5.1 INTRODUCTION

As mentioned in Chapter I the presence of chaos is an undesirable phenomenon in many nonlinear systems. For example, it may lead to violent vibrations and irregular operations in mechanical systems. Thus one may wish to avoid chaos and convert the actual chaotic dynamics of the system into a desired periodic motion. Chaos can be controlled by stabilizing a desired unstable periodic orbit embedded in the chaotic attractor by feedback methods [63-66,68,70-72,75] or can be eliminated by bringing the system to some periodic motion by adding an appropriate nonfeedback periodic perturbation [67,69,73,74]. Alternatively, when a chaotic and a periodic attractors coexist with different basins of attraction, defined as the set of all initial conditions of a system for which the trajectories representing that system in the phase space will asymptotically approach a particular attractor. Then chaos can be eliminated by means of migration control. That is, if the system is initially on the chaotic attractor then it can be directed to the coexisting periodic attractor.

In the previous Chapter we studied the occurrence of periodic and chaotic dynamics in the two coupled Duffing oscillators. For a range of parametric values we found the coexistence of more than one attractor, periodic as well as chaotic. Thus, it is important to study the possibility of switching the system
motion from one attractor to another, particularly from chaotic to periodic. Motivated by the above, in this Chapter we investigate migration from one attractor to another with special emphasis on chaotic to periodic by employing various control algorithms. In our study we consider the following control methods:

1. Open-plus-closed-loop control method (OPCL)[70,75],
2. Adaptive control algorithm (ACA)[64,65],
3. Continuous feedback method of Chen and Dong [71] and

The migration dynamics from chaos to a coexisting periodic attractor in the above methods are compared by investigating

(i) range of stable control region,
(ii) dependence of recovery time on the amplitude $\epsilon$ of the controller and
(iii) variation of the required perturbation with time.

From the detailed investigation we found that the four methods considered here are suitable for migrating the dynamics of the system from chaos to a coexisting periodic orbit. Migration is possible for a range of values of amplitude of the perturbation. The correction signal and the dependence of recovery time on $\epsilon$ show different characteristic features in the various control algorithms.

In this Chapter, first we give a brief account of the above mentioned four control algorithms. Then, we study migration from a chaotic attractor to a coexisting periodic attractor by OPCL method. The dependence of recovery time on the amplitude $\epsilon$ of the parameter is investigated. Next, we illustrate the
migration strategy in ACA. Then, migration control by the continuous feedback method of Chen and Dong is studied. Finally, we show that control is possible by a discrete feedback method of Singer et al. where the perturbation is either $+\epsilon$ or $-\epsilon$.

5.2 ALGORITHMS FOR MIGRATION CONTROL

In this section, we briefly describe the salient features of various migration control algorithms. Consider a general $n$-dimensional nonlinear dynamical system

$$\dot{X} = \frac{dX}{dt} = F(X; p; t),$$

(5.1)

where $X = (x_1, x_2, ... x_n)^T$ represents the $n$ state variables and $p$ is a control parameter. We assume that the system has coexistence of more than one attractor for a given value of the control parameter. If all the coexisting attractors are periodic then one can speak of migration from one periodic attractor to another suitable coexisting periodic attractor. On the other hand, when a periodic attractor coexists with a chaotic attractor, we can direct the motion of the system from chaos to the periodic attractor. In most of the control algorithms either the parameter $p$ is slightly perturbed or an appropriate additional perturbation is added to the right hand side of eq.(5.1). The salient features of the prominent methods are as follows.

5.2.1 Open-Plus-Closed-Loop (OPCL) Control Method [70,75]

In the OPCL control method, perturbation is added to the right hand side of eq.(5.1). The perturbation is assumed with a switching function $S(t)$, $S(t) = 0$ ($t < 0$) and $S(t) = 1$ ($t > 0$). The time $t = 0$ refers to the time at which control is
activated. Equation (5.1) with OPCL control is given by

$$\dot{X} = F(X) + K(g, X, t),$$  \hfill (5.2a)

where

$$K(g, X, t) = S(t) \left[ \dot{g} - F(g) + (F'(g) + A)(X - g) \right]$$  \hfill (5.2b)

with prime denoting differentiation with respect to $g$. $A$ is a constant matrix whose eigenvalues all have negative real parts. One may choose its elements as $a_{ij} = a_{ij}$. The function $g(t)$ is a goal dynamics towards which $X(t)$ would tend. That is, with the control, in the long time limit we have

$$\lim_{t \to \infty} \|X(t) - g(t)\| = 0. \hfill (5.3)$$

Here perturbation vanishes once the goal dynamics is achieved. To implement this method the knowledge of the evolution equation is necessary.

5.2.2 Adaptive Control Algorithm (ACA) [64,65]

In the ACA, the system motion $X$ is setback to a desired state $\bar{X}$ by adding an additional dynamics on the parameter $p$ or addition of parameter introduced through the evolution equation

$$\dot{p}(t) = \epsilon G(X(t) - \bar{X}(t)), \quad \epsilon << 1, \hfill (5.4)$$

where the function $G$ is proportional to the difference between the actual orbit $X$ and the desired goal orbit $\bar{X}$ and $\epsilon$ indicates the stiffness of the control. The function $G$ could be linear or nonlinear. The parameter $p$ evolves according
to (5.4) and adjusts its value until the desired state is reached. This method has been used to stabilize desired unstable periodic orbit in certain maps and differential equations [64,65], Bonhoeffer-van der Pol oscillator [76,147] and FitzHugh-Nagumo equation [74].

5.2.3 Method of Chen and Dong [71]

In the Chen and Dong method the feedback is simply proportional to \( X(t) - \bar{X}(t) \). The method has been applied to many theoretical model equations [71,76]. The advantage of this method is that the evolution equation of the system is not required. The desired control is possible for a certain range of values of \( \epsilon \).

5.2.4 Singer-Wang-Bau Feedback Method [68]

Introducing a small perturbation to an external parameter of a dynamical system, the sign of which depends on the sign of the deviation of the actual output to a preset Singer et al. [68] demonstrated the conversion of a chaotic flow into a stationary solution in the Lorenz system and in an experimental thermal convection loop. With the addition of feedback the eq.(5.1) is written as

\[
\dot{X} = F(X, p) + \epsilon \text{sgn}(X(t) - \bar{X}(t)),
\]

where \( \text{sgn}(y) \) denotes sign of \( y \). In this method the perturbation is either \(+\epsilon\) or \(-\epsilon\). Control is possible only for a certain range of values of \( \epsilon \).
Figure 5.1 The coexisting (a) chaotic attractor and (b) periodic attractor of the two coupled Duffing oscillators.
5.3 MIGRATION CONTROL IN TWO COUPLED DUFFING OSCILLATORS[148]

For our study we rewrite the two coupled Duffing oscillators equations as

\[ \dot{x} = y, \quad (5.6a) \]
\[ \dot{y} = -dy + \alpha_1 x - \beta_1 x^3 - \delta xu^2 + f \cos \omega t, \quad (5.6b) \]
\[ \dot{u} = v, \quad (5.6c) \]
\[ \dot{v} = -dv + \alpha_2 u - \beta_2 u^3 - \delta ux^2 + f \cos \omega t. \quad (5.6d) \]

We fix the parameters at \( \alpha_1 = -1, \beta_1 = -4, \alpha_2 = -1.1, \beta_2 = -3.9, \delta = 0.002, \omega = 0.526, d = 0.4 \) and \( f = 0.1147 \). For this choice both chaotic and periodic attractors coexist. They are shown in fig.(5.1). Assume that the system is in the chaotic state. Suppose we wish to convert the actual chaotic dynamics (orbit \( a \) in fig.5.1) to the coexisting periodic attractor (orbit \( b \) in fig.5.1). First, we study the applicability of the OPCL method.

5.3.1 Migration Control by Open-Plus-Closed-Loop Method

The two coupled Duffing oscillators eqs.(5.6) with OPCL control is written as

\[ \dot{x} = y + K_x, \quad (5.7a) \]
\[ \dot{y} = -dy + \alpha_1 x - \beta_1 x^3 - \delta xu^2 + f \cos \omega t + K_y, \quad (5.7b) \]
\[ \dot{u} = v + K_u, \quad (5.7c) \]
\[ \dot{v} = -dv + \alpha_2 u - \beta_2 u^3 - \delta ux^2 + f \cos \omega t + K_v. \quad (5.7d) \]
Figure 5.2 Migration from the chaotic motion to the coexisting periodic orbit in the two coupled Duffing oscillators. The controlled system is (5.7).
Figure 5.3 The variation of the required perturbation in the controlled two coupled Duffing oscillators for migration from the chaotic motion to the chosen periodic motion. In the subplot (a) continuous and dashed curves represent the perturbations $K_x$ and $K_y$ respectively. In the subplot (b) they represent $K_u$ and $K_v$ respectively.
where $K_x$, $K_y$, $K_u$ and $K_v$ are the perturbations given by eq.(5.2b) introduced for migration. That is,

$$
K_x = S(t)\left[\dot{g}_x - g_y + a(x - g_x) + (y - g_y)\right],
$$

(5.7e)

$$
K_y = S(t)\left[\dot{g}_y - (-dg_y + a_1g_x - \beta_1g_x^2 - \delta g_xg_u^2) + (\alpha_1 - 3\beta_1g_x^2 - \delta g_u^2)(x - g_x) + (a - d)(y - g_y)\right],
$$

(5.7f)

$$
K_u = S(t)\left[\dot{g}_u - g_u + a(u - g_u) + (v - g_v)\right],
$$

(5.7g)

$$
K_v = S(t)\left[\dot{g}_v - (-dg_v + a_2g_u - \beta_2g_u^3 - \delta g_u^2g_v^2) + (\alpha_2 - 3\beta_2g_u^2 - \delta g_v^2)(u - g_u) + (a - d)(v - g_v)\right].
$$

(5.7h)

We select the goal dynamics $g(t)$ as the coexisting periodic orbit and fix $a = -0.5$. Figure (5.2) illustrates the migration from chaotic motion to the chosen goal orbit. In the absence of the control, the system is integrated using a fourth-order Runge-Kutta method with time step $t = (2\pi/\omega)/100$ with the initial condition $X(0) = (x(0), y(0), u(0), v(0)) = (0, 0.35, 0, 0.3)$. The system is allowed to evolve in a chaotic state. Control is switched on at $t = 80 \times 2\pi/\omega$, with $S(t) = 1$. Figures (5.3a) and (5.3b) show the required perturbations. The perturbations are found to vanish once the migration to $g(t)$ is achieved. This is because the goal orbit is a particular solution of the uncontrolled system.

The response of the system is studied with the switching function

$$
S(t) = 1 - \exp(-\lambda t),
$$

(5.8)

where $\lambda$ is a constant. The desired migration is achieved for $\lambda > 0$. The efficacy
Figure 5.5 Migration from the chaotic motion to the coexisting periodic orbit by ACA.
Figure 5.4 The recovery time $R_T$ versus $\lambda$ for OPCL method.
Figure 5.6 The variation of the required perturbations $p(t)$ for the ACA.
of the OPCL control has been studied by calculating the recovery time \( R_T = t'_0 - t_0 \) where \( t_0 \) and \( t'_0 \) are the times at which control is initiated and after which \( \| X(t) - g(t) \| \) is always less than a preassumed accuracy, say \( 10^{-3} \), respectively. \( R_T \) is calculated for 200 initial conditions chosen on the chaotic attractor and then the average value is obtained. Figure (5.4) shows the dependence of \( R_T \) on \( \lambda \). As \( \lambda \) is increased from zero, \( R_T \) decreases rapidly and approaches a constant value for higher values of \( \lambda \).

### 5.3.2 Adaptive Control Algorithm

The two coupled Duffing oscillators equations with ACA are written as

\[
\begin{align*}
\ddot{x} &= y, \\
\dot{y} &= -\alpha_1 x - \beta_1 x^3 - \delta x v^2 + f \cos \omega t + p(t), \\
\dot{u} &= v, \\
\dot{v} &= -\alpha_2 u - \beta_2 u^3 - \delta u x^2 + f \cos \omega t, \\
\dot{p} &= \epsilon \left[ (x + y - u - v) - (\bar{x} + \bar{y} - \bar{u} - \bar{v}) \right].
\end{align*}
\]

where \( \bar{X}(t) = (\bar{x}(t), \bar{y}(t), \bar{u}(t), \bar{v}(t)) \) is the desired goal orbit. We choose \( \bar{X} \) as the coexisting period-T attractor. Figure (5.5) shows the migration from chaotic attractor to the coexisting period-T attractor for \( \epsilon = 0.002 \). The variation of the perturbation \( p(t) \) is plotted in fig.(5.6). The control is switched on at \( t = 80 \times 2\pi/\omega \). The parameter \( p(t) \) evolves according to eq.(5.9e) and adjusts its value so that the desired state is reached. Once the desired migration is achieved \( p(t) \) vanishes. The control can be switched off if the condition \( X = \bar{X} \) is realized.
Figure 5.7 The recovery time $R_T$ versus $\varepsilon$ for ACA.
Figure 5.8 Poincaré map data $x$ showing the migration from the chaotic motion to the coexisting periodic orbit by Chen and Dong method where $\varepsilon = -0.0055$. Here $t$ is in steps of period of the driving force.
In general, the control mechanism is sensitive to the value of \( \epsilon \) and the form of the function \( G \). In eqs.(5.9) stable control to the chosen coexisting attractor is found to occur for \( \epsilon \) values in the interval \((-0.0035, -0.0018)\) and \((0.00023, 0.004)\). Figure (5.7) shows the dependence of recovery time on \( \epsilon \).

5.3.3 Chen and Dong Method

Migration control is possible by the addition of a continuous feedback of the form \((X - \bar{X})\) to the eqs.(5.6). The two coupled Duffing oscillators equations with the addition of the feedback term introduced by Chen and Dong is

\[
\begin{align*}
\dot{x} &= y, \quad (5.10a) \\
\dot{y} &= -dy + \alpha_1 x - \beta_1 x^3 - \delta x u^2 + f \cos \omega t - \epsilon (x - \bar{x}), \quad (5.10b) \\
\dot{u} &= v, \quad (5.10c) \\
\dot{v} &= -dv + \alpha_2 u - \beta_2 u^3 - \delta u x^2 + f \cos \omega t - \epsilon (u - \bar{u}), \quad (5.10d)
\end{align*}
\]

where \( \epsilon (x - \bar{x}) \) and \( \epsilon (u - \bar{u}) \) are the perturbations. The system is allowed to evolve in the chaotic state. The approximate location of the chosen periodic attractor in Poincaré map is \( X(\bar{x}, \bar{y}, \bar{u}, \bar{v}) = (0.1597, 0.0264, 0.1379, 0.0196) \). Figure (5.8) shows the migration dynamics of the system to the chosen orbit. The values of \( x \) collected at every period of the driving force are plotted in this figure. The desired migration is achieved for \( \epsilon > \epsilon_c = 0.0039 \). We have calculated the recovery time \( R_T \) for a range of values of \( \epsilon \). Figure (5.9) shows the dependence
Figure 5.9 The recovery time $R_T$ versus $(\varepsilon - \varepsilon_c)$ for Chen and Dong method. The dots represent numerical data and continuous line is the best curve fit.
Figure 5.10  Poincaré map data $x$ showing migration from the chaotic motion to the coexisting periodic orbit by Chen and Dong method where $\varepsilon = -0.01349$. The controlled system is (5.12).
Figure 5.11 The recovery time $R_T$ versus $(\varepsilon - \varepsilon_c)$ for Chen and Dong method. The controlled system is (5.12).
of $R_T$ (in units of $2\pi/\omega$) on $\epsilon$. $R_T$ is found to vary as

$$R_T = 0.3846(\epsilon - \epsilon_c)^{-0.546}. \quad (5.11)$$

That is, power-law dependence of $R_T$ on $(\epsilon - \epsilon_c)$ is found.

Also, we considered the feedback terms of the form $(x + y - \bar{x} - \bar{y})$ and $(u + v - \bar{u} - \bar{v})$. The two coupled Duffing oscillators equations with these feedback terms is written as

\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -dy + \alpha_1 x - \beta_1 x^3 - \delta x u^2 + f \cos \omega t - \epsilon (x + y - \bar{x} - \bar{y}), \\
\dot{u} &= v, \\
\dot{v} &= -dv + \alpha_2 u - \beta_2 u^3 - \delta u x^2 + f \cos \omega t - \epsilon (u + v - \bar{u} - \bar{v}).
\end{align*} \quad (5.12)

Figure (5.10) illustrates the migration to the periodic attractor for $\epsilon = 0.01349$. For eqs.(5.12), the desired controlling is achieved for $\epsilon > \epsilon_c = 0.01349$. Figure (5.11) shows the dependence of $R_T$ on $\epsilon$. The recovery time (in units of $2\pi/\omega$) scales as

$$R_T = 0.4265(\epsilon - \epsilon_c)^{-0.340}. \quad (5.13)$$

5.3.4 Singer-Wang-Bau Method

The two coupled Duffing oscillators equations with the addition of a feedback
Figure 5.12 Poincaré map data \( x \) showing migration from the chaotic motion to the coexisting periodic orbit by Singer et al. method where \( \varepsilon = 0.045 \). The controlled system is (5.14).
Figure 5.13 The recovery time $R_T$ versus $(\varepsilon_c - \varepsilon)$ for Singer et al. method. The controlled system is (5.14).
term introduced by Singer-Wang-Bau is written as

\[
\dot{x} = y, \quad (5.14a)
\]

\[
\dot{y} = -dy + \alpha_1 x - \beta_1 x^3 - \delta x u^2 + f \cos \omega t - \epsilon \sgn(x - \bar{x}), \quad (5.14b)
\]

\[
\dot{u} = v, \quad (5.14c)
\]

\[
\dot{v} = -dv + \alpha_2 u - \beta_2 u^3 - \delta u x^2 + f \cos \omega t - \epsilon \sgn(u - \bar{u}). \quad (5.14d)
\]

Stable migration is found for \( \epsilon \) in the range of \((0.01 - 0.059)\) and \((0.09 - 0.1)\). Figure (5.12) shows the migration dynamics for \( \epsilon = 0.045 \). Figure (5.13) shows the dependence of numerically calculated \( R_T \) (in units of \( 2\pi/\omega \)) on \( \epsilon \). The recovery time is found to vary as

\[
R_T = 8.345 \exp[-39.787(\epsilon_c - \epsilon)], \quad (5.15)
\]

where \( \epsilon_c = 0.059 \).

Also, we considered feedback terms of the form \((x + y - \bar{x} - \bar{y})\) and \((u + v - \bar{u} - \bar{v})\). The two coupled Duffing oscillators equations with these feedback terms is written as

\[
\dot{x} = y, \quad (5.16a)
\]

\[
\dot{y} = -dy + \alpha_1 x - \beta_1 x^3 - \delta x u^2 + f \cos \omega t - \epsilon \sgn(x + y - \bar{x} - \bar{y}), \quad (5.16b)
\]

\[
\dot{u} = v, \quad (5.16c)
\]

\[
\dot{v} = -dv + \alpha_2 u - \beta_2 u^3 - \delta u x^2 + f \cos \omega t - \epsilon \sgn(u + v - \bar{u} - \bar{v}). \quad (5.16d)
\]
Figure 5.14 Poincaré map data $x$ showing migration from the chaotic motion to the coexisting periodic orbit by Singer et al. method where $\varepsilon = 0.00415$. 
Figure 5.15 The recovery time $R_T$ versus $(\varepsilon - \varepsilon_c)$ for Singer et al. method. The controlled system is (5.16).
Figure (5.14) illustrates the migration dynamics of the system to the chosen orbit. For eqs.(5.16), desired controlling is achieved for $\epsilon > \epsilon_c = 0.00415$. Figure (5.15) shows the dependence of $R_T$ on $\epsilon$. The recovery time (in units of $2\pi/\omega$) scales as

$$R_T = 0.2129(\epsilon - \epsilon_c)^{-0.339}.$$  

(5.17)

5.4 CONCLUSION

In this Chapter we have studied the migration from a chaotic attractor to a coexisting periodic attractor in the two coupled Duffing oscillators. The desired migration is achieved by the four control methods: OPCL, ACA, Chen & Dong method and Singer et al. method. Thus the simultaneous presence of periodic orbits in a chaotic system is beneficial. The algorithms considered here have several common and different characteristic features. In all the control methods the required perturbation is found to vanish once the desired goal orbit is reached.

In the OPCL method migration to a desired coexisting attractor is always guaranteed. In contrast to this, in the case of other feedback methods stable control is observed only for a certain range of values of the stiffness parameter $\epsilon$. Therefore, the desired value of $\epsilon$ has to be determined either by linear stability analysis or experimentally before implementing the specific control algorithm. The feedback methods considered in our study are primarily designed to stabilize the unstable periodic orbits embedded in the chaotic attractor. However, our analysis shows that the desired orbit need not be embedded in the chaotic attractor.