CHAPTER 2

REGULARITY IN RINGS

In this chapter we shall study regularity and one-sided regularity (i.e., left and right regularity) for elements of a ring. Some basic results concerning one-sided regularity are collected in §1; 2-finite rings are defined and studied in §2; normal, 2-finite rings are studied in §3; sufficient conditions for $\text{Reg}(R)$ to be closed under multiplication are given in §4.

§1. One-sided regularity in rings.

A main result of this section (Proposition 2.6(a)) is in [A:54]; see also Proposition 1(1) in [H:78]. However, our proof, which uses the concept of the idealizer, is different and some intermediate results are used later.

The terminology of §B of Chapter 0 will be followed; $i$ denotes a non-negative integer and $m, n$ denote natural numbers. The letters $a, b, c$ denote, as usual, elements of a ring. The notation $0.6$ for annihilators will be followed.
2.1. Let \( a \) be right regular with \( a^2b = a \). Then:

(a) \( 1(a^2) = 1(a) \).

(b) For each \( i \), \( a^{i+1}b^i = a \) holds. (This is clear for \( i = 1 \) (and for \( i = 0 \)). For \( i > 2 \) the equation \( a^ib^i = a \) yields \( a^ib^i = a^{i+1}b^i = a^ib^i-1 \).

(c) Suppose that \( i + 1 \leq n \). Then we have

\[
a^nb^i = a^{n-1}a^{i+1}b^i = a^{n-1}a = a^{n-1},
\]

using Part (b).

(d) Substituting \( i = m \) and \( n = 2m \) in (c) we get

\[
a^{2m} = a^m.
\]

Thus \( a^m \) is right regular for each \( m \).

Although it is possible to deduce the next result from Lemma 3 of [A:54] we give a direct proof.

2.2. Proposition. Let \( a \) be right regular and let \( a^m \) be left regular for some \( m \). Then \( a \) is left regular.

Proof. Let \( a^2b = a \) and \( ca^m = a^m \) for some \( b, c \).

Then using (b) and (d) of 2.1 we get

\[
a = a^{m}b^{m-1} = ca^{-2m}b^{m-1} = ca^{-2}a^{m-1} = (ca^{-m-1})a^2.
\]

This shows that \( a \) is left regular.

Next we recall a well-known concept.

2.3. Let \( A \) be a right ideal of a ring \( R \). Let \( \mathcal{T}(A) \) be the set of all elements \( t \) of \( R \) satisfying \( tA \subseteq A \). Then
\( \mathcal{I}(A) \) is a subring of \( R \), called the **idealizer of** \( A \).

(in \( R \)); \( \mathcal{I}(A) \) contains \( A \) as an ideal, being the largest subring which does so.

2.4. **Proposition.** Suppose that \( a \) is right regular with \( a^2b = a \). Assume further that \( r(a^2) = r(a) \). then:

(a) \( b \in \mathcal{I}(r(a)) \), the idealizer of \( r(a) \).
(b) \( ab^n a = ab^{n-1} \) for each \( n \).
(c) \( c = ab^2 \) is a semi-inverse of \( a \); in particular, \( a \) is left regular.
(d) \( ab = ac \) is an idempotent.

**Proof.** (a) Since \( a^2b \cdot r(a) = a \cdot r(a) = 0 \) we have \( b \cdot r(a) \in r(a^2) = r(a) \).

(b) Since \( \mathcal{I}(r(a)) \) is a subring of \( R \), by Part (a) we get \( b^{n-1} \in \mathcal{I}(r(a)) \) for each \( n \). Also, \( a^2b = a \) yields \( a^2(ba - 1) = 0 \) and so \( ba - 1 \in r(a^2) = r(a) \). So we get \( b^{n-1}(ba - 1) \in r(a) \). This implies that \( ab^n a = ab^{n-1} \) for each \( n \).

(c) Part (b) yields, on putting \( n = 1, 2, 3 \) successively,

(2.5) \( aba = a, ab^2 a = ab, ab^3 a = ab^2 \).
Next, using 2.5, we get,

\[ ac = a^2b^2 = ab = ab^2a = ca, \quad c^2a = ab^2ab^2a = ab.b^2a \]
\[ = ab^3a = ab^2 = c, \quad \text{and} \quad a^2c = a^3b^2 = a \quad \text{(using 2.1 (b))}. \]

This shows that \( c \) is a semi-inverse of \( a \).

Therefore \( a \) is left regular.

(d) This follows from 2.5 and (c). 

The proof of Part (c) of the above proposition yields a different proof of (most of) the following proposition due to Azumaya; we record 2.6 since we need it later.

2.6. Proposition (Azumaya). (a) Let \( a \) be right regular.

Then \( a \) is left regular if and only if \( r(a^2) = r(a) \).

(b) An element \( a \) is strongly regular if and only if \( a \) is a semi-unit. In this case \( a \) has a unique semi-inverse \( z \). Moreover \( C(a) = C(z) \) (in the notation of 0.7).

2.7. Proposition. Let \( a \) be an element of a ring \( R \), which is a subring of \( R' \). Suppose that \( a \) is \( R \)-right regular and \( R' \)-left regular (see 0.18). Then \( a \) is \( R \)-left regular.
Proof. By hypothesis, there exists an element $t$ of $R'$ such that $a = ta^2$. Therefore $r_R(a) = r_R(a^2)$. Hence, by 2.6(a), $a$ is $R$-left regular.

§ 2. 2-finite rings.

Let $K$ be a field and $M_n(K)$, the ring of $n \times n$ matrices over $K$. Let matrices $A, B$ satisfy $A = A^2B$. It can be shown using linear algebra that there exists a matrix $C$ such that $A = CA^2$. Thus in $M_n(K)$ every right regular element is left regular and the dual property also holds. Thus $M_n(K)$ is left and right 2-finite in the sense of the following definition. (These properties of $M_n(K)$ also follow as a special case of 2.19 or 2.21 below.)

2.8. Definition. A ring $R$ is right 2-finite if every right regular element is left regular.

2.9. Remarks. (i) A ring $R$ has been called $n$-finite in the literature if $M_n(R)$ is a directly finite ring; see [L:70]. It is hoped that our usage of the term "2-finite", in a different sense, will not cause confusion.
(ii) Although the term "2-finite" is not used by Azumaya [A:54] his Theorem 1 asserts that every ring of bounded index is left and right 2-finite. Apart from this result we have not come across any study of 2-finite rings in the literature.

2.10. Remark. Let \( a \) be right invertible and left regular. The equations \( ca^2 = a \) and \( ab = 1 \) yield

\[
1 = ab = ca^2b = ca.ab = ca
\]

It follows that \( c = b \) and hence \( a \) is invertible.

2.11. Proposition. Every 2-finite ring is directly finite. (See 0.13.)

Proof. Let \( R \) be right 2-finite. Suppose that \( a, b \in R \) satisfy \( ab = 1 \). Then \( a = a^2b \) shows that \( a \) is right regular. By hypothesis \( a \) is left regular. It follows by Remark 2.10 that \( a \) must be invertible. So by 0.13 \( R \) is directly finite.

2.12. Remark. We do not have an example of a ring which is directly finite but not 2-finite. The results of this section may be of some use for settling the question of the existence of such rings.
In the rest of this section we give sufficient conditions for the 2-finiteness of a ring. These results are analogues of known (or easy) results concerning the direct finiteness of a ring. However many of the proofs are different. In 2.13 - 2.15 we deduce the 2-finiteness of a ring from the 2-finiteness of certain rings related to it.

2.13. Proposition. Let $R$ be a subring of a right 2-finite ring $R'$. Then $R$ is right 2-finite.

Proof. Let $a$ be $R$-right regular (and hence $R'$-right regular). As $R'$ is right 2-finite, $a$ is $R'$-left regular. Hence, by 2.7, $a$ is $R$-left regular.

2.14. Remark. Let $\{R_i\}_{i \in I}$ be a family of rings. Then their direct product $\prod R_i$ is right 2-finite if and only if each $R_i$ is right 2-finite. (This can be verified easily.)

2.15. Proposition. Let $R$ be a ring with centre $C$. Suppose that for each maximal ideal $v$ of $C$ the localization $R_v$ is a right 2-finite ring. Then $R$ is right 2-finite.
Proof. Let $R'$ be the direct product of the rings $\mathbb{R}_g$.

By 2.14 the ring $R'$ is right 2-finite. There is a canonical injective ring homomorphism from $R$ to $R'$.

Hence, by 2.13 $R$ is also right 2-finite.

2.16. Remark. Note that 2.13 - 2.15 have easily verifiable analogues for directly finite rings.

2.17. Azumaya [A:54] called an element $a$ of a ring right $\pi$-regular if some suitable power of $a$ is right regular; $R$ is right $\pi$-regular if so is every element of $R$; an element (or a ring) is strongly $\pi$-regular if it is left and right $\pi$-regular. Dischinger [D:76] proved that every left $\pi$-regular ring is right $\pi$-regular; Zöschinger [D:76] and Hirano [H:78] have given simplified proofs of this fact. Thus there is no distinction between left $\pi$-regularity, right $\pi$-regularity and strong $\pi$-regularity for rings. Clearly strongly regular rings (0.23) and artinian rings are strongly $\pi$-regular. Commutative $\pi$-regular rings have been characterised in Lemma 5.6 of Storrer [S:68]. They are precisely the zero-dimensional rings.

It is easy to see that strongly $\pi$-regular rings are directly finite.
2.18. Remark. Let $R$ be a strongly $\pi$-regular ring. Let $a$ be a right regular element of $R$. By the left $\pi$-regularity of $R$ the element $a^m$ is left regular for some $m$. Hence by 2.2 $a$ is left regular. This shows that $R$ is right (and, similarly, left) 2-finite. 

It is known that left (or right) noetherian rings are directly finite. In the following theorem we consider larger classes of rings.

2.19. Theorem. Let $R$ be a ring which satisfies the ascending chain condition on right annulets (i.e. right ideals of the form $r(J)$). Then $R$ is right 2-finite.

Proof. Let $a$ be right regular. Consider the chain $r(a) \subseteq r(a^2) \subseteq \ldots$. By the hypothesis on $R$ this chain terminates and so $r(a^m) = r((a^m)^2)$ for some $m$. By 2.1(d), the element $a^m$ is right regular. Hence by 2.6(a) $a^m$ must be left regular. It follows, by an application of 2.2, that $a$ must be left regular.

2.20. Among rings satisfying the hypothesis of 2.19 are right noetherian rings, left artinian rings (their subrings) and domains.
2.21. Proposition. Let $R$ be finitely generated as a module over its centre $C$. Then $R$ is left and right 2-finite. (It is known that such rings are directly finite; see Corollary 3 of Orzech [0:71].)

Proof. Let $x_1, x_2, \ldots, x_r$ be a set of generators of $R$ as a module over $C$. Let $a$ be a right regular element of $R$ with $a^2b = a$. Let $S_0$ be a finite set of elements of $C$ which appear as coefficients of the $x_i$ when $a, b, x_jx_k$ and $x_jx_k$ ($1 \leq j, k \leq r$) are expressed as $C$-linear combinations of the $x_i$ ($1 \leq i \leq r$).

Let $C_0$ be the subring of $C$ generated by $S_0$ and let $R_0$ be the $C_0$-subalgebra of $R$ spanned by the elements $x_i$. Then $R_0$ is left and right noetherian and $a, b \in R_0$. We are now through by Theorem 2.19.

Next we recall a result of Peterson ([P:75], Proposition 2.3.)

2.22. Proposition. Let $R$ be a ring integral over its centre $C$. Then $R$ is directly finite.

By adapting Peterson's argument we prove:
2.23. Proposition. Let $R$ be integral over its centre $C$. Then $R$ is left and right 2-finite.

Proof. Assume that $a^2b = a$. Write

$$b^{n+1} = \sum_{j=0}^{n} c_j b^j,$$

with $c_j \in C$ for each $j$. Then, using 2.1(b), we get,

$$a = a^{n+2}b^{n+1} = \sum_{j=0}^{n} c_j a^{n+2}b^j = \sum_{j=0}^{n} c_j a^{n+2-j} = \left( \sum_{j=0}^{n} c_j a^{n-j} \right)a^2$$

This shows that $a$ is left regular.

2.24. Remark. Although 2.21 can be deduced from 2.23 both proofs have been given since each appeared to be of independent interest.

§3. Normal, 2-finite rings.

The main result of this section characterises normal 2-finite rings. Notation 0.8 for sets of idempotents will be followed.
2.25. Theorem. The following conditions are equivalent for a ring $R$ with centre $C$.

(1) $R$ is normal and right 2-finite.
(2) If $a = a^2b$, then $ab \in C$.
(3) If $a = a^2b$, then $a = ba^2$.
(4) If $a = a^2b$, then $ab$ and $ba$ are central idempotents.

Proof. (1) $\Rightarrow$ (2). Let $a = a^2b$ hold for elements $a, b$ of $R$. Since $R$ is right 2-finite, $a$ is left regular; equivalently, $r(a) = r(a^2)$. By 2.4(d) $ab \in i(R)$. As $R$ is normal, $ab \in C$.

(2) $\Rightarrow$ (3). Let $a = a^2b$. Then, by hypothesis, $ab \in C$ and therefore $a = aab = aba$. Hence $ba = (ba)^2.1$ implies (again invoking the hypothesis) $ba.1 \in C$.

Therefore $a = aba = baa = ba^2$.

(3) $\Rightarrow$ (1). Let $e = e^2, y \in R$. Write $n: = (1 - e)ye$. Then $n^2 = 0, en = 0$ and $ne = n$. Hence $e^2(e + n) = e$, which implies that $(e + n)e^2 = e$. Therefore $n = 0$.

It follows that $ye = eye$ for each idempotent $e$. Replacing
e by \((1 - e)\) in the last equation we get
\[y(1 - e) = (1 - e)y(1 - e)\]
which yields \(ey = ye\).
Thus \(ey = ye = ye\) for each \(y \in R\), showing that
\(e \in C\). This proves that \(R\) is normal. It is trivially right 2-finite.

\((2) \Rightarrow (4)\). This follows by the proof of \((2) \Rightarrow (3)\);
\(a = aba\) yields the idempotency of \(ab\).
\((4) \Rightarrow (2)\) is trivial.

This completes the proof of the theorem.

In Proposition 2.26 it will be shown that
semi-commutative rings are left and right 2-finite. It will then follow from 0.14 that all left or right duo rings are left and right 2-finite.

2.26. Proposition. Semi-commutative rings are (normal and)
left and right 2-finite.

Proof. The normality of semi-commutative rings (0.14)
will be used in the proof of this proposition. We shall verify Condition (2) of Theorem 2.25. Let \(a = a^2b\),
i.e. \(a(1 - ab) = 0\). As \(R\) is semi-commutative
ar(l - ab) = 0 for each r ∈ R. In particular, ab(l - ab) = 0 showing that ab ∈ I(R) ⊂ C, by the normality of R.

2.27. Examples. Normal, left and right 2-finite rings need not be semi-commutative. This is shown by the following examples.

(1) The ring of matrices of the form \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
where a + d, b and c are even integers is an example of a normal ring which is not semi-commutative.

(See Example 5.5 in [1].) As this ring is a subring of $M_2(\mathbb{Q})$ it is left and right 2-finite by 2.13.

(2) Let $G$ be a finite group and $Z(G)$ its integral group ring. It is well-known (see, e.g., p.35 of [1]) that $Z(G)$ has no non-trivial idempotents; it is therefore a normal ring. As $Z(G)$ is finitely generated as a $\mathbb{Z}$-module, it is left and right 2-finite, by 2.21. Since $Q(G)$ is the localization of $Z(G)$
with respect to the set of non-zero integers, it is easily seen that \( Z(G) \) is semi-commutative if and only if \( Q(G) \) is semi-commutative. By Maschke's theorem \( Q(G) \) is semi-simple, and therefore, by Wedderburn's structure theorem, a finite product of matrix rings over division rings. It is easy to give examples of groups \( G \) such that \( Q(G) \) is not reduced (equivalently, because of the above argument, not semi-commutative). For these groups \( Z(G) \) is normal, left and right 2-finite, but not semi-commutative.

§ 4. Closure properties.

In this section we shall give sufficient conditions on a ring \( R \) which ensure that the subset \( \text{Reg}(R) \) of all regular elements of \( R \) is closed under multiplication.

2.28. Definition. A ring \( R \) has property (RC) if the subset \( \text{Reg}(R) \) is closed under multiplication.

2.29. Remark. If \( R \) is commutative then trivially \( R \) has property (RC): \( a = aba \) and \( c = cdc \) implies that \( ac = aba \) \( cdc = acbd \) \( ac \).
2.30. Remark. If $R$ is regular then $\text{Reg}(R) = R$ shows that $R$ has property (RC).

2.31. Example. Let $R'$ be a subring of a ring $R$. Then $R$ has property (RC) does not imply $R'$ has property (RC), as shown by the following example.

The ring $M_n(D)$, where $D$ is a division ring, and $n \geq 2$ is regular and therefore has property (RC). Let $R' = UT_n(D)$, the upper triangular matrix ring, a subring of $M_n(D)$. The elements $E_{11} + E_{12}$ and $E_{22}$ are idempotents and so belong to $\text{Reg}(R')$.

But $(E_{11} + E_{12})E_{22} = E_{12} \not\in \text{Reg}(R')$.

This example also shows the following:

(i) A left and right artinian ring need not have property (RC).

(ii) $R/\text{Rad}(R)$ has property (RC) does not imply that $R$ has property (RC). In the above example $R'/\text{Rad}(R')$ is regular, being a finite product of copies of the division ring $D$. 

2.32. Remark. Let \( \{ R_i \}_{i \in I} \) be a family of rings and let \( R = \prod_{i \in I} R_i \). Then \( \text{Reg}(R) = \prod_{i \in I} \text{Reg}(R_i) \), with natural identification. (This generalises the well-known result that \( R \) is regular if and only if each \( R_i \) is regular.)

2.33. Remark. It follows from 2.32 that the ring \( \prod_{i \in I} R_i \) has property \((RC)\) if and only if each \( R_i \) has property \((RC)\).

The main theme of this chapter is `regularity in rings'. However it seems natural to deduce a sufficient condition for the \((RC)\) property to hold (2.36) from a proposition valid in modules.

2.34. Proposition. Let \( R \) be a ring with centre \( C \) and \( M \) a left \( R \)-module. Let \( a \in R \) and \( m \in M \). Assume that there is an element \( b \in R \) such that \( a = aba \) and \( ba \in C \). Then we have:

(i) If \( m \) is e-regular, then \( am \) is e-regular.
(ii) If \( m \) is regular, then \( am \) is regular.

Proof. Write \( e = ba \). Then \( e^2 = e \in C \), and \( Ra = Re \).
(i) By assumption, $R_m$ is a direct summand of $M$. Since $e$ is central, $R_m = R_{em} \oplus R(1 - e)m$. Thus $R_{am} = R_{em}$ is a direct summand of $M$. Hence $am$ is $e$-regular in $M$.

(ii) Since $m$ is regular there exists $f \in \text{Hom}_R(M, R)$ such that $(mf)m = m$.

Now $am = (aba)(mf)m = (am)(fba)m$, as $bae C$.

Therefore, $am = (am)g(am)$ where $g = fbe \text{Hom}_R(M, R)$.

It follows that $am$ is regular in $M$. #

2.35. Corollary. Let $R$ be a normal ring and $M$ a left $R$-module. If $a$ is regular in $R$ and $m$ is $e$-regular (resp. regular) in $M$, then $am$ is $e$-regular (resp. regular) in $M$.

2.36. Corollary. If $R$ is a normal ring, then $R$ has property (RC). (This extends 2.29.)

2.37. Remarks. (i) Let $\text{Reg}_l(R)$ (resp. $\text{Reg}_r(R)$) denote the set of all left regular (resp. right regular) elements of $R$. Trivially, $\text{Reg}_l(R)$ and $\text{Reg}_r(R)$ are closed under multiplication in commutative rings and in strongly regular rings (0.22).
(ii) Let $\mathcal{P} = \text{Reg}_{1}(R) \cap \text{Reg}_{r}(R)$, the set of all strongly regular elements of $R$. Then remarks similar to 2.32 and 2.33 can be made about the sets $\text{Reg}_{1}(R)$, $\text{Reg}_{r}(R)$ and $\mathcal{P}$.

(iii) Let $D$ be a division ring and $R = M_n(D)$, with $n \geq 2$. Let $A = E_{11} + E_{12}$ and $B = E_{22}$. Then $A$, $B$ are idempotents and therefore belong to $\mathcal{P}$. However $AB = E_{12}$ is a non-zero nilpotent element; so $AB$ is neither left regular nor right regular.

In the final result of this section we give a sufficient condition for the strong regularity of $a_1$, $a_2$ to imply the strong regularity of $a_1 a_2$. (As seen in 2.37 (iii) this does not always happen.)

2.38. Theorem. Let $a_1$, $a_2$ be strongly regular elements of a ring $R$. If $a_1$ and $a_2$ commute then $a_1 a_2$ is strongly regular.

Proof. We shall use results of §1 of this chapter, especially 2.6(b). Let $z_i$ be the semi-inverse of $a_i$ for $i = 1, 2$. Then $C(a_i) = C(z_i)$ for $i = 1, 2$. 

Now \( a_1 \in C(a_2) \) and \( a_2 \in C(a_1) \) by hypothesis. So 
\[ a_1 \in C(z_2) \text{ and } a_2 \in C(z_1). \]
Hence \( z_1 \in C(a_2) = C(z_2) \).

Thus the elements \( a_1, z_1, a_2, z_2 \) commute with each other.

Therefore we have: \( (a_1a_2)(z_1z_2) = (z_1z_2)(a_1a_2) \),
\[ (a_1a_2)^2(z_1z_2)^2 = a_1^2z_1^2a_2^2z_2^2 = a_1a_2 \text{ and finally,} \]
\[ (a_1a_2)(z_1z_2)^2 = a_1z_1^2a_2z_2^2 = z_1z_2. \]

These computations show that \( a_1a_2 \) is a strongly regular element of \( R \).