4.1 INTRODUCTION

The discovery of chaotic dynamics in dissipative dynamical systems has influenced traditional subjects ranging through all the physical and biological sciences, many of the social sciences, mathematics and engineering. In the last two Chapters we investigated the behaviour of integrability of a force-free and damped two coupled Duffing oscillators. It is important to investigate the dynamics of the system, particularly the onset of chaos, in the nonintegrable region.

It is well known that for chaotic behaviour to occur, a fixed point of a dissipative continuous dynamical system must undergo a Hopf bifurcation thereby developing a limit cycle motion. As the control parameter is varied, this limit cycle may bifurcate further leading to chaotic motion. Using linear stability analysis we found that the fixed points of eqs.(2.7) do not undergo Hopf bifurcation for any nonzero values of the parameters. Thus the system (2.7) cannot show chaotic behaviour. However, when the system is subjected to external periodic forces, a variety of interesting behaviours such as period doubling phenomenon, onset of chaos, coexistence of multiple attractors and merging of attractors occur. In this Chapter first we numerically investigate the occurrence of regular and chaotic dynamics for some specific values of pa-
rameters in the periodically driven and damped two coupled Duffing oscillators eqs.(1.2).

The study of occurrence of chaotic behaviour in different parameter spaces of the system (1.2) is of great importance and this can be carried out using the numerical tools such as bifurcation diagram, Lyapunov exponents, power spectra analysis and so on. For a practical application it is very useful to have at least a basic knowledge of the nature of the chaos boundary in the parameter space. For a general dynamical system there seems to exist no exact solution to this problem at the present time. However, within the perturbation theory, there is an analytical approach, namely the Melnikov-technique [7,11]. The method essentially gives a criterion for the onset of horseshoe chaos, which is the transverse intersection of the stable and unstable manifolds of a saddle. The orbits created by the horseshoe mechanism display an extremely sensitive dependence on initial conditions and possibly exhibit either a chaotic transient before settling to a periodic orbit or a strange attractor [7,11]. In many dynamical systems [137-141] the onset of chaos has been found to occur near the Melnikov threshold curve. Motivated by the above, we study the application of Melnikov method to the periodically driven and damped two coupled Duffing oscillators.

This Chapter is organized as follows. Section 4.2 is devoted to the study of the influence of external periodic force. The dynamics is numerically investigated by varying the amplitude of the force for three physically interesting potentials for fixed values of the other parameters. We show the coexistence of more than one periodic attractor, period doublings of the coexisting attractors and the onset of chaos. In section 4.3 we apply the Melnikov method to the two

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coupled Duffing oscillators. The onset of horseshoe chaos in the perturbed system is investigated. The nature of flow on the perturbed manifold is studied by an averaging procedure. The dimensionality of the stable and unstable manifolds of various fixed points of the averaged equations is studied. The analytical prediction is found to be in good agreement with the numerical estimation of the onset of chaos. Finally, section 4.4 contains conclusion.

4.2 Chaotic Dynamics in the Two Coupled Duffing Oscillators

4.2.1 Potential Wells of the System

The two coupled Duffing oscillators eqs.(1.2) can also be written as

\[
\begin{align*}
\ddot{x} &= -d\dot{x} - \frac{\partial V(x, y)}{\partial x} + f_1 \cos \omega_1 t, \\
\ddot{y} &= -d\dot{y} - \frac{\partial V(x, y)}{\partial y} + f_2 \cos \omega_2 t,
\end{align*}
\]  

(4.1)

where the potential function \(V\) is given by

\[V(x, y) = A_1 x^2 + \alpha_1 x^4 + A_2 y^2 + \alpha_2 y^4 + \delta x^2 y^2.\]  

(4.2)

Throughout our analysis we assume that \(A_2\) and \(\alpha_2\) have the same signs as \(A_1\) and \(\alpha_1\) respectively. Further, we assume that the coupling is weak. The shape of the potential varies with the signs of \(A_1, A_2, \alpha_1, \alpha_2\) and is summarized in table 4.1.
Figure 4.1 Potential $V$, given by eq. (4.2) for (a) $A_1, A_2, \alpha_1, \alpha_2 > 0$, (b) $A_1, A_2 < 0, \alpha_1, \alpha_2 > 0$, (c) $A_1, A_2 > 0, \alpha_1, \alpha_2 < 0$ and (d) $A_1, A_2, \alpha_1, \alpha_2 < 0$. 
Figure 4.1 Continued...
Table 4.1 Nature of the potential for different values of the parameters $A_1, A_2, \alpha_1$ and $\alpha_2$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Sign of</th>
<th>Type of the potential</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+</td>
<td>+ + + +</td>
<td>Single well with infinite height potential</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>- - + +</td>
<td>Potential with a hump at the centre</td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>+ - - -</td>
<td>Single well with finite height hump potential</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>- - - -</td>
<td>Inverted single well potential</td>
</tr>
</tbody>
</table>

The nature of the solution of the system (4.1) depends on the shape of the potential. In cases (1) and (2) there exists globally bounded solutions since $V(x, y) \rightarrow \infty$ as $|x|$ and $|y| \rightarrow \infty$. However, for case (3) we may have unbounded solutions of exploding amplitudes for the choices of sufficiently large initial values of $(x, y)$ since $V(x, y) \rightarrow -\infty$ as $|x|$ and $|y| \rightarrow \infty$. Finally, the potential with $A_1, A_2, \alpha_1$ and $\alpha_2 < 0$ is physically uninteresting since the system has an exploding amplitude $V(x, y) \rightarrow -\infty$ as $|x|$ and $|y| \rightarrow \infty$. In the following we numerically study the occurrence of regular and chaotic dynamics in (4.1) for the first three potentials listed in table 4.1.

4.2.2 Single Well With Infinite Height Potential [132]

We fix the parameters at $A_1 = 0.005$, $A_2 = 0.01$, $\alpha_1 = 10.0$, $\alpha_2 = 10.0$, $\delta = 0.05$, $d = 3\sqrt{A_1}$ and $\omega_1 = \omega_2 = \omega = 1$. We choose $f_1 = f_2 = f$. The forcing amplitude $f$ is then varied from a small value. From our numerical study we find the following.

Figure (4.2a) shows the occurrence of period doubling bifurcations, chaotic motions and window regions. For characterizing the regular and chaotic mo-
Figure 4.2 (a) Bifurcation diagram illustrating period doubling route to chaos. (b) Maximal Lyapunov exponent ($\lambda$) against the parameter $f$ corresponding to (a).
Figure 4.3 Phase portrait of (a) a symmetrical orbit for $f = 0.2$ and (b) an asymmetrical orbit for $f = 0.8$. 
Figure 4.4 Poincaré map of the chaotic attractor for $f = 1.7$. 
tions we have calculated the maximal Lyapunov exponent (λ) employing the algorithm of Wolf et al. [142]. Variation of λ against f is plotted in fig.(4.2b).

For small values of f the system exhibits a symmetrical orbit. A typical orbit is shown in fig.(4.3a) for f = 0.2. As the parameter f is varied the symmetrical orbit loses its stability and experiences a symmetry breaking bifurcation. The resulting asymmetric orbit is shown in fig.(4.3b) for f = 0.8. When f is further increased the asymmetrical attractor undergoes period doubling cascade leading to chaos. Period-T (= 2π/ω), 2T, 4T and 8T orbits are found in the interval f ∈ (0.0 – 0.896), (0.896 – 1.016), (1.016 – 1.04) and (1.04 – 1.052) respectively. The onset of chaos is found at f = 1.064. A feature of chaotic regime is the presence of windows of periodic solutions interspersed throughout the range of their existence. For example, a period-3T window occurs for f ∈ (1.364 – 1.652) in which there is no chaotic behaviour. The developed chaos disappears at f ≈ 2.072 and a period-T limit cycle occurs. In the chaotic regime as the parameter f is increased the size of the attractor increases gradually as shown in the bifurcation diagram (4.2a). In fig.(4.4) the Poincaré map of the chaotic attractor in x – ẍ plane is plotted for f = 1.7. Occurrence of other routes to chaos such as intermittency and quasiperiodic are also possible. However, we have not investigated these.

4.2.3 Potential With A Hump At The Centre [132]

We now fix A₁ = −0.5, A₂ = −0.055, α₁ = 0.25, α₂ = 0.025, d = 0.025,  

d = 0.4 and ω₁ = ω₂ = 1. For small values of f coexistence of four limit cycle orbits occur one in each potential well. As the parameter f is increased these orbits exhibit a cascade of period doubling leading to chaotic motion. Each
Figure 4.6 (a)-(b) Phase portrait and (c)-(d) Poincaré map of the chaotic attractor for $f = 0.6$. 
Figure 4.5 Successive bifurcations of two coexisting period-1 attractors.
coexisting attractor possesses its own basin of attraction, defined as the set of initial conditions from which the system evolves to a particular orbit. Figures (4.5a) and (4.5b) show the successive bifurcations of two coexisting attractors. Interestingly, the attractors underwent bifurcations at the same values of $f$. For example, the period-$T$ orbits bifurcate to period-$2T$ orbits at $f \approx 0.255$ and period-$4T$ at $f \approx 0.26475$. Chaotic motion is first observed at $f = 0.267$. The chaotic attractors disappeared at $f = 0.281$. Due to a crisis period-$T$ limit cycles are found. Moreover, these period-$T$ orbits underwent period doubling which is clearly seen in fig.(4.5). Further, we note a sudden expansion in the size of the chaotic attractors at $f \approx 0.2715$ and $0.3088$. For $f > 0.3165$ cross-well chaos is observed. Here, the chaotic attractors merge into a single attractor. This is shown in fig.(4.6) for $f = 0.6$.

4.2.4 Single Well With Finite Height Hump Potential [132]

Next we consider the case $A_1, A_2 > 0$ and $\alpha_1, \alpha_2 < 0$. The dynamics of the system has been investigated for the following fixed parameters $A_1 = 0.5$, $A_2 = 0.55$, $\alpha_1 = -1.0$, $\alpha_2 = -0.975$, $\delta = 0.025$, $d = 0.4$ and $\omega_1 = \omega_2 = 0.526$ thereby varying $f$ as done in the other two potential well cases. In this potential well also we have found coexistence of more than one stable periodic orbit and period doubling bifurcations.

In the case of potential with a hump at the centre considered in the previous subsection we found period doubling bifurcations of each of the four coexisting period-$T$ orbits at the same values of $f$. In contrast to this, in single well with finite height hump potential the attractors were found to undergo period doubling bifurcations at different values of $f$. Figure (4.7) shows the
Figure 4.7 Phase portraits of three different coexisting period-1 attractors for $f = 0.1143$. The initial conditions $(x(0), \dot{x}(0), y(0), \dot{y}(0))$ used for the orbits $a$, $b$ and $c$ are $(0.0, 0.35, 0.0, 0.0)$, $(-0.06, 0.35, 0.0, 0.0)$ and $(0.1, 0.0, 0.0, 0.0)$ respectively.
Figure 4.8 Period doubling bifurcations culminating in chaos of two coexisting attractors.
Figure 4.9 (a)-(b) Poincaré maps of the two coexisting chaotic attractors for $f = 0.11466$. (c) Enlargement of the attractor shown in (b). (d) Poincaré map of the chaotic attractor for $f = 0.11482$. 
phase portrait of the three coexisting period-T orbits for $f = 0.1143$. When $f$ is increased the limit cycle labelled as $c$ in fig.(4.7) alone persists while the other two become unstable at certain $f$ values and undergo cascades of period doubling bifurcations. Bifurcations of the period-T orbit $a(b)$ to period-2T occur at $f = 0.1143375(0.11448)$; to period-4T at $f = 0.114481(0.1146225)$; to period-8T at $f = 0.1145184(0.114645)$.

Figure (4.8) shows successive period doubling process leading to chaotic motion of the coexisting period-T attractors. The Poincaré map of the two coexisting chaotic attractors at $f = 0.11466$ is given in fig.(4.9). For comparison, the same scales in $x$ and $\dot{x}$ coordinates are used. For clarity, in fig.(4.9c) we show the magnification of the chaotic attractor shown in fig.(4.9b). As the parameter $f$ is increased beyond a certain critical value of $f$, both the chaotic attractors merge and form a single large chaotic attractor. This is caused by a crisis in which both the attractors fuse together and form a large attractor. Figure (4.9d) shows the Poincaré map of such a chaotic attractor. By comparing figs.(4.9a), (4.9b) and (4.9d) we note that the large attractor is indeed a mixture of the two coexisting attractors found at lower values of $f$. The coexistence of limit cycle $c$ along with the chaotic attractors provides a mode of physical regulation as it allows switching over to a periodic regime upon suitable perturbation which will be discussed in the next Chapter.

4.3 ANALYTICAL PREDICTION OF HORSESHOE CHAOS IN (1.2)

In this section first we describe the notion of horseshoe chaos and the Melnikov approach. Then we study the application of the method to the two coupled Duffing oscillators.
Figure 4.10 Stable and unstable orbits of the saddle fixed point \( X_s \).
(a) Unperturbed system where \( X_0 \) is the centre; the orbits smoothly join. (b)-(d) Perturbed system (b). The unstable orbit lies outside the stable orbit. (c) The unstable orbit lies inside the stable orbit. (d) The unstable and stable orbits intersect.
4.3.1 Horseshoe Chaos and Melnikov Method

For simplicity and illustrative purpose we consider a system of two ordinary differential equations of the form

\[
\frac{dX}{dt} = h_0(X) + \epsilon h_1(X, t, \epsilon), \quad (4.3)
\]

where \( X = (x_1, x_2)^T \), \( h_0 = (f_0, g_0)^T \) and \( h_1 = (f_1, g_1)^T \). For the system (4.3) the following conditions are assumed to be satisfied [7]:

(i) When \( \epsilon = 0 \), the system has an equilibrium point \( X_0 \), which is a centre, and a saddle fixed point \( X_s \).

(ii) The function \( f_0(X) \) and \( g_0(X) \) are analytical in \( X \) in a sufficiently large neighbourhood of the point \( X_0 \).

(iii) For \( \epsilon = 0 \), (4.3) possesses a homoclinic orbit, \( X_h(t) = (x_{1h}(t), x_{2h}(t))^T \), an orbit connecting the saddle to itself.

(iv) The function \( f_1(X, \epsilon, t) \) and \( g_1(X, \epsilon, t) \) are analytical in \( X \) in a sufficiently large neighbourhood of \( X_0 \). They are continuous and periodic in \( t \), \( f_1(t + 2\pi) = f_1(t) \), \( g_1(t + 2\pi) = g_1(t) \).

The system is illustrated in the phase space \((x_1, x_2)\) in fig.(4.10). In the fig.(4.10a), \( X_s^s \) is the stable manifold or orbit of the saddle \( X_s \) and it is the set of points \( X(0) \) such that \( X(t) \) approaches \( X_s \) as \( t \to \infty \). The unstable manifold \( X_u^u \) is the set of points whose trajectory approaches \( X_s \) in reverse time. The stable and unstable manifolds of the unperturbed system smoothly join each other.

When the system is perturbed, \( \epsilon \neq 0 \), the stable and unstable orbits do
Figure 4.11 Global consequences of homoclinic bifurcation.
not in general smoothly join together. The perturbed homoclinic orbit splits up into a stable orbit $X^s(t, t_0)$ defined in the interval $t_0 \leq t \leq \infty$ and an unstable orbit $X^u(t, t_0)$ defined in the interval $\infty < t \leq t_0$. For dissipative perturbation the orbits take one of the three possibilities shown in fig.(4.10b) - (4.10d). In fig.(4.10b) the unstable orbit always lies outside the stable orbit while in fig.(4.10c) the unstable orbit always lies inside the stable orbit. In fig.(4.10d) the stable and unstable orbits intersect transversely leading to an infinite number of intersections. The point of intersection is called a homoclinic point [7,11].

Figure (4.11) illustrates the evolution of a small rectangular area covering two homoclinic points. Areas near homoclinic points undergo repeated stretching and folding leading to a horseshoe type map. This map is a simplified version of a map first studied by Smale [143] and due to the shape of the image of the domain of the map, it is called a Smale – horseshoe. We know that the repeated action of stretching and folding of orbits leads to sensitive dependence on initial conditions. Thus, the presence of transverse intersections of stable and unstable orbits implies that the Poincaré map has the so-called horseshoe chaos. Even though the orbits created by the horseshoe mechanism are unstable they can exert a dramatic influence on the behaviour of orbits which bars close to the point of intersection. These orbits will display an extremely sensitive dependence on initial conditions and exhibit a chaotic transient before settling to a periodic or strange attractors. The appearance of transverse intersections of homoclinic orbits can be predicted analytically by the Melnikov technique. The essence of this method is to calculate the Melnikov function, which is proportional to the distance $d(t)$ between the stable and unstable orbits of a saddle. In figs.(4.10b) and (4.10c), $d(t)$ does not change sign. On the
other hand, in fig.(4.10d) the sign of \( d(t) \) oscillates. This criterion is used to predict the occurrence of horseshoe chaos. In the following we apply the Melnikov method to the two coupled Duffing oscillators eqs.(1.2) and calculate the Melnikov function analytically.

4.3.2 Calculation Of Melnikov Function For Two Coupled Duffing Oscillators [144]

To apply the Melnikov method as stated above the equation of motion is to be rewritten in the standard form (4.3) and the unperturbed system of it \((\epsilon = 0)\) should contain at least one saddle fixed point and a centre fixed point and an integrable separatrix solution passing through the saddle point. For convenience we rewrite the eqs.(1.2) as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -dx_2 + A_1 x_1 - \alpha_1 x_1^3 - \delta x_1 x_3^2 + f \cos \theta, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -dx_4 + A_2 x_3 - \alpha_2 x_3^3 - \delta x_1^2 x_3 + f \cos \theta, \\
\dot{\theta} &= \omega.
\end{align*}
\]

In eq.(4.4), if damping and forcing terms are chosen as the perturbations, then the unperturbed part is integrable [127,145] for four specific parametric choices only. Further, in one integrable case, the equations of motion were found to be separable and hence the choice is equivalent to the study of uncoupled Duffing oscillators. This has been noted earlier by Holmes and Marsden [146]. If the damping, coupling terms and external forces are treated as perturbations, then
one can easily verify that the Melnikov function is independent of the parameter $\delta$. Alternatively, we choose the subsystem

$$\ddot{x} = A_1 x - \alpha_1 x^3 - \delta xy^2$$

(4.5)

as the unperturbed part. In the standard form of (4.3), eq.(4.4) can be rewritten as

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= A_1 x_1 - \alpha_1 x_1^3 - \delta x_1 x_3^2 + \epsilon(-dx_2 + f\cos\theta), \\
\dot{x}_3 &= \epsilon x_4, \\
\dot{x}_4 &= \epsilon(A_2 x_3 - \alpha_2 x_3^3 - \delta x_1^2 x_3 - dx_4 + f\cos\theta), \\
\dot{\theta} &= \omega.
\end{align*}$$

(4.6)

where $\epsilon$ is a small parameter. That is, in eq.(4.6), one of the two oscillators is considered as weak compared to the other.

The Melnikov analysis starts with the identification of a saddle fixed point and a separatrix solution in the unperturbed system. The fixed points of eq.(4.6) with $\epsilon = 0$ are $(x_1, x_2 = 0, x_3, x_4)$, where the roots of the equation

$$x_1(A_1 - \alpha_1 x_1^3 - \delta x_3^2) = 0,$$

(4.7)

are the values of $x_1$ while $x_3$ and $x_4$ are arbitrary. The roots of eq.(4.7) are

$$x_1 = 0, \pm[(A_1 - \delta x_3^2)/\alpha_1]^{1/3}.$$

(4.8)
Figure 4.12 Homoclinic orbits of the unperturbed system (4.5) ($\varepsilon = 0$). Solid vertical line and solid dotted curve represent the solution of eq.(4.6).
For $x_3 \in \left(-\sqrt{A_1/\delta}, \sqrt{A_1/\delta}\right)$, there are three real roots with an intermediate root corresponding to a hyperbolic and the other two being elliptic fixed points. For $|x_3| > \sqrt{A_1/\delta}$, there exists only one elliptic fixed point $(0,0,x_3,x_4)$. For $|x_3| < \sqrt{A_1/\delta}$, the fixed point $\Phi(0,0,x_3,x_4)$ is connected to itself by a pair of homoclinic orbits which satisfy the relation

$$\frac{1}{2} \left[ x_2^2 - A_1x_1^2 + \frac{1}{2} \alpha_1 x_1^4 + \delta x_1^2 x_3^2 \right] = 0. \quad (4.9)$$

The phase space of (4.6) appears as in fig.(4.12), where the component $x_4$ is suppressed for clarity. The entire picture holds for any value of $x_4$. The unperturbed system has a hyperbolic invariant manifold $M$ with boundary $\Phi(x_1(x_3),0,x_3,x_4,\theta_0)$, where $x_3 \in \left(-\sqrt{A_1/\delta}, \sqrt{A_1/\delta}\right)$ and $\theta_0 \in (0,2\pi/\omega)$.

In the perturbed system, assume that $M$ persists as invariant manifold $M_\epsilon$ given by

$$M_\epsilon = \Phi(x_1(x_3),0,x_3,x_4) + \sigma(\epsilon,\theta_0). \quad (4.10)$$

The nature of stable and unstable manifolds in the perturbed system and the significances of transverse intersections of stable and unstable manifolds are not well understood [11]. However, the presence of orbits homoclinic to fixed points and periodic orbits have dramatic dynamical consequences. In particular, the homoclinic orbits can provide the mechanism for the folding of the phase space. Further, the invariant sets such as fixed points and periodic orbits to which the orbit is homoclinic can provide the mechanism for the stretching and contraction which are essential for producing chaotic motion. Thus, it is necessary to know the flow on the normally hyperbolic manifold $M_\epsilon$ under the perturbation. The nature of flow on the perturbed $M_\epsilon$ can be studied by
the averaging procedure. If periodic orbits exist on \( M_\epsilon \) then the appropriate Melnikov function can be computed to determine whether or not the stable and unstable manifolds of the periodic orbits intersect transversely. Thus, the next step is to determine whether \( M_\epsilon \) contains any periodic orbits.

We consider the perturbed equations restricted to \( M_\epsilon \) given by eqs.(4.6c - 4.6e). Periodic orbits of (4.6c - 4.6e) in a suitable Poincaré surface of section or Poincaré map become a fixed point. So we consider the averaged equations

\[
\dot{x}_3 = \left( \frac{\epsilon}{2\pi} \right) \int_0^{2\pi} x_4 d\theta = \epsilon x_4, \quad (4.11a)
\]

\[
\dot{x}_4 = \left( \frac{\epsilon}{2\pi} \right) \int_0^{2\pi} (-dx_4 + A_2x_3 - \alpha_2x_3^3 - \delta x_1^2 x_3 + f \cos \theta) d\theta
\]  
\[= \epsilon \left( -dx_4 + A_2x_3 - \alpha_2x_3^3 - \delta x_1^2 x_3 \right). \quad (4.11b)
\]

The fixed points of the averaged eq.(4.11) correspond to the periodic orbits of (4.6c - 4.6e) of period \( 2\pi/\omega \), having the same stability type as the fixed points of the averaged equations. Further, the periodic orbits on \( M_\epsilon \) become fixed points of the four-dimensional Poincaré map of eq.(4.6) formed by fixing \( \theta = \tilde{\theta}(=2\pi/\omega) \). The fixed points of the averaged equations are

\[
(x_3, x_4) = (0, 0), (\pm \sqrt{I}, 0), \quad (4.12)
\]

where \( I = (A_2 - \delta x_1^2)/\alpha_2 \). When \( x_1 = 0 \), we obtain \( I = A_2/\alpha_2 \). The stability determining eigenvalues are obtained from

\[
\lambda^2 + \epsilon d\lambda - \epsilon^2 \left[ A_2 - 3\alpha_2x_3^2 - \delta x_1^2 - 2\delta x_1x_3 \left( \frac{dx_1}{dx_3} \right) \right] = 0. \quad (4.13)
\]
Figure 4.13 The geometry of the Poincaré map in a three-dimensional phase space by ignoring the one dimension of the state manifold for (a) $d^2 > 8A_2$ and (b) $d^2 < 8A_2$. 
The fixed point \((x_1, x_2, x_3, x_4) = (0, 0, 0, 0)\) has a two-dimensional stable and two-dimensional unstable manifolds. The fixed points \((0, 0, \pm \sqrt{t}, 0)\) are of saddle-node type. They have one-dimensional unstable and three-dimensional stable manifolds.

**Case 2: \(d^2 < 8A_2\)**

The saddle point \((0, 0, 0, 0)\) has a two-dimensional stable and two-dimensional unstable manifolds. The fixed points \((0, 0, \pm \sqrt{t}, 0)\) are saddle-focus and possess three-dimensional stable manifolds (spiralling in the \(x_3, x_4\) directions) and one-dimensional unstable manifolds.

Figure (4.13) shows the geometry of the Poincaré map where the coordinate \(x_2\) is suppressed for clarity. For the system of the form

\[
\begin{align*}
    \dot{x}_1 &= f_1(x_1, x_2, y_1, y_2) + \epsilon g_1(x_1, x_2, y_1, y_2, t), \\
    \dot{x}_2 &= f_2(x_1, x_2, y_1, y_2) + \epsilon g_2(x_1, x_2, y_1, y_2, t), \\
    \dot{y}_1 &= \epsilon G_1(x_1, x_2, y_1, y_2, t), \\
    \dot{y}_2 &= \epsilon G_2(x_1, x_2, y_1, y_2, t),
\end{align*}
\]

the Melnikov function is given by [11]

\[
M(t_0) = \int_{-\infty}^{\infty} \left[ < D_X H(X, Y), g > + < D_Y H(X, Y), G > \right](X(\tau), Y, \tau) \, d\tau
\]

\(64\)
\[- <D_Y(X(Y), Y), \int_{-\infty}^{\infty} G(X, Y, \tau) d\tau >, \quad (4.15)\]

where \(D_Z\) denotes differentiation with respect to \(Z\), \(X = (x_1, x_2)\), \(Y = (y_1, y_2)\). \(H\) is the Hamiltonian of the unperturbed system and \(\bar{Y}\) is the fixed point of the averaged eq. (4.11). Here, \(<f, g>\) represents the inner product of \(f\) and \(g\). The homoclinic trajectories \((x_{1h}, x_{2h})\) of the unperturbed system of (4.6) are given by the following analytical expressions.

**Case 1: \((x_3, x_4) = (0, 0)\)**

\[
x_{1h}(\tau) = \pm \sqrt{\frac{2A_1}{\alpha_1}} \text{sech}\sqrt{A_1} \tau, \quad (4.16a)
\]
\[
x_{2h}(\tau) = \mp \sqrt{\frac{2}{\alpha_1}} A_1 \text{sech}\sqrt{A_1} \tau \tanh\sqrt{A_1} \tau. \quad (4.16b)
\]

where \(\tau = (t - t_0)\).

**Case 2: \((x_3, x_4) = (\pm \sqrt{I}, 0)\)**

\[
x_{1h}(\tau) = \pm \sqrt{\frac{2(A_1 - \delta I)}{\alpha_1}} \text{sech}\sqrt{A_1 - \delta I} \tau, \quad (4.17a)
\]
\[
x_{2h}(\tau) = \mp \sqrt{\frac{2}{\alpha_1}} (A_1 - \delta I) \text{sech}\sqrt{A_1 - \delta I} \tau \tanh\sqrt{A_1 - \delta I} \tau. \quad (4.17b)
\]

Using the homoclinic orbits (4.16) and (4.17) in (4.15) and evaluating the integral we obtain the Melnikov function.

**Case 1: \((x_3, x_4) = (0, 0)\)**

\[
M(t_0) = \left[ -4dA_1^{3/2}/(3\alpha_1) \right] \pm f\pi \sqrt{\frac{2}{\alpha_1}} \text{sech} \left[ \pi \omega/\left(2\sqrt{A_1}\right) \right] \sin \omega t_0. \quad (4.18)
\]
Figure 4.14 Threshold curves for horseshoe chaos in the $(\omega, f)$ plane for (a) $\delta = 0.1$ and (b) $\delta = 0.05$. The other parameters are fixed at $\alpha_1 = 1, \alpha_2 = 0.1, A_1 = 1, A_2 = 0.11$ and $d = 0.4$. 
Case 2: \((x_3, x_4) = (\pm \sqrt{I}, 0)\)

\[ M(t_0) = A + f(B\cos \omega t_0 \pm C\sin \omega t_0), \]  

\[ (4.19a) \]

where

\[ A = 4(A_1 - \delta I)^{3/2} \left[ -d\alpha_1 - 4\delta I + 3\delta I\alpha_1 (A_2 - \alpha_2 I) / (A_1 - \delta I) \right] / (3\alpha_1^2), \]

\[ B = \delta \sqrt{I} \pi \omega \cosech \left[ \pi \omega / (2\sqrt{A_1 - \delta I}) \right] / \alpha_1, \]

\[ C = \sqrt{2} \pi \omega \sech \left[ \pi \omega / (2\sqrt{A_1 - \delta I}) \right] / \alpha_1. \]  

\[ (4.19b) \]

The case \((x_3, x_4) = (0, 0)\) corresponds to the uncoupled Duffing oscillator [11]. Now we shall analyse case 2. The necessary condition for the intersection of stable and unstable orbits is obtained as

\[ f \geq f_M = |A| \sqrt{B^2 + C^2}. \]  

\[ (4.20) \]

The sufficient condition requires the existence of simple zeros of \(M(t_0)\). For \(f > f_M\), \(M(t_0)\) oscillates between positive and negative values indicating that the stable and unstable manifolds intersect transversely. The value of \(f_M\) corresponds to the homoclinic tangency. The prediction of \(f_M\) from eq.(4.20) is plotted in fig.(4.14) in \((\omega, f)\) parameter space. Horseshoe chaos occurs in the region above the threshold curve.

4.3.3 Numerical Simulation

The existence of horseshoe does not imply that the typical trajectories
Figure 4.15 Bifurcation diagram as a function of $f$ for $\omega = 1$ and $\delta = 0.1$ for the coupled oscillators.
Figure 4.16 Melnikov threshold curves for the coupled systems (continuous curve) for \( \delta = 0.1 \) and uncoupled system (dashed curve).
will be asymptotically chaotic. However, it can exert a dramatic influence on the behaviour of orbits which pass close to it. In many dynamical systems, the presence of horseshoe was shown to be the starting point over which the systems underwent some of the possible routes to chaos. Consequently, the Melnikov threshold curve is considered as a lower threshold for the onset of asymptotic chaos. In view of this, we have numerically investigated the onset of chaos in eq.(4.6). For $\delta = 0.1$, the Melnikov threshold value $f_M$ is 0.305. The other parameter values are fixed as $d = 0.4$, $\alpha_1 = 1$, $\alpha_2 = 0.1$, $A_1 = 1$, $A_2 = 0.11$ and $\omega = 1.0$. Figure (4.15) shows the bifurcation phenomenon as a function of the parameter $f$. Period doubling phenomenon leading to chaotic motion is found to occur when the parameter $f$ is varied from a small value. We denote $f_c$ as the critical value of the parameter $f$ at which onset of chaos where trajectory jumps between positive and negative values of $x_1$ and $x_3$ occurs. Numerically, the onset of chaos is found to occur at $f_c = 0.32$.

To know the influence of the second oscillator (4.6c - 4.6d) on the onset of chaotic dynamics of the coupled Duffing oscillators, we have studied the onset of chaos in the uncoupled Duffing oscillator

$$x_1 = x_2, \quad (4.21a)$$

$$\dot{x}_2 = A_1 x_1 - \alpha_1 x_1^3 - \epsilon(-dx_2 + f\cos\theta). \quad (4.21b)$$

In eq.(4.21) the parameter values are fixed at $A_1 = 1$, $\alpha_1 = 1$, $d = 0.4$ and $\omega = 1$, the same as those used in the coupled oscillators. Figure (4.16) shows the threshold curves for the onset of chaos in both the uncoupled and coupled oscillators. The continuous and dashed curves represent $f_M$ for the coupled oscillator.
Figure 4.17 Poincaré map of the chaotic attractor of the uncoupled system at the onset of chaos.
Figure 4.18 Chaotic attractors of the coupled oscillators.
and uncoupled systems respectively. From this figure, we note that for \( \omega \) less than a critical value \( \omega_c \) the \( f_M \) of the coupled oscillators is lower than that of the uncoupled system. However, for \( \omega > \omega_c \) the \( f_M \) value of the coupled systems is higher than that of the uncoupled oscillator (4.21). This is further verified by numerical experiment. In the uncoupled case onset of chaos is found at \( f_c = 0.307 \) (\( f_M = 0.3013 \)). This \( f_c \) value can be compared with the value \( f_c = 0.32 \) of the coupled oscillators. That is, onset of chaos is delayed in the coupled oscillators.

The nature of the chaotic attractor at \( f_c \) is also studied in the coupled and uncoupled oscillators. Figure (4.17) shows the Poincaré map of the chaotic attractor at \( f_c = 0.307 \) for the uncoupled system. Figure (4.18) shows the attractor of the coupled systems for \( f_c = 0.32 \). The influence of the coupling term and second oscillator on the structure of the chaotic attractor can be clearly seen. In the uncoupled system, the chaotic attractor consists of thin layer structures. The geometrical structure of the attractors of both the coupled and uncoupled systems appear almost similar. However, in the coupled systems due to coupling term, the points in the \( x_1 - x_2 \) plane are distributed in the neighbourhood of the layers. On the other hand, the geometrical structure of the coupled systems in the \( x_3 - x_4 \) plane is highly different. This is due to the small values of the parameters of the second oscillator. A detailed analysis of the prediction of onset of chaos has been performed for different sets of \( \omega \) and \( d \). The results are summarized in table (4.2). From this table, the analytical prediction is found to be in good agreement with the numerical analysis of the system. Since the second oscillator is treated as weak, the present analysis is applicable for small values of \( A_2, \alpha_2, d \) and \( f \).
Table 4.2 Critical values of $f_M$ and $f_c$ for various values of $d$ and $\omega$ for the coupled Duffing oscillators.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$d$</th>
<th>$f_M$</th>
<th>$f_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.25</td>
<td>0.201</td>
<td>0.237</td>
</tr>
<tr>
<td></td>
<td>0.30</td>
<td>0.236</td>
<td>0.245</td>
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<tr>
<td></td>
<td>0.35</td>
<td>0.271</td>
<td>0.278</td>
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<tr>
<td></td>
<td>0.40</td>
<td>0.305</td>
<td>0.320</td>
</tr>
<tr>
<td></td>
<td>0.45</td>
<td>0.341</td>
<td>0.350</td>
</tr>
<tr>
<td>1.5</td>
<td>0.40</td>
<td>0.451</td>
<td>0.580</td>
</tr>
<tr>
<td>1.8</td>
<td>0.40</td>
<td>0.631</td>
<td>0.705</td>
</tr>
<tr>
<td>2.0</td>
<td>0.40</td>
<td>0.760</td>
<td>0.810</td>
</tr>
</tbody>
</table>

4.4 CONCLUSION

In this Chapter, we have studied the occurrence of chaotic motion in the Duffing oscillators eqs.(1.2) for the three physically interesting potential wells for specific parametric values. When the system is subjected to external periodic forces a variety of interesting behaviours such as period doubling phenomenon, coexistence of multiple attractors, chaotic motions and merging of attractors occur. By varying the forcing amplitude $f$ from a small value, the above mentioned behaviours were studied for three types of potential wells.

Also we have applied the Melnikov-analytical technique to predict onset of chaos. Even though the scaling of eq.(4.6) seems to be artificial, it is done in order to obtain the Melnikov function in terms of all the parameters including $\delta$. Interestingly, the calculated Melnikov function indeed depends on all the parameters of the system which clearly justifies the scaling introduced in
The influence of the second oscillator (4.6c–4.6d) on the onset of chaotic dynamics of the coupled Duffing oscillators, has also been studied. The analytical prediction is found to be in good agreement with the numerically predicted onset of chaos in the two coupled Duffing oscillators.