$, $A_1 = 4A_2$,
iv) $\alpha_1 = 8x_2$, $\delta = 6x_2$, $A_1 = 4A_2$.

However, when the damping term is added the system is found to be integrable only for $\alpha_1 = x_2$, $\delta = 6x_1$, $A_1 = A_2$ and $d = \pm 3\sqrt{A_1}$. Further, we have studied the occurrence of chaotic motion for the three potential wells for specific parameter values. Approximate
theories of nonlinear oscillation can be used to get much insight into the occurrence of chaotic motion and locate the chaotic regions in the various parameter space of (51). Chaotic attractor of (51) at critical bifurcations such as onset of chaos, band merging, crisis and intermittency can be characterized by the dynamical structure functions [26,27]. It is also of interest to study the effect of the coupling parameter \(\delta\) on both the regular and chaotic dynamics of uncoupled systems. These will be investigated in future.

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[25] S Rajasekar and S Paul Raj (submitted for publication)
Analytical prediction of horseshoe chaos in a two coupled Duffing oscillators

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Abstract: A periodically driven two coupled Duffing oscillators is considered. The unperturbed system has homoclinic orbits. Onset of horseshoe chaos in the perturbed system is investigated using Melnikov-analytical technique. The nature of flow on the perturbed manifold is studied by an averaging procedure. The dimensionality of the stable and unstable manifolds of various fixed points of the averaged equations is studied. Then, analytical threshold condition for horseshoe chaos is obtained. The analytical prediction is found to be in good agreement with the numerical estimation of onset of chaos.

Keywords: Two coupled Duffing oscillators, chaos and Melnikov method
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1. Introduction

The study of occurrence of chaotic behaviour in different parameter space of a nonlinear deterministic dynamical system is of great importance and this can be carried out using the numerical tools such as bifurcation diagram, Lyapunov exponents, power spectra analysis and so on. Such a computer based analysis requires large amount of time. However, for weakly perturbed systems the Melnikov method [1,2] is used for the prediction of horseshoe chaos analytically. In this paper, the homoclinic bifurcation in the two coupled Duffing oscillators

\[
\begin{align*}
\dot{x} &= -d_x + A_1 x - \alpha_1 x^3 - \delta x y^2 + f \cos \omega t, \\
\dot{y} &= -d_y + A_2 y - \alpha_2 y^3 - \delta x^2 y + f \cos \omega t,
\end{align*}
\]

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is studied using Melnikov method. Eq. (1) has wide range of applications [3–8]. Recently, using Painlevé analysis we have identified the parametric choices for which the system (1) is integrable and constructed exact analytical solution for the integrable cases [9].

In general, to apply the Melnikov method the given equation of motion is rewritten in the standard form

$$\dot{X} = f(X) + \varepsilon g(X,t),$$

where $X = (x_1, x_2, \ldots, x_n)$, $f = (f_1, f_2, \ldots, f_n)$, $g = (g_1, g_2, \ldots, g_n)$ and $g$ is periodic in $t$ with period $T$. The unperturbed system of (2) should contain at least one saddle and centre fixed points and an integrable separatrix solution passing through the saddle point. In eq. (1), if damping and forcing terms are chosen as the perturbations, then the unperturbed part is integrable [10,11] for four specific parametric choices only. Further, in one integrable case, the equations of motion were found to be separable and hence the choice is equivalent to the study of uncoupled Duffing oscillators. This has been noted earlier by Holmes and Marsden [12]. If the damping, coupling terms and external forces are treated as perturbations, then one can easily verify that the Melnikov function is independent of the parameter $\delta$. Alternatively, in the present paper the subsystem

$$\ddot{x} = A_1 x - \alpha_1 x^3 - \delta xy^2,$$

is considered as the unperturbed part. Accordingly, in the standard form of (2), eq. (1) can be written as

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= A_1 x_1 - \alpha_1 x_1^3 - \delta x_1 x_2^3 + \varepsilon(-dx_2 + f \cos \Theta), \\
\dot{x}_3 &= \varepsilon x_4, \\
\dot{x}_4 &= \varepsilon(A_2 x_3 - \alpha_2 x_3^3 - \delta x_1^2 x_3 - dx_4 + f \cos \Theta), \\
\dot{\Theta} &= \omega,
\end{align*}$$

where $\varepsilon$ is a small parameter. In eq. (3), one oscillator is considered as weak compared to the other. Later, we show that the Melnikov function indeed depend on all the parameters of the system including $\delta$.

The paper is organised as follows. In Section 2, first we study the nature of flow on the perturbed manifold by an averaging procedure. The dimensionality of the stable and unstable manifolds of various fixed points of the averaged equations is studied. Then, analytical threshold condition for horseshoe chaos is obtained. In Section 3, analytical prediction is compared with numerical results. Finally, Section 4, contains summary and conclusions.
Analytical prediction of horseshoe chaos etc

2. Calculation of Melnikov function

The Melnikov analysis starts with the identification of saddle fixed point and separatrix solution in the unperturbed system. The fixed points of (3) with \( \varepsilon = 0 \) are \( x_1(x_3), x_2 = 0, x_3(x_4) \), where \( x_1(x_3) \) is the roots of the equation

\[
x_1(A_1 - \alpha_1 x_1^2 - \delta x_3^2) = 0
\]

and \( x_3, x_4 \) are arbitrary. The roots of (4) are

\[
x_1 = 0, \pm \left( (A_1 - \delta x_3^2) / \alpha_1 \right)^{1/2}
\]

For \( x_3 \in (-\sqrt{A_1 / \delta}, \sqrt{A_1 / \delta}) \), there are three real roots with intermediate root corresponding to a hyperbolic and the other two are elliptic fixed points. For \( |x_3| > \sqrt{A_1 / \delta} \), there exists only one elliptic fixed point \((0, 0, x_3, x_4)\). For \( |x_3| < \sqrt{A_1 / \delta} \), the fixed point \((0, 0, x_3, x_4)\) is connected to itself by a pair of homoclinic orbits which satisfy

\[
\frac{1}{2} \left[ x_2^2 - A_1 x_1^2 + (\alpha_1 / 2) x_1^4 + \delta x_3^2 x_4^2 \right] = 0.
\]

The phase space of (3) appears as in Figure 1, where the component \( x_4 \) is suppressed for clarity. The entire picture holds for any value of \( x_4 \). The unperturbed system has hyperbolic

![Figure 1. Homoclinic orbits of the unperturbed system (3) (\( \varepsilon = 0 \)). Solid vertical line and solid dotted curve represents the solution of (4).](image-url)

invariant manifold \( M \) with boundary \( \phi(x_1(x_3), 0, x_3, x_4, \Theta_0) \), where \( x_3 \in (-\sqrt{A_1 / \delta}, \sqrt{A_1 / \delta}) \) and \( \Theta_0 \in (0, 2\pi/\omega) \).
In the perturbed system, assume that \( M \) persists as invariant manifold \( M_e \) given by

\[
M_e = (\phi(x_1(x_3)), 0, x_3, x_4) + \sigma(\varepsilon, \Theta_0).
\]  

(7)

In the unperturbed system, \( x_3 = x_4 = 0 \) and hence the manifold \( M \) is invariant and therefore, no orbits cross the boundary. In the perturbed system, \( x_3 \) and \( x_4 \) need not be zero so all orbits may leave the perturbed manifold by crossing the boundary. In this case, the nature of stable and unstable manifolds in the perturbed system and the significances of transverse intersections of stable and unstable manifolds are not well understood [2]. However, the presence of orbits homoclinic to fixed points and periodic orbits have dramatic dynamical consequences. In particular, the homoclinic orbits can provide the mechanism for the folding of the phase space. Further, the invariant sets such as fixed points and periodic orbits to which the orbit is homoclinic, can provide the mechanism for the stretching and contraction which are essential for producing chaotic motion. Thus, it is necessary to know the flow on the normally hyperbolic manifold \( M_e \) under the perturbation. The nature of flow on the perturbed \( M_e \) can be studied by the averaging procedure. If periodic orbits exist on \( M_e \), then the appropriate Melnikov integral can be computed to determine whether or not the stable and unstable manifolds of the periodic orbits intersect transversely. Thus, the next step is to determine whether \( M_e \) contains any periodic orbits. For this purpose, we consider the perturbed equations restricted to \( M_e \) given by eqs. (3c–3e). Periodic orbits of (3c–3e) in a suitable Poincaré surface of section or Poincaré map become a fixed point. So, we consider the averaged equations

\[
\dot{x}_3 = (\varepsilon / 2\pi) \int_0^{2\pi} x_4 d\Theta = \varepsilon x_4,
\]

(8a)

\[
\dot{x}_4 = (\varepsilon / 2\pi) \int_0^{2\pi} (dx_4 + A_2 x_3 - \alpha_x x^3_3 - \delta x^2_1 x_3 + f \cos \Theta) d\Theta
\]

\[
= \varepsilon(-dx_4 + A_2 x_3 - \alpha_x x^3_3 - \delta x^2_1 x_3).
\]

(8b)

The fixed points of the averaged eq. (8) corresponds to the periodic orbits of (3c–3e) of period \( 2\pi/\omega \) having the same stability type as the fixed points of the averaged equations. Further, the periodic orbits on \( M_e \) become fixed points of the four dimensional Poincaré map of eq. (3) formed by fixing \( \Theta = \Theta (= 2\pi/\omega) \). The fixed points of the averaged equations are \((x_3, x_4) = (0, 0), (\pm \sqrt{l}, 0)\), where \( l = (A_2 - \delta x^2) / \alpha_x \). When \( x_1 = 0 \), we obtain \( l = A_2 / \alpha_x \). The stability determining eigenvalues are obtained from

\[
\lambda^2 + \varepsilon \lambda = \varepsilon^2 (A_2 - 3 \alpha_x x^2_3 - \delta x^2_1 - 2 \delta x_1 x_3 (dx_1 / dx_3)) = 0.
\]

(9)

d\(x_1 / dx_3 \) can be obtained from eq. (4). From eq. (9), we found that the Poincaré map has the following structure:
(i) $d^2 > 8A_2$:

The fixed point $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ has a two dimensional stable and two dimensional unstable manifolds. The fixed points $(0, 0, \pm \sqrt{1}, 0)$ are of saddle-node type. They have one dimensional unstable and three dimensional stable manifolds.

(ii) $d^2 < 8A_2$:

The saddle point $(0, 0, 0, 0)$ has a two dimensional stable and two dimensional unstable manifolds. The fixed points $(0, 0, \pm \sqrt{1}, 0)$ are saddle-focus and possess three dimensional stable manifolds (spiralling in the $x_3, x_4$ directions) and one dimensional unstable manifolds.

Figure 2 shows the geometry of the Poincaré map where the coordinate $x_2$ is suppressed for clarity.

![Figure 2. The geometry of the Poincaré map in a three dimensional phase space by ignoring the one dimension of the state manifold for (a) $d^2 > 8A_2$ and (b) $d^2 < 8A_2$.](image)

For the system of the form

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, y_1, y_2) + \varepsilon g_1(x_1, x_2, y_1, y_2, t), \\
\dot{x}_2 &= f_2(x_1, x_2, y_1, y_2) + \varepsilon g_2(x_1, x_2, y_1, y_2, t), \\
\dot{y}_1 &= \varepsilon G_1(x_1, x_2, y_1, y_2, t), \\
\dot{y}_2 &= \varepsilon G_2(x_1, x_2, y_1, y_2, t),
\end{align*}
\]

the Melnikov function is given by [2]

\[
M(t_0) = \int_{-\infty}^{\infty} \left[ \langle DxH(X,Y), g \rangle + \langle D_fH(X,Y), G \rangle \right] (X(\tau), Y, \tau) d\tau \\
- \langle D_f(X(Y), Y), \int_{-\infty}^{\infty} G(X,Y, \tau) d\tau \rangle,
\]
where $D_z$ denotes differentiation with respect to $Z$, $X = (x_1, x_2)$, $Y = (y_1, y_2)$, $H$ is the Hamiltonian of the unperturbed system and $\bar{Y}$ is the fixed point of the averaged eq. (8). Here, $\langle f, g \rangle$ represents inner product of $f$ and $g$. The homoclinic trajectories of the unperturbed system of (3) are given by the following analytical expressions.

**Case 1.** $(x_3, x_4) = (0, 0)$:

\[ x_{1b}(\tau) = \pm \sqrt{2}\alpha_1 \text{sech}\sqrt{A_1} \tau, \quad (12a) \]
\[ x_{2b}(\tau) = \pm \sqrt{2/A_1} \alpha_1 \text{sech}\sqrt{A_1} \tau \text{tanh}\sqrt{A_1} \tau. \quad (12b) \]

**Case 2.** $(x_3, x_4) = (\pm \sqrt{7}, 0)$:

\[ x_{1b}(\tau) = \pm \sqrt{2}\alpha_1 \frac{(A_1 - \delta l)}{\alpha_1} \text{sech}\sqrt{(A_1 - \delta l)}\tau, \quad (13a) \]
\[ x_{2b}(\tau) = \pm \sqrt{2/A_1} \alpha_1 (A_1 - \delta l) \text{sech}\sqrt{(A_1 - \delta l)}\tau \text{tanh}\sqrt{(A_1 - \delta l)}\tau. \quad (13b) \]

Using the homoclinic orbits (12) and (13) in (11) and evaluating the integral, we find

**Case 1.** $(x_3, x_4) = (0, 0)$:

\[ M(t_0) = [-4aA_1^{3/2} / (3\alpha_1)] \pm f\pi \alpha_1 \sqrt{2/A_1} \text{sech}(\pi \omega (2\sqrt{A_1})) \sin \omega t_0. \quad (14) \]

**Case 2.** $(x_3, x_4) = (\pm \sqrt{7}, 0)$:

\[ M(t_0) = A + f(B \cos \omega t_0 \pm C \sin \omega t_0), \quad (15a) \]

where

\[ A = 4(A_1 - \delta l)^{3/2} [-d\alpha_1 - 4\delta^2 l + 3\delta l \alpha_2 (A_2 - \alpha_2) / (A_1 - \delta l)] / (3\alpha^2_1), \]
\[ B = [\delta \sqrt{l} \pi \omega \text{cosech}(\pi \omega (2\sqrt{A_1 - \delta l}))/\alpha_1, \]
\[ C = [\sqrt{2} \pi \omega \text{sech}(\pi \omega (2\sqrt{A_1 - \delta l}))] / \alpha_1. \quad (15b) \]

The case $(x_3, x_4) = (0, 0)$ corresponds to the uncoupled Duffing oscillator [1]. Now we shall analyse the Case 2. The necessary condition for the intersection of stable and unstable manifolds is obtained as

\[ f \geq f_M = |A| / \sqrt{B^2 + C^2}. \quad (16) \]

The sufficient condition requires the existence of simple zeros of $M(t_0)$. For $f > f_M$, $M(t_0)$ oscillates between positive and negative values indicating that the stable and unstable manifolds intersect transversely producing local chaos. The value of $f_M$ corresponds to the homoclinic tangency. The prediction of $f_M$ from eq. (16), is plotted in Figure 3 in $(\omega, f)$ parameter space. Horseshoe chaos occurs in the region above the threshold curve.

### 3. Numerical simulations

In general, the existence of horseshoe does not imply that the typical trajectories will be asymptotically chaotic. However, it can exert a dramatic influence on the behaviour of
orbits which pass close to it. In many dynamical systems, the presence of horseshoe was shown to be the starting point over which the systems undergone some of the possible routes to chaos. Consequently, the Melnikov threshold curve is considered as a lower threshold for the onset of asymptotic chaos. In view of this, we have numerically investigated the onset of chaos in eq. (3). For \( \delta = 0.1 \), the Melnikov threshold value \( f_M \) is 0.305. The other parameters values are fixed as \( d = 0.4, \alpha_1 = 1, \alpha_2 = 0.1, A_1 = 1, A_2 = 0.1 \) and \( \omega = 1.0 \). Figure 4 shows the bifurcation phenomenon as a function of the parameter \( f \).

Period doubling phenomenon leading to chaotic motion is found to occur when the parameter \( f \) is varied from small value. We denote \( f_c \) as the critical value of the parameter \( f \),

\[ \text{Analytical prediction of horseshoe chaos etc} \]
at which onset of chaos where trajectory jumps between positive and negative values of $x_1$ and $x_3$, occurs. Numerically, the onset of chaos is found to occur at $f_c = 0.32$.

To know the influence of the second oscillator (3c–3d) on the onset of chaotic dynamics of the coupled Duffing oscillators, we have studied the onset of chaos in the uncoupled Duffing oscillator

\begin{align}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= A_1 x_1 - \alpha_1 x_3 - \epsilon(-d x_1^2 + f \cos \Theta).
\end{align}

In eq. (17) the parameter values are fixed at $A_1 = 1$, $\alpha_1 = 1$, $d = 0.4$ and $\omega = 1$, the same as those used in the coupled oscillators. Figure 5 shows the threshold curves for the onset of chaos in both the uncoupled and coupled oscillators. The continuous and dashed curves represent $f_M$ for the coupled and uncoupled systems, respectively. From this figure, we note that for $\omega$ less than a critical value $\omega_c$, the $f_M$ of the coupled oscillators is lower than the uncoupled system. However for $\omega > \omega_c$, the $f_M$ value of the coupled system is higher than that of the uncoupled systems. This is further varified by numerical experiment. In the uncoupled oscillators, onset of chaos is found at $f_c = 0.307$ ($f_M = 0.3013$). This $f_c$ value can
be compared with the value $f_c = 0.32$ of the coupled oscillators. That is, onset of chaos is delayed in the coupled oscillators. In the uncoupled system, also period doubling bifurcation culminating in chaos is observed.

The nature of the chaotic attractor at $f_c$ is also studied in the coupled and uncoupled oscillators. Figure 6 shows the Poincaré map of the chaotic attractor at $f_c = 0.307$ for the uncoupled system. Figure 7 shows the attractor of the coupled systems for $f_c = 0.32$. The influence of the coupling term and second oscillator on the structure of the chaotic attractor can be clearly seen. In the uncoupled system, chaotic attractor consists of thin layer structures. The geometrical structure of the attractors of both the coupled and uncoupled systems, appear almost similar. However, in the coupled systems due to the coupling term, the points in the $x_1-x_2$ plane are distributed in the neighbourhood of the layers. On the other hand, the geometrical structure of the coupled systems in the $x_3-x_4$ plane is highly different. This is due to the small values of the parameters of the second oscillator. A detailed analysis of the prediction of onset of chaos has been formed for different sets of $\omega$ and $d$.

The results are summarized in Table 1. From this table, the analytical prediction is found to be in good agreement with the numerical analysis of the system. Since the second oscillator is treated as weak, the present analysis is applicable for small values of $A_2$, $\alpha_2$, $\delta$, $d$ and $f$.

4. Summary and conclusions

In this paper, we have applied Melnikov-analytical technique to a two coupled Duffing oscillators to predict onset of chaos. Even though the scaling of eq. (3) seems to be artificial, it is done in order to obtain the Melnikov function in terms of all the parameters including $\delta$. Interestingly, the calculated Melnikov function indeed depend on all the parameters of the system which clearly justifies the scaling introduced in (3). The influence of the second oscillator (3c–3d) on the onset of chaotic dynamics of the coupled Duffing oscillators, has also been studied. The analytical prediction is found to be in good agreement with the numerical onset of chaos in the two coupled Duffing oscillators.
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References


Migration control in two coupled Duffing oscillators

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In this paper we study the migration from one attractor to another coexisting attractor in two coupled Duffing oscillators by an open-plus-closed-loop control method and adaptive control algorithm. Suppression of chaos by these methods is also investigated. [S1063-651X(97)06904-3]
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Even though chaos is a robust phenomenon exhibited by nonlinear systems, recent investigations clearly show that chaotic motion can be controlled or directed towards a desired regular orbit by means of preassigned and small perturbations, either to the system parameters or through the addition of weak external forces. The control methods can be broadly classified into (i) feedback [1-5] and (ii) nonfeedback [6-10] methods. The feedback methods mainly aim to stabilize the suitable unstable periodic orbit (UPO) embedded in the chaotic attractor of the system. Very recently, Jackson [II] showed that directing the system from one attractor to another coexisting attractor is possible by an open-plus-closed-loop (OPCL) control. This is called migration.

In this paper we study the migration control in the two coupled Duffing oscillators

\[ x = y, \]
\[ \dot{y} = -dy + \alpha_1 x - \beta_1 x^3 - \delta x u^2 + f \cos(\omega t), \]
\[ \dot{u} = v, \]
\[ \dot{v} = -dv + \alpha_2 u - \beta_2 u^3 - \delta u x^2 + f \cos(\omega t). \]

Equation (1) has been used to model Soret-Bénard convection [12], vibration of a stretched string [13], motions of nonlinear circular plates [14], and so forth.

In Eq. (1) coexistence of multiple attractors are found for a range of values of the parameters. We illustrate the migration from one periodic orbit to another coexisting periodic orbit, and chaotic orbit to a coexisting periodic orbit. We study the migration control using OPCL and adaptive control algorithms.

For a system of the form

\[ \dot{X} = F(X), \]

where \( X = (x_1, x_2, \ldots, x_N) \) and \( F = (F_1, F_2, \ldots, F_N) \), the OPCL control is given by

\[ \dot{X} = F(X) + K(g,X,t). \]

with

\[ K(g,X,t) = S(t) \cdot g - \hat{F}(g) + [F'(g) + A](X - g). \]

where the prime denotes differentiation with respect to \( g \) and \( S(t) \) is the switching function, \( S(t) = 0 \) \((t < 0)\) and, for example, \( S(t) = 1 \) \((t > 0)\). The time \( t = 0 \) refers to the time at which control is activated. The matrix \( A \) is a constant whose eigenvalues all have negative real parts. For simplicity one can choose its elements as \( a_{ij} = a \delta_{ij} \). The function \( g(t) \) is a goal dynamics towards which \( X(t) \) would tend, that is, with the control in the long time limit we have

\[ \lim_{t \to -} \|X(t) - g(t)\| = 0. \]

If \( g(t) \) is an attractor of Eq. (2) then \( \hat{g} = F(g) \) in Eq. (3b) vanishes and, consequently, the factor \( F'(g) = dF/dg \) alone, is specifically related to the system.

The two coupled Duffing oscillators with OPCL control are written as

\[ \dot{x} = y + K_x, \]
\[ \dot{y} = -dy + \alpha_1 x - \beta_1 x^3 - \delta x u^2 + f \cos(\omega t) + K_y, \]
\[ \dot{u} = v + K_u, \]
\[ \dot{v} = -dv + \alpha_2 u - \beta_2 u^3 - \delta u x^2 + f \cos(\omega t) + K_v. \]

where \( K_x, K_y, K_u, \) and \( K_v \) are the perturbations given by Eq. (3b), introduced for migration. When the external periodic force is included, for small values of amplitude \( f \), of the force two orbits with period \( T = 2\pi/\omega \) coexist. For example, \( \alpha_1 = 1, \beta_1 = 1, \alpha_2 = 0.14, \beta_2 = 0.1, \delta = 0.05, \omega = 1, d = 0.4, \) and \( f = 0.25 \) for two period-T orbits to coexist. Figure 1 shows the transfer of the system dynamics from the limit cycle \( X_+ \) to \( X_- \), where \( X = (x,y,u,v) \).

As an interesting case we next consider the migration from chaotic motion to a coexisting periodic motion. For \( \alpha_1 = -1, \beta_1 = -4, \alpha_2 = -1.1, \beta_2 = -3.9, d = 0.4, \delta = 0.002, f = 0.1147, \) and \( \omega = 0.526 \) both chaotic and periodic attractors coexist. Suppose the system is in the chaotic state. We select the goal dynamics \( g(t) \) as the coexisting periodic orbit and fix \( a = -0.5 \). Figure 2 illustrates the migration from chaotic motion to the chosen goal orbit. In the absence of the control the system is integrated using a fourth-order Runge-Kutta method with time step \( \tau = (2\pi/\omega)/100 \) with the initial condition \( X(0) = (0.035,0.03) \). The system is allowed to evolve in a
chaotic state. Control is switched on at $t = 80(2\pi/\omega)$, with $S(t) = 1$. Figures 3(a) and 3(b) show the required perturbations. The perturbations are found to vanish once the migration to $g(t)$ is achieved. This is because the goal orbit is a particular solution of the uncontrolled system.

The system dynamics is studied with the switching function

$$S(t) = 1 - \exp(-\lambda t),$$

where $\lambda$ is a constant parameter. Desired migration is achieved for $\lambda > 0$. The efficacy of the OPCL control has been studied by calculating the recovery time $R_T = t_f - t_0$, where $t_0$ and $t_f$ are the times at which control is initiated and after which $|X(t) - g(t)|$ is always less than $10^{-3}$, respectively. $R_T$ is calculated for 200 initial conditions chosen on the chaotic attractor and then its average value is obtained. Figure 4 shows the dependence of $R_T$ on $\lambda$. As $\lambda$ is increased from zero $R_T$ decreases rapidly and approaches a constant value for higher values of $\lambda$. Migration from one attractor to another attractor can also be achieved by the adaptive control algorithm (ACA) [2,9]. The two coupled Duffing oscillators equation with the ACA is written as

$$\dot{x} = y,$$

$$\dot{y} = -dy + \alpha_1 x - \beta_1 x^3 - \delta x u^2 + f\cos(\omega t) + p(t),$$

$$\dot{u} = v,$$

where $\delta$ is a constant parameter and $p(t)$ is a constant perturbation.

FIG. 1. Migration dynamics from the limit cycle $X_+$ to $X_-$ of the two coupled Duffing oscillators by the OPCL method.

FIG. 2. Migration from chaotic motion to a coexisting periodic orbit in Eq. (1). The controlled equation is Eq. (5).

FIG. 3. Variation of the required perturbations in the controlled two coupled Duffing oscillators (5) for migration from chaotic motion to a periodic motion. In (a) continuous and dashed curves represent the perturbations $K_1$ and $K_2$, respectively. In (b) they represent $K_1$ and $K_2$, respectively.

FIG. 4. Recovery time $R_T$ vs $\lambda$ for the OPCL method.
FIG. 5. Migration from chaotic motion to a coexisting periodic orbit by the ACA.

\[ \dot{x} = -d(u + 2u - \beta u^3 - \delta x^2 + f(\omega t)), \quad (7d) \]

\[ \dot{p} = \epsilon[(x + y - u - v) - (\bar{x} + \bar{y} - \bar{u} - \bar{v})] \]

\[ = \epsilon G(x - \bar{x}), \quad (7e) \]

where \( p(t) \) is the perturbation added for migration, \( \bar{X} = (\bar{x}, \bar{y}, \bar{u}, \bar{v}) \) is the desired orbit, \( \epsilon \) is the stiffness parameter of the control, and \( G \) is a function proportional to \( (X - \bar{X}) \). The function \( G \) can be linear or nonlinear. Here we consider the linear form of \( G \). To illustrate the migration from chaotic dynamics to a coexisting periodic motion we choose \( a_1 = -1, b_1 = -4, a_2 = -1.1, b_2 = -3.9, d = 0.4, \delta = 0.002, \omega = 0.526, \) and \( f = 0.11474 \). Figure 5 shows the migration from chaotic attractor to the coexisting limit cycle for \( \epsilon = 0.002 \). The variation of the perturbation \( p(t) \) is plotted in Fig. 6. The control is switched on at \( t = 80(2\pi/\omega) \). The parameter \( p(t) \) evolves according to Eq. (7e) and adjusts its value until the desired state is reached. Once the desired migration is achieved \( p(t) \) vanishes and the control can be switched off if the condition \( X = \bar{X} \) is realized.

In general, the control mechanism is sensitive to the value of \( \epsilon \) and the form of the function \( G \). In Eq. (7) stable control to the coexisting limit cycle attractor is found to occur for \( \epsilon \) values in the interval \((-0.0035, -0.0018)\) and \((0.00023, 0.004)\). Figure 7 shows the dependence of recovery time on \( \epsilon \). We add that the recovery time shows a different characteristic behavior in the ACA (Fig. 7) compared to the OPCL (Fig. 4) method. In the two coupled Duffing oscillators, instead of \( p \), any other parameters can also be chosen for migration control.

In summary, we studied the transfer from one attractor to another coexisting attractor in the two coupled Duffing oscillators. Interestingly, migration from chaos to periodic motion is possible by both OPCL and ACA methods. Thus, the simultaneous presence of periodic orbits in a chaotic system is of great use for bringing the system from chaos to order. In the OPCL and ACA methods the required perturbation vanishes once the desired goal orbit is reached. The other existing feedback methods \([1,2,4,5]\) are primarily designed to stabilize the unstable periodic orbits embedded in the chaotic attractor, whereas, as to implement the OPCL, the desired attractor need not be embedded in the chaotic attractor. As shown in the two coupled Duffing oscillator the actual dynamics can be directed towards a goal orbit which is far away from the actual orbit. In the OPCL method migration from one attractor to a desired coexisting attractor is always guaranteed. In the case of ACA and other feedback methods \([2,4,5]\) stable control is possible only for certain range of values of the stiffness parameter \( \epsilon \), and it has to be determined either by linear stability analysis or experimentally before implementing the specific control algorithm. Further, in contrast to the linear feedback methods, where control function must be on forever, the migratory controls (OPCL and ACA) require control actions for only a limited time. That is, the control can be switched off once the system trajectory reaches the basin of attraction of the goal dynamics.

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Noise-induced jumps in two coupled Duffing oscillators

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Abstract

We consider two coupled Duffing oscillators with the coexistence of four attractors. Switching between the coexisting attractors by noises is investigated. Three types of noises such as Gaussian, amplitude dependent and dichotomous are used. We observed noise-induced extrinsic intermittency. Intermittent dynamics is characterized using power spectrum, average and relative residence times on each precoupled attractors and probability distribution of time interval between successive switchings. Variation of probability distribution of state variables in Poincaré map is studied as a function of noise strength. The effect of various noises is compared. The occurrence of crisis-induced and type-I intermittencies are also presented. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Two coupled Duffing oscillators; Noise-induced intermittency

1. Introduction

Coexistence of many different steady-state solutions has been observed in nonlinear dynamical systems for the same control parameters. It is important to know the local stability of such solutions in the presence of infinitesimal perturbations. This is because dramatic changes in the dynamics of nonlinear systems in the presence of noise can occur. A number of studies have been carried out to quantify the effect of weak external noise in nonlinear systems and many fascinating phenomena were observed. Particularly, shift in the bifurcation points [1,2], occurrence of
chaos [3,4], suppression of chaos [5–8], stochastic resonance [9–12], stochastic synchronization [13–15] switching between the coexisting attractors [16–18] have been reported.

For systems possessing a periodic attractor and being close to a saddle-node bifurcation, a transition to intermittent chaos can be induced by a small amount of external noise [3]. Stabilization of flip-flop process in the Lorenz system [7], diffusion-like spread along a limit cycle attractor in Bonhoeffer-van der Pol oscillator [19], amplification of fluctuations in Ikeda model [20] and control of a nonchaotic system [21] by external noise have been found. Arecchi et al. [17] investigated the effect of noise in a forced Duffing oscillator and found noise-induced transitions between different basins of attraction obeying simple kinetic equations. Kautz [18] observed thermally induced escape from the basin of attraction in a DC-biased Josephson junction. Kapitaniak [22] has studied the behaviour of the probability density function obtained from Fokker–Plank equation of a nonlinear system driven by random and periodic forces. Gwinn and Westervelt [16] investigated the role of multiple basins of attraction and their boundaries in the production of intermittency in a damped, driven pendulum. Jaeger and Kantz [23] studied the effect of noise on the invariant measures. Recently, Loreto et al. [24] reported intermittency in logistic map where the control parameter is switched randomly into one of the two chosen values. Nonchaotic-intermittent dynamics is observed in a Sinai problem where the mapping function describing dynamical evolution of an initial probability is chosen randomly from two nonlinear functions [25]. Noise-induced new solutions in Brusselator have been found [27].

In this paper we study noise-induced intermittent switching between the coexisting attractors in two coupled Duffing oscillators. We consider three types of noises, namely, Gaussian, amplitude dependent and dichotomous. Also, we show the occurrence of crisis-induced and Pomeau–Manneville type-I intermittencies. In the two coupled Duffing oscillators for noise strength less than a certain critical value four chaotic attractors coexist. If the initial conditions are chosen within the basin of attraction of an attractor then the long-time behaviour of the system is on the associated attractor and the lifetime of the system on it is infinite. However, at a critical noise strength intermittent switching between all the coexisting attractors is found. Intermittent dynamics is observed for a range of values of noise strength. We characterize the intermittent dynamics using power spectrum, average residence time and relative residence time on each precoupled attractors and probability distribution of state variables in Poincaré map. The power spectrum is found to scale as $1/\omega^2$ in the intermittency region for all noises. After the intermittency power spectrum scales as $1/\omega$. Average residence time on each attractors decreases exponentially with the strength of the noises.

The plan of the paper is as follows. In Section 2 we investigate the effect of additive Gaussian noise and show the occurrence of intermittent hopping between the coexisting attractors for a range of noise strength. We characterize the intermittent dynamics using various quantities mentioned above. The influence of additive dichotomous noise is considered in Section 3. Section 4 is devoted to the study of intermittency induced by amplitude dependent noise (multiplicative uniform noise). In a noise free system
we found Pomeau–Manneville type-I and crisis-induced intermittencies. In Section 5 we briefly show the occurrences of these. Finally, Section 6 contains summary and conclusions.

2. Two coupled Duffing oscillators driven by additive Gaussian noise

We consider the two coupled Duffing oscillators driven by external periodic forces and noise, namely,
\[
\begin{align*}
\ddot{x} &= -dx - \frac{\partial V(x,y)}{\partial x} + f_1 \cos \omega_1 t + \eta(t), \\
\ddot{y} &= -dy - \frac{\partial V(x,y)}{\partial y} + f_2 \cos \omega_2 t + \eta(t),
\end{align*}
\]
where the potential function \( V(x,y) \) is given by
\[
V(x,y) = A_1 \frac{x^2}{2} + A_2 \frac{x^4}{4} + A_3 \frac{y^2}{2} + A_4 \frac{y^4}{4} + \delta \frac{x^2 y^2}{2}.
\]

Eq. 1a, 1b, 1c models a variety of physical systems [27–30]. Various nonlinear phenomena have been studied in the noise free system. The noise and force free system \((f_1 = f_2 = 0, \eta = 0)\) is integrable for specific sets of parameters value [31]. For the integrable cases we have constructed explicit analytical solutions. Elliott [28] studied the resonance behaviour and Nabergoj et al. [26] investigated the stability of nonoscillating solution in Eq. 1a, 1b, 1c with \( \eta = 0 \) and \( f_2 = 0 \). Applying the method of multiple scales Nayfeh and Vakakis [32] analysed subharmonic frequency response curves. The interaction between high- and low-frequency modes is analysed by Nayfeh and Nayfeh [29]. Torus doubling in the system driven by quasiperiodic force has also been investigated [33]. Recently, we studied the migration from a chaotic attractor to a coexisting periodic attractor in Eq. 1a, 1b, 1c with \( \eta = 0 \) [34].

The shape of the potential function (1c) depends on the sign of the parameters. For \( A_1, A_2 < 0, x_1, x_2 > 0 \) the potential has four minima. We fix the parameters at \( A_1 = -0.2, A_2 = -0.25, x_1 = 0.5, x_2 = 0.5 \) and \( \delta = 0.025 \). For these parametric values the potential is a four well potential. In this study we choose \( f_1 = f_2 = f \) and \( \omega_1 = \omega_2 = \omega \). For \( \eta = 0, \omega = 1, d = 0.4 \) and for small values of \( f \) four limit cycle attractors coexist – one in each potential wells. As the parameter \( f \) is varied these attractors undergo period doubling bifurcations leading to chaotic motion. We fix \( f \) at 0.25 and include the noise term \( \eta(t) \). First, we choose \( \eta(t) \) as a Gaussian noise with mean zero and standard deviation \( \sigma \). We investigate the effect of the noise by varying \( \sigma \).

Eq. 1a, 1b, 1c is integrated using a fourth-order Runge–Kutta method with time step \( \Delta t = (2\pi/100) \) and noise is added at every time step \( \Delta t \). A Gaussian random number generator is used to represent the noise source in the numerical integration. For \( \sigma \) less than a critical value four chaotic attractors coexist with one in each potential wells. At \( \sigma = \sigma_c \) trajectories switch intermittently between the four attractors. For convenience
we denote the attractor lying in the potential well with \( x > 0, \ y > 0 \) as \( a_{++} \); \( x > 0, \ y < 0 \) as \( a_{+-} \); \( x < 0, \ y > 0 \) as \( a_{-+} \); \( x < 0, \ y < 0 \) as \( a_{--} \). In the following, we study the nature of the dynamics in the Poincaré map.

Fig. 1 shows the time series plots for three values of \( \sigma \). In this figure \( n \) denotes the drive cycles. Intermittent hopping between the attractors is clearly seen in Fig. 1a and b. At \( \sigma = \sigma_c = 0.053 \) four chaotic attractors merged and formed a single attractor via noise-induced coupling. In this intermittency region the trajectory stays on a precoupled attractor over a long time and make a jump to another precoupled attractor and so on. Consequently, as shown in Fig. 1 for \( \sigma = 0.065 \) and 0.09 state variables \( x \) and \( y \) switch irregularly between noisy bands, which correspond to the precoupled attractors.

The mechanism for the observed intermittent jumping between the attractors is that in the presence of noise system initially on an attractor is forced by the Gaussian noise to leave the attractor when the system motion is taken across the boundary of its basin of attraction. Then the system moves towards one of the other three attractors and stays there for some time and jumps to another attractor when the trajectory crosses its basin boundary and so on.

From Fig. 1a and b we note that the switching between the four chaotic bands increases with the increase in \( \sigma \). For \( \sigma = 0.16 \) intermittent dynamics disappears and the
system jumps erratically among the four potential wells. To characterise the noise-induced intermittency called extrinsic intermittency [16] we calculated the power spectrum. Different scaling behaviour of the power spectrum is found for intermittency and fully developed chaos. The numerically computed power spectrum is displayed in Fig. 2. For $\sigma = 0.09$ corresponding to intermittency the power spectrum has an approximate $1/\sigma^2$ dependence. In contrast, the power spectrum for $\sigma = 0.16$ (Fig. 2b) corresponding to Fig. 1c scales as $1/\sigma$.

Fig. 3 displays the variation of probability distribution $P(x, y)$ of the state variables $x, y$ in the Poincaré map for three values of $\sigma$. The $x$-$y$ phase space region $x \in (-1, 1), y \in (-1, 1)$ is divided into $50 \times 50$ cells. $2 \times 10^5$ data points are collected. The number
of times the trajectory visits each cell is counted. The relative probability $P_i$ of each cell is equal to $n_i/n$, where $n_i$ and $n$ are the number of points in the $i$th cell and the total number of points respectively. In Fig. 3 the four bands in the distribution correspond to the four attractors which are coupled by noise-induced jumping. For $\sigma = 0.065$, $P$ of the chaotic bands $a_{+-}$ and $a_{--}$ are relatively large compared to the other two. This is an indication that the system spends relatively longer time on $a_{+-}$ and $a_{--}$ than on $a_{++}$ and $a_{--}$. However, increase in $\sigma$ changes the probability distribution as shown in Fig. 3b and c. The $P$ of $a_{+-}$ and $a_{--}$ decrease while that of $a_{++}$ and $a_{--}$ increase with noise strength. The noise-induced coupling between the attractors added new regions of low probability that connect the four attractors. These low-probability regions are not visible in Fig. 3a for $\sigma = 0.065$, however, are clearly seen in Fig. 3b and c for $\sigma = 0.09$ and $\sigma = 0.16$, respectively. The probability of added regions increases with increase in $\sigma$. The distribution $N(P)$ of the probability [16] is also studied. To calculate $N(P)$ the region $x \in (-1, 1)$ and $y \in (-1, 1)$ is now divided into $100 \times 100$ cells. Fig. 4 shows the weighted distribution $PN(P)$ versus $P$. $PN(P)$ would be a single narrow peak at the average value of $P$ if the probability distribution is uniform on the attractor. For $\sigma = 0.065$ and 0.09 $PN(P)$ is spread over a wide range of $P$. For
$\sigma = 0.16$ the distribution of probability on the attractor is shifted to low values of $P$ and increase in $PN(P)$ is observed for low $P$.

As shown in Fig. 1 the trajectory in the Poincaré map randomly switches between the different chaotic bands. To characterize this we consider the residence time defined as the time spend by the trajectory on a chaotic band before switching to another. Before the noise-induced intermittency the residence time on each coexisting attractor is infinite. However, in the intermittency region the residence time on each chaotic bands depend on the noise strength and is randomly distributed. We calculated the average residence time $RT$ by averaging over $10^4$ residence times. Another important quantity is the relative residence time $RRT$. $RRT$ on an $i$th precoupled attractor is defined as the ratio of the total residence time on it and sum of the total residence time on all the precoupled attractors. $RRT$ is computed over $5 \times 10^6$ drive cycles. Fig. 5 shows $RT$ versus $1/(\sigma - \sigma_c)$ in $\ln-\ln$ plot. Circles represent numerical result and continuous line is the best straight line fit to the data. $RT$ of $a_{+1}$ is relatively low compared to the other chaotic bands for a range of values.
Fig. 5. The residence time $RT$ versus $1/(\sigma - \sigma_c)$ in the ln-ln scale. The plots (a), (b), (c) and (d) correspond to the chaotic bands $a_+, a_-, a_+$ and $a_-$, respectively.

Fig. 6. Relative residence time $RRT$ versus $(\sigma - \sigma_c)$.

of $\sigma$. $RT$ behaves as $\sim a \exp[b/(\sigma - \sigma_c)]$ where $a$ and $b$ are constants. We found that on $a_-, RT \sim 5.57 \exp[0.1936/(\sigma - \sigma_c)]$; on $a_+, RT \sim 0.91 \exp[1.26/(\sigma - \sigma_c)]$; on $a_-, RT \sim 2.51 \exp[0.67/(\sigma - \sigma_c)]$; on $a_-, RT \sim 2.15 \exp[0.76/(\sigma - \sigma_c)]$. Fig. 6 display the dependence of relative residence time on $\sigma$. $RRT$ on $a_-$ and $a_+$ decrease with increase in $\sigma$ while it increase on $a_+$ and $a_-$ with $\sigma$. Because of these, in Fig. 3b and c we observed decrease in $P$ on the chaotic bands $a_+$ and $a_-$ and increase in
P on the other two chaotic bands. In Fig. 5 we see that the average residence time on each chaotic bands decreases with increase in $\sigma$. On the other hand, from Fig. 6 we note that $RRT$ decreases on two chaotic bands while increases on the other two. This different behaviour is due to the relatively higher rate of decrease of average residence time on $a_{++}$ and $a_{+-}$. Also, we have calculated the time interval ($T_s$) between successive switching into a chaotic band. Fig. 7 shows the probability distribution of $T_s$ of the chaotic band $a_{++}$ for $\sigma = 0.09$. Power law distribution is found for $a_{++}$ and also for the other chaotic bands. For the chaotic bands $a_{++, a_{+-}, a_{--}}$ and $a_{--}$ the probability distributions of $T_s$ are found to scale as $0.104 \exp(-0.004T_s)$, $0.046 \exp(-0.005T_s)$, $0.075 \exp(-0.006T_s)$ and $0.135 \exp(-0.029T_s)$, respectively.

3. Intermittent dynamics induced by additive dichotomous noise

In this section we study the system (1) with the noise term $\eta(t)$ chosen as an additive dichotomous noise of the coin-toss square wave type [35-40]. In the presence of dichotomous noise stochastic relaxation [37] and exits and nonoccurrence of exits from a potential well [38] in a damped Duffing oscillator have been studied.
Broussell et al. [39] observed noise-induced transitions in transmittance curve in an experiment with a ZnSe interference filter subjected to dichotomous fluctuations.

The dichotomous noise is generated using the expression [38]

$$\eta(t) = Da_{n}, \quad [x + (n - 1)]t_0 < t \leq (x + n)t_0,$$

where $n = \ldots, -2, -1, 0, 1, 2, \ldots$ is the set of integers, $x$ is a random variable uniformly distributed between 0 and 1, $a_{n}$ are independent random variables that take on the values $-1$ and $1$ with equal probability $1/2$, $t_0$ is a parameter of the process $\eta(t)$ and $D$ is the amplitude of the noise. In our numerical simulation we fixed $t_0 = 10\Delta t$ where $\Delta t$ is the time step used in the numerical integration of Eq. 1a, 1b, 1c.

The influence of dichotomous noise in the system (1) is similar to the Gaussian noise. Intermittent switching between the co-existing chaotic attractors occurs for a range of values of $D$. Fig. 8 shows the nature of the evolution of Eq. 1a, 1b, 1c for different values of $D$. For $D = 0.02$ and $0.025$ switching dynamics is observed (Fig. 8a and b). Erratic jumping between the attractors without intermittency is found for $D = 0.05$ (Fig. 8c).

In the intermittent region, the power spectrum shown in Fig. 9a for $D = 0.025$ has $1/\omega^2$ scaling. For $D = 0.05$, (Fig. 9b) corresponding to fully developed chaos the power spectrum exhibits $1/\omega$ dependence. The probability distribution $P(x, y)$ is
Fig. 9. Power spectrum of the x-component of the system (1) with (a) $D = 0.025$ and (b) $D = 0.05$.

plotted in Fig. 10. Similarities between the Fig. 3 (with Gaussian noise) and 10 (dichotomous noise) can be clearly seen. In the dichotomous noise-induced intermittency also $P$ is relatively low on the chaotic bands $a_{++}$ and $a_{--}$ just after the critical value of $D$. However, $P$ of $a_{++}$ and $a_{--}$ increases where as that of $a_{+-}$ and $a_{-+}$ decreases with increase in the noise amplitude $D$. The variation of the average residence time is found to scale on the chaotic bands $a_{++}$, $a_{+-}$, $a_{-+}$ and $a_{--}$ as $0.754 \exp[(0.798/(D - D_c))], 1.51 \exp[(0.613/(D - D_c))], 1.23 \exp[(0.68/(D - D_c))]$ and $5.14 \exp[(0.15/(D - D_c))]$, respectively, where $D_c = 0.018$. The relative residence time is found to increase with $D$ in the intermittent region on $a_{++}$ and $a_{--}$ and decrease on the other two chaotic bands. The probability distribution of the time duration $T_S$ between successive switching into $a_{++}, a_{+-}, a_{-+}$ and $a_{--}$ is found to scale as $0.118 \exp(-0.0027T_S), 0.036 \exp(-0.00287T_S), 0.07 \exp(-0.00357T_S)$ and $0.122 \exp(-0.02237T_S)$, respectively, for $D = 0.025$. 
4. Effect of amplitude dependent noise

We consider the amplitude dependent noise (multiplicative uniform noise) given by
\[ \eta(t) = \sqrt{|x|\xi}, \]
where \( \xi \) is a random number, equally distributed in the interval \( \xi \in [-\xi_0, \xi_0] \), and \( \xi_0 \) is called the noise amplitude. The effect of the noise (3) is studied by varying the amplitude \( \xi_0 \). Since the influence of the amplitude-dependent noise is similar to the two noises considered in the previous sections, we give only a brief summary of our numerical simulation.

For \( \xi_0 < \xi_{0c} = 0.16 \) intermittent dynamics is not observed. At \( \xi_{0c} \) switching between the four attractors is initiated. Fig. 11 show the trajectory plots for various values of \( \xi_0 \). Intermittent dynamics is seen for \( \xi_0 = 0.16 \) and 0.19. In the present case also we observed decrease in \( P \) over \( a_{++} \) and \( a_{--} \) while increase in \( P \) over \( a_{+} \) and \( a_{-} \) with the noise amplitude. Further, the power spectrum is found to show \( 1/\omega^2 \) and \( 1/\omega \) scalings in the intermittency and fully developed chaotic regions, respectively.
The coupled attractors, smeared out more and more with the increase in the amplitude of the noise. The average residence time on the chaotic bands $a_{++}$, $a_{+-}$, $a_{-+}$ and $a_{--}$ as $1.083 \exp(0.08/(\xi_0 - \xi_0))$, $7.736 \exp(0.446/(\xi_0 - \xi_0))$, $8.683 \exp(0.33/(\xi_0 - \xi_0))$ and $8.48 \exp(0.42/(\xi_0 - \xi_0))$, respectively. The probability distribution of the time interval $T_S$ between successive switching into $a_{++}, a_{+-}, a_{-+}$ and $a_{--}$ found to scale as $0.113 \exp(-0.0016T_S)$, $0.0255 \exp(-0.0018T_S)$, $0.0661 \exp(-0.0027T_S)$ and $0.1714 \exp(-0.0257T_S)$, respectively, for $\xi_0 = 0.19$.

5. Intrinsic intermittencies

So far we have studied the intermittent dynamics in the system (1) driven by external noises. Such noise-induced hopping between coexisting attractors is called extrinsic intermittency [16]. On the other hand, intrinsic intermittency occurs in the absence of external noise and is associated with the bifurcations of nonlinear systems as parameters values are varied. This includes Pomeau–Manneville and crisis-induced intermittencies. In addition to the extrinsic intermittent dynamics the occurrence of intrinsic intermittency has also been studied in the system (1). In this section we give an example of two intrinsic intermittencies in system (1) with $\eta = 0$. 

![Time series plots of the system (1) with amplitude-dependent noise. The values of $\xi_0$ are (a) $\xi_0 = 0.16$, (b) $\xi_0 = 0.19$ and (c) $\xi_0 = 0.35$.](image-url)
Fig. 12. Time series plots of the system (1) without external noise but driven by crisis-induced intermittency. The parameter values are $A_1 = -0.2$, $A_2 = -0.25$, $z_1 = z_2 = 1$, $\delta = 0.025$, $d = 0.4$ and $\omega = 1$ and (a) $f = 0.2198$, (b) $f = 0.22$ and (c) $f = 0.25$. Here $x$ value always negative and $y$ value jumps between positive and negative values.

We consider the system (1) with $A_1 = -0.2$, $A_2 = -0.25$, $z_1 = z_2 = 1$, $\delta = 0.025$, $d = 0.4$ and $\omega = 1$. For $f$ values just below $f_c = 0.2187$ four chaotic attractors coexist. At $f = f_c$, crisis-induced coupling between the attractors is observed. Fig. 12 shows the trajectory plots for three values of $f$. In all the Fig. 12a–c $x$ value is always positive while $y$ value jumps between positive and negative values. Here the initial conditions is chosen on $a_{++}$ attractor. Fig. 13 display the trajectory plots where the initial conditions is taken on $a_{--}$ attractor. We observe that, in contrast to the previous case, $x$ is now always negative and $y$ value jumps randomly between positive and negative values. We observe the hoppings between $a_{++}$ and $a_{+-}$ and $a_{+-}$ and $a_{--}$ alone. That is, all the coexisting attractors are not coupled, only pair of the attractors are coupled by the crisis. In the crisis-induced intermittency region ($f = 0.2198$) also the power spectrum is found to scales as $\sim 1/\omega^2$, $1/\omega$ dependence is found after the intermittency region ($f = 0.25$).

Pomeau–Manneville type-I intermittency is also observed in the system (1) in the absence of noise. For $f$ values just below $f_c = 0.42266$ the system (1) has a period-3 attractor. At $f = f_c$ intermittent dynamics is developed. Fig. 14 depict $y$ versus drive cycle for three values of $f$. For $f = 0.42266$ laminar region with period-3 interrupted by chaotic bursts occurs. In the chaotic region we can see two chaotic bands one with $y < 0$. 

[Diagram of time series plots and trajectories]
and another with $y>0$. That is, type-I as well as crisis-induced intermittent coupling of chaotic bands occurs. As shown in Fig. 14b for $f=0.424$ period-3 laminar regions disappear, however, coupling between two chaotic bands still present. At $f=0.45$ Fig. 14c the trajectory moves only on one of the coexisting attractors.

6. Summary and conclusions

In this paper we have studied the influence of various noises in a two coupled Duffing oscillators with specific emphasis on noise-induced intermittent hopping between four coexisting attractors. The effect of three noises considered are found to be similar. We have shown the occurrence of intermittent dynamics for a range of values of strength of the noises. In the system (1) switching from one chaotic band to another corresponds to switching from one potential well to another.

We found many similarities in the dynamics of Eq. 1a, 1b, 1c induced by three noises. In all the cases we observed the following:

(i) The switching between four attractors is random and the probability distribution of the time interval between successive switching on to an attractor is Poisson.

(ii) The power spectrum show $1/\omega^2$ in the intermittent dynamics and $1/\omega$ in the fully developed chaos.
(iii) The probability distribution $P(x, y)$ becomes more and more inhomogeneous with increase in the strength of the noises.

(iv) The mean residence time on each potential well decreases exponentially with noise strength.

(v) The relative residence time on $a_{+-}$ and $a_{-+}$ decreases with noise strength while it increases in the other two chaotic bands.

In an one-dimensional cubic map [41] two period-3 attractors were coupled only through a long transient. In contrast to this, in the system (1) driven by the noises the transient is very short as seen from the Figs. 1, 8 and 11. However, transient evolution is observed in the intrinsic intermittent dynamics (Figs. 12, 13).

We have shown the occurrence of an interior crisis in the absence of noise where two coexisting chaotic attractors merged and formed a single attractor on which the
trajectories moved intermittently. Joint occurrence of type-I and crisis-induced intermittencies is also shown.

The effect of Gaussian noise in nonlinear dynamical systems is widely studied. The influence of other noises are not investigated in detail. Our present study in the system (I) shows that replacing Gaussian noise by dichotomous and amplitude-dependent noises lead to similar dynamics. Previously, Gammaitoni et al. [37] studied Duffing oscillator driven by noises and showed that replacing Gaussian noise with dichotomous noise of comparable intensity does not affect the stochastic relaxation. A detailed study on the effect of various noises in nonlinear systems may provide interesting results.

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