CHAPTER 1

REGULARITY IN MODULES

In this chapter we shall study regularity in modules. Definitions of regular (elements in) modules in the sense of Zelmanowitz and Elliger are given in §1. Some basic results concerning these are also recorded in this section. In §2 McCoy's lemma for rings is extended to modules; this result is then applied to yield, inter alia, another proof of the fact that $A$, $B$ regular modules implies $A \oplus B$ regular. We devote §3 to the construction of pairs of commutative rings $R$, $U$ such that $R$ is regular but $R$ is not regular in $R_U$. Localizations of regular modules are studied in §4.

§ 1. Regularity and e-regularity.

In this section we define regular and e-regular elements and modules. Easy consequences of these definitions are derived and examples given. Many of these results may be known. However, we have not seen them recorded in the literature.

This section is divided into five parts. The heading of each will make its contents clear.
A. Basic definitions

1.1. An element $m$ of a module $\mathbb{R}^M$ is regular in $M$ (when there is no possibility of confusion, regular) if there exists an element $f$ in $\text{Hom}(M, \mathbb{R})$ such that $(mf)m = m$.

1.2. A subset $B$ of $M$ is regular if each element of $B$ is regular.

1.3. A module is regular if each of its elements is regular.

The following proposition can be extracted from the proof of Theorem 2.2 of [Z:72].

1.4. Proposition. Let $\mathbb{R}^M$ be a module and let $m \in M$. Then $m$ is regular if and only if $Rm$ is a projective direct summand of $M$.

Remarks concerning the proof. In view of the fundamental nature of this proposition, we point out the following:

"Only if" part. A direct proof can be found in the proof of Lemma 1.1 of [N:76].

"If" part. Since we assume (in contrast to Zelmanowitz) that $R$ has an identity element, a shorter proof can be given: Consider the map $h: R \to Rm$ defined by $rh = rm$. As $Rm$ is projective, $h$ is split by a map
g:Rm \to R satisfying goh = 1_{Rm}. As Rm is a direct summand of M, there exists an extension f:M \to R of g. Now we have m = (mg)h = (mf)h = (mf)m, which proves the regularity of m.

Elliger (§ 4, [E:71]) has defined regular modules differently; they have been called as e-regular by us. Below we give his definition, preceded by an element-wise definition.

1.5. An element m of a module M is e-regular in M (or e-regular) if Rm \subseteq M.

1.6. A subset B of M is e-regular if each element of B is e-regular.

1.7. A module is e-regular if each of its elements is e-regular.

1.8. Notation. For a module D, Reg(D) (respectively, eReg(D)) will denote the set of all regular (respectively, e-regular) elements of D. When the ring is clear from the context we shall drop the letter R.

1.9. Remark. Reg(D) \subseteq eReg(D); this follows from 1.4 and 1.5. Conditions on D which ensure that eReg(D) = Reg(D) are given in 1.17. If D is an e-regular non-regular module clearly we have eReg(D) \neq Reg(D). Examples of such modules are given in 1.18 and 1.24(iii).
1.10. Shrikhande [S:73], called a module hereditary if every submodule is projective and semi-hereditary if every finitely generated submodule is projective. A ring $R$ is left hereditary (semi-hereditary) if $R^R$ is hereditary (semi-hereditary). Evans [E:72] called a module a C.P.module if each of its cyclic submodules is projective. Hill [H:85] called a module a P.P.module if it is a projective C.P. module. A ring $R$ is a left p.p.ring if $R^R$ is a C.P.module (equivalently, a P.P.module). We have the implications:

\[
\begin{array}{c}
\text{hereditary} \\
\text{semi-hereditary} \\
\text{projective} \\
\text{P.P.} \\
\text{projective}
\end{array}
\Rightarrow
\begin{array}{c}
\text{regular} \\
\text{semi-hereditary} \\
\text{C.P.}
\end{array}
\]

Of these, regular $\Rightarrow$ semi-hereditary is a part of Theorem 2.2 of [Z:72]. The rest are obvious.

Torsion free modules over commutative domains are C.P.modules. Thus the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Z}^N$ are non-projective C.P. modules. The ring $R = \mathbb{Z}/4\mathbb{Z}$ is not a p.p.ring; so $R^R$ is a non-C.P. module which is projective.

1.11. We shall follow Bourbaki [BII8] in the use of semi-simple. A ring is semi-primitive if its Jacobson radical is zero; such rings have been called as (Jacobson-)semi-simple by some.

Some of the above terminology will be used in the following remarks; the proofs are straight-forward.
1.12. Remark (Elliger). Semi-simple modules are e-regular.

1.13. Remark. Semi-simple, projective modules are regular.

1.14. Remark. Let $\mathcal{D}$ be a module such that $\text{Hom}_R(\mathcal{D}, R) = 0$. Then $\text{Reg}(\mathcal{D}) = 0$.

1.15. Remark. An element $d$ of a module $\mathcal{D}$ is regular if and only if it is e-regular and $Rd$ is projective.

1.16. Remark. A module is regular if and only if it is e-regular and a C.P.module.

1.17. Remark. Assume that either (1) $\mathcal{D}$ is projective or (2) $\mathcal{D}$ is a C.P.module. Then $e\text{Reg}(\mathcal{D}) = \text{Reg}(\mathcal{D})$.

1.18. Remark. Let $p$ be a prime. Since $\text{Hom}_Z(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$ we have $\text{Reg}(\mathbb{Z}/p\mathbb{Z}) = 0$ (by 1.14). In particular, the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z}$ is non-regular. Since it is simple, it is e-regular (over $\mathbb{Z}$ as well as $\mathbb{Z}/p\mathbb{Z}$), by 1.12.

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B. The classical case

The following (known) facts connect the above concepts with the 'classical' case of regularity in a ring.

1.19. Remark. Let $M = {}_R R$; then $\text{Hom}_R(M, R) = R$. Hence the condition in 1.1 reduces to: there exists $b$ in $R$ such that $aba = a$, recovering the definition (0.16) of a regular element in a ring. Since $\mathcal{R}$ and
1.20. Proposition. The following conditions are equivalent for an element \( a \) of a ring \( R \).

1. \( a \) is regular in \( R_R \).
2. \( a \) is regular in \( R_R \).
3. \( a \) is \( e \)-regular in \( R_R \).
4. \( a \) is \( e \)-regular in \( R_R \).
5. There exists an element \( b \) in \( R \) such that \( aba = a \).

1.21. Remark. It follows from 1.20 that \( \text{Reg}(R_R) = \text{Reg}(R_R) = \text{eReg}(R_R) = \text{eReg}(R_R) \) is the set of all regular elements of \( R \). This set will be denoted by \( \text{Reg}(R) \).

C. Change of ring

Let \( \phi: R \to R' \) be a ring homomorphism, \( D \) a left \( R' \)-module. Then \( D \) has a natural \( R \)-module structure via \( \phi \).

1.22. Proposition. Suppose \( \phi \) is onto. Then we have

1.22(*) \[ \text{eReg}(R_D) = \text{eReg}(R,D) \]

Proof. Let \( d \) be an element of \( D \). Since \( \phi \) is onto we have, (1) \( Rd = R'd \) and (2) the equivalence of the conditions "\( Rd \) is an \( R \)-direct summand of \( D \)" and "\( R'd \)
is an $R'$-direct summand of $D'$. The desired conclusion follows.

1.23. Corollary. Let $R$ be a regular ring and let $A$ be an ideal of $R$. Then $R/A$ is e-regular as a left (as well as a right) $R$-module.

**Proof.** Let $R' = R/A$, $D = R'$ and $\varphi: R \rightarrow R'$ the canonical ring homomorphism. Now $R'$ is a regular ring and is therefore (as seen in §1B) e-regular as a left module over itself. Hence $R'$ is e-regular as a left $R$-module.

1.24. Examples. The following examples may help understand the concepts introduced here. They will show that when $\varphi$ is not onto 1.22(*) may not hold.

(i) Let $R = \mathbb{Z}$, $R' = \mathbb{Q}$ and $\varphi$ the canonical inclusion. Let $D = \mathbb{Q}$. Then $e\text{Reg}(Q) = \{0\}$ but $e\text{Reg}(\mathbb{Q}) = \mathbb{Q}$ shows that $e\text{Reg}(D) \subseteq e\text{Reg}(R, D)$ in this situation.

(ii) Let $R = K$, a field and $R' = K[x]$ and $\varphi$ the canonical inclusion. Let $D = K[x]$. Then $e\text{Reg}(K[x]) = K[x]$ and $e\text{Reg}(\mathbb{K}[x]) = K$ shows that $e\text{Reg}(D) \supseteq e\text{Reg}(R, D)$ in this situation.

(iii) Let $K$ be a field, $U = K^N$, $A = K^{(N)}$; $A$ is an ideal of the regular ring $U$ and $U/A$ is also a regular ring (by 0.25). It follows, by 1.23, that the $U$-module $U/A$ is e-regular.
Now let \( f: U/A \to U \) be \( U \)-linear and \( I = (U/A)f \). Then \( AI = 0 \) implies that \( I \subseteq \text{Ann}(U/A) = 0 \). This shows that \( \text{Hom}_U(U/A, U) = 0 \) and therefore, by 1.14, \( \text{Reg}(U/U/A) = 0 \); in particular, \( U/A \) is not regular as a \( U \)-module.

There are natural ring homomorphisms \( K \to U \) and \( U \to U/A \). Thus \( U \) is "sandwiched" between \( K \) and \( U/A \). However, we have, using the fact that \( U/A \) is a \( K \)-vector space,

\[
\text{Reg}(U/U/A) = U/A \supset \text{Reg}(U/U/A) = \{0\} \subset \text{Reg}(U/U/A) = U/A.
\]

This example and 1.18 show that there is no regular analogue of 1.22.

1.25. Remark. Cheatham and Enochs [CE:81] have called a ring \( R \) (left) quasi-perfect if every finitely generated flat left \( R \)-module is projective. They have shown (Part(c) of Theorem 2) that over left quasi-perfect rings factor modules of regular modules are regular.

Elsewhere in that paper they have asserted that left coherent rings are left quasi-perfect. Notice that the ring \( U \) of Example 1.24(iii) is coherent, since it is regular. However the regular \( U \)-module \( U \) has \( U/A \) as a non-regular factor module. Therefore \( U \) cannot be quasi-perfect. This can also be seen directly (the \( U \)-module \( U/A \) is flat but non-projective) as was pointed out in
another context by 

D. Module pairs

In what follows, \((B, D)\) denotes a module pair \((0.2)\).

1.26. Remark. Let \(b\) be an element of \(B\) which is e-regular in \(D\). Since \(Rb\) is a direct summand of \(D\), it is also a direct summand of \(B\). This shows \(B \cap \text{eReg}(D) \subseteq \text{eReg}(B)\).

1.27. Remark. As in 1.26 we have \(B \cap \text{Reg}(D) \subseteq \text{Reg}(B)\).

1.28. Remark. Over \(R = \mathbb{Z}\) we have \(\text{Reg}(Q) = \text{eReg}(Q) = \{0\}\) and \(\text{Reg}(Z) = \text{eReg}(Z) = \{0, 1, -1\}\). Thus strict inclusions can hold in 1.26 and 1.27.

1.29. Remarks. (i) Suppose that \(D\) is an e-regular module. Then \(D = \text{eReg}(D)\) yields (using 1.26) that \(B = \text{eReg}(B)\). Thus we get Elliger's result (Proposition 4.5 of [E:71]) that submodules of e-regular modules are e-regular.

(ii) The regular analogue of this result due to Zelmanowitz ([1.4] of [Z:72]) can be similarly derived from 1.27.

Now we give a condition which ensures that \(B \cap \text{Reg}(D) = \text{Reg}(B)\).

1.30. Definition. A module pair \((B, R)\) satisfies condition (\(C_1\)) if each \(R\)-homomorphism from \(B\) to \(R\) can be extended to a \(R\)-homomorphism from \(D\) to \(R\).
(equivalently, the natural map \( \text{Hom}_R(D, R) \to \text{Hom}_R(B, R) \)
is onto).

1.31. Remark. The pair \((B, D)\) satisfies \((C_1)\) in each of the following cases:

1. \( R \) is left self-injective, and
2. (\( R \) is any ring and) \( B \) is a direct summand of \( D \).

As an immediate consequence we have

1.32. Remark. Assume that \((B, D)\) satisfies \((C_1)\). Then \( B \cap \text{Reg}(D) = \text{Reg}(B) \). Hence \( B \) is regular if and only if \( B \) is regular in \( D \).

1.33. Remark (\( e \)-regular analogue of 1.32).

Assume that \( B \leq D \). Then \( B \cap e\text{Reg}(D) = e\text{Reg}(B) \). Hence \( B \) is \( e \)-regular if and only if \( B \) is \( e \)-regular in \( D \).

E. Other definitions

While we shall not study them in this thesis it should be noted that there are some other definitions of regular modules. Fieldhouse calls a module regular ("\( f \)-regular") if every submodule is pure [F:69]. Ware has studied regular projective ("\( w \)-regular") modules [W:71]. For projective modules the various classes (regular, \( e \)-regular, \( f \)-regular and \( w \)-regular) coincide; in particular each concept yields von Neumann regularity

§2. Absolutely regular modules.

We shall not be interested in e-regularity in the rest of this thesis (except in a few remarks). Thus, hereafter, regularity means regularity in the sense of Zelmanowitz (1.1 - 1.4).

In this section we shall prove the existence of commutative rings R and U satisfying the following conditions: R is a regular subring of U, but R is not regular in U. The following definitions are suggested by this situation:

1.34. Definition. A module B is absolutely regular if for each overmodule D, B is regular in D.

1.35. Definition. A ring R is left absolutely regular if the module _R is absolutely regular. (In the commutative case we drop the adjective 'left'.)
We also introduce the following definition.

1.36. Definition. A module B has the weak extension property if for each overmodule D, the pair (B, D) satisfies (C_1). (See 1.30.)

1.37. Remark. It follows from 1.31 that \( R_B \) has the weak extension property in each of the following cases.

   (1) R is left self-injective and B arbitrary.

   (2) (R is any ring and) B is injective.

1.38. Remark. Clearly, every absolutely regular module is regular.

1.39. Proposition. Every regular module with the weak extension property is absolutely regular.

Proof. Let B be a regular module with the weak extension property. Let D be an overmodule of B. Then, by assumption, (B, D) satisfies (C_1). This implies, by 1.32, that B is regular in D. Hence B is absolutely regular.

1.40. Corollary. Every regular, left self-injective ring is left absolutely regular.

1.41. Corollary. Semi-simple rings (1.11) are left and right absolutely regular.
The rest of this section is devoted to the construction (for each field of characteristic zero) of a class of regular rings which are not absolutely regular. These results arose from a consideration of Example 4.4 of [Z.-H.:76].

1.42. Let $K$ be a commutative ring, $U = K^N$, $A_i = K^{(N)}$ and $R_0$ a subring of $U$ containing $A_i$. Clearly $A$ is a large $R_0$-submodule of $U$. If $R_0 \neq U$, then $R_0$ cannot be a $R_0$-direct summand of $U$, and in particular cannot be self-injective. Therefore with a suitable choice of $K$ and $R_0$, we can hope to get the examples we want. (See 1.40.)

Assume that $K$ is a field of characteristic zero. (In fact, some results can be proved under the assumption that $K$ is any infinite field.) Let $R$ be the regular subring of $U$ consisting of sequences $(a_i)_{i \in N}$ for which the set $\{ a_i \mid i \in N \}$ is finite.

1.43. Proposition. Let $R$ be any subring of $R$ containing $A$ (so that $U$ becomes an $R$-module). Let $f \in \text{Hom}_R(U, R)$. Then $Uf \subseteq A$.
Proof. Suppose, if possible, $tf \notin A$ for some $t = (t_i)_{i \in N} \in U$. Assume $tf = (a_i)_{i \in N}$. Then $J = \{i \in N \mid a_i \neq 0\}$ is an infinite set. Now define an element $u$ of $U$ as follows. If $i \in J$, write $u_i = i t_i / a_i$, if $i \notin J$ write $u_i = 0$; finally set $u = (u_i)_{i \in N}$.

We shall identify $K$ with a subring of $R$ via the map $a \mapsto (a, a, a, \ldots)$. Write $e^i = (\delta^i_j)_{j \in N}$, where $\delta^i_j$ denotes Kronecker delta. Then $e^i \in A \subseteq R$. Let $i \in J$. Note that $e^i u = (i / a_i) e^i.$ Using the $R$-linearity of $f$, we have: the $i$th co-ordinate of $uf$ equals

$$((e^i u) f)_i = i / a_i, \qquad [(e^i f)]_i = i / a_i (tf)_i = i.$$ 

Since $K$ is a field of characteristic zero, and $J$ is an infinite set, this contradicts the requirement that $uf \in R \subseteq R$.

1.44. Corollary. $\text{Reg}(\mathcal{U}) = A$; therefore the module $\mathcal{U}$ is not regular.

Proof. Let $t \in \text{Reg}(\mathcal{U})$. Then $t = (tf)_t$ for some $f \in \text{Hom}_R(U, R)$. Hence $t \in \text{At} \subseteq A$. Next let $a \in A$. Then there exists $a' \in A$ such that $aa' a = a$. Consider $f : U \to R$ defined by $tf = ta'$ for $t \in U$. Then $a = aa' a = (af)a$ shows that $a \in \text{Reg}(\mathcal{U})$. 
1.45. Examples. Let $K$ be a field of characteristic zero. Set

$$R_K = \{ R \mid R \text{ is a regular subring of } R \}$$

Then any ring in $R_K$ is regular but not absolutely regular (by 1.44). Examples of rings in $R_K$ are:

1. $R$ itself, and
2. the ring of sequences $a$ in $U$ such that all but finitely many entries of $a$ are equal.

1.46. Remark. The ring $U$ is regular and self-injective (see Corollary 5.2 of [SV:74] for a proof of self-injectivity). Therefore $U$ is absolutely regular by 1.40. The examples in 1.45 show that regular subrings of absolutely regular rings need not be absolutely regular.

§3. McCoy's lemma for modules.

The following lemma due to McCoy has been used in the study of regular rings ([KI], pp. 111-115).

1.47. Lemma. Let $R$ be a ring and $a, d$ be elements of $R$. If $dad - d$ is regular, then $d$ is regular.

If $D$ is a left $R$-module, $\text{Hom}_R(D, R)$ has the natural structure of a right $R$-module; see §C of Chapter 0.
This structure will be exploited to extend 1.47 to modules:

1.48. Proposition ("McCoy's lemma for modules"). Let $d \in_D D$ and $q \in \text{Hom}_R(D, R)$ such that $(dq)d - d$ is regular (in $D$). Then $d$ is regular (in $D$).

Proof. By hypothesis, there exists $g \in \text{Hom}_R(D, R)$ such that $(dq)d - d = [(dq)d - d]g$.

On expanding and rearranging this equation we get $d = (dh)d$ where

$$h = q - g \circ q(dg) + g(dq) - q(dg)(dq) \in \text{Hom}_R(D, R).$$

1.49. Remark. On putting $D = R$ in Proposition 1.48 we get 1.47.

The rest of this section is devoted to some applications of 1.48.

1.50. Proposition. Let $D, M$ be left $R$-modules and $k: D \to M$ be an $R$-homomorphism with kernel $B$. Assume that $B$ is regular in $D$. If $d \in D$ is such that $dk$ is regular in $M$, then $d$ is regular in $D$.

Proof. Since $dk$ is regular in $M$ there exists $f \in \text{Hom}_R(M, R)$ satisfying $((dk)f)dk = dk$. Write
q: = kf ∈ \text{Hom}_R(D, R). The element \((dq)d - d\) belongs to \(B\) and is therefore regular in \(D\). By 1.48 the element \(d\) is regular in \(D\).

1.51. Corollary. Let \(B \triangleleft D\) and assume that \(B\) is regular in \(D\). (This happens if \(B\) is regular and \((B, D)\) satisfies \((C_1)\).) If \(d\) in \(D\) is such that \(d\) is a regular element of the factor module \(D/B\), then \(d\) is regular in \(D\). If \(D/B\) is a regular module, then \(D\) is a regular module.

1.52. Corollary. If \(B \triangleleft D\), if \(B\) is absolutely regular, and if \(D/B\) is a regular module, then \(D\) is a regular module.

1.53. Definition. Let \(a\) be an element and \(B\) a submodule of a module \(D\). The pair \((a, B)\) satisfies condition \((C_2)\) if there exists \(f:D \to R\) such that \((af)a = a\) and \(Bf = 0\).

1.54. Remark. The seemingly artificial condition \((C_2)\) applies in the following situation: Suppose that \(D = A \bigoplus B\), and \(a \in A\). Then the pair \((a, B)\) satisfies \((C_2)\) if and only if \(a\) is regular in \(A\).
1.55. Proposition. If the pair \((a, B)\) satisfies \((C_2)\) in \(D\), if \(B\) is regular in \(D\), and if \(b\) is an element of \(B\), then \(a + b\) is regular in \(D\).

Proof. By hypothesis, there exists \(f: D \rightarrow R\) such that \((af)a = a\) and \(Bf = 0\). Write

\[z := [(a + b)f] (a + b) - (a + b)\] .

Now \(z = (af)b - b\) is an element of \(B\) and is therefore regular in \(D\). It follows, by 1.48, that \(a + b\) is a regular element of \(D\). 

1.56. Corollary. Let \(D = A \bigoplus B\). Then \(D\) is a regular module if and only if \(A\) and \(B\) are regular modules.

Proof. By 1.29 (ii) it is sufficient to prove the "if part". Let \(a + b\) (with \(a \in A\) and \(b \in B\)) be an element of \(D\). By 1.54 the pair \((a, B)\) satisfies \((C_2)\). Hence by 1.55 \(a + b\) is regular in \(D\).

1.57. Remark. The result proved in 1.56 is a crucial step in the proof of Theorem 2.8 of \([2:72]\). However, the proof given here is different.
§4. Localizations of regular modules.

Central localizations of regular rings have been studied by Armendariz, Fisher and Steinberg [AFS:74]. In this section we shall study central localizations of regular modules.

1.58. Preliminaries. Let (as before) R be a ring, C a subring of \( \text{Centre}(R) \), M a left R-module, \( N = \text{Hom}_R(M, R) \) (a C-module) and \( m \) a fixed element of M. The set \( mN = \{ mf \mid f \in N \} \) has been of interest to commutative algebraists, who have often denoted it as \( \mathcal{O}_M(m) \) (see, e.g., (2.1) in [BR:82]). In view of the right R-module structure on N, \( mN \) is a right ideal of R. The set \( (mN)m = mmN \) is a C-submodule of \( Rm \).

A multiplicatively closed subset of C will be denoted by T. Then \( T^{-1}C \) is a subring of \( \text{Centre}(T^{-1}R) \), and \( T^{-1}M \) is a left \( T^{-1}R \)-module. If \( p \in \text{Spec}(C) \), we shall use the usual notations \( C_p \), \( R_p \) and \( M_p \); similar notation will be used when \( v \in \text{Max}(C) \). For basic properties of these rings and modules of fractions we refer to [AM] and [JII].

1.59. Proposition. Let \( m \) be a regular element of \( R^M \) and \( t \) an element of T. Then \( m/t \) is a regular element of \( T^{-1}R T^{-1}M \).
Proof. Let \( m = (mf)m \) for some \( f \in \text{Hom}_R(M, R) \).

Consider \( tf/l \in \text{Hom}_{T^{-1}R}(T^{-1}M, T^{-1}R) \). (Note that \( t \in T \subset C \subset \text{spec}(R) \).)

Since \( [(m/t)tf/l] m/t = (mf)m/t = m/t \), we get the desired result.

1.60. Corollary. Let \( {}_R M \) be a regular module. Then \( T^{-1}M \) is a regular \( T^{-1}R \)-module for each \( T \); in particular, \( M_v \) is a regular \( R_v \)-module for each \( v \in \text{Max}(C) \).

1.61. Remark. On putting \( M = R \) in 1.60 we get the well-known result that if \( R \) is a regular ring then so is \( T^{-1}R \) for each multiplicatively closed subset \( T \) of the centre of \( R \).

1.62. Example. Recall Example 1.24(iii). The ring \( U \) is regular but \( U/A \) is not a regular \( U \)-module. For each \( v \in \text{Max}(U) \), \( U_v \) is a field; therefore \( (U/A)_v \) is a regular \( U_v \)-module (being a vector space over \( U_v \)). This example shows that even when \( M_v \) is a regular \( R_v \)-module for each \( v \in \text{Max}(C) \), \( M \) need not be a regular \( R \)-module.
A finiteness condition on \( M \) which ensures the regularity of \( M \) will be given in Theorem 1.64. Note that this theorem extends (a) \( \rightarrow \) (c) of Theorem in [AFS:74], since \( R^R \) is trivially finitely presented.

1.63. Remark. The following notation will be used in the proof of Theorem 1.64. The \( C \)-module \( (C_m + mNm)/mNm \) will be denoted by \( W(m) \). Clearly \( m \) is regular \( \iff \) \( C_m + mNm \) and \( m = 0 \). (Thus \( W(m) \) is a "test module" for the regularity of \( m \) in \( M \).)

1.64. Theorem. Let \( M \) be a left \( R \)-module. Consider the following conditions.

\[
(1) \quad R^M \text{ is regular} \\
(2) \quad T^{-1}_R T^{-1} M \text{ is regular for each } T. \\
(3) \quad \overset{M}{R_p} \text{ is regular for each } p \in \text{Spec}(C). \\
(4) \quad \overset{M}{R_v} \text{ is regular for each } v \in \text{Max}(C). 
\]

Then \( (1) \iff (2) \iff (3) \iff (4) \). If, further, \( M \) is a finitely presented \( R \)-module then the four conditions are equivalent.
Proof. (1) implies (2) was seen in 1.60 and (2) implies (1) is trivial (choose T = [1]). The assertions (2) implies (3) and (3) implies (4) are also clear. We shall now prove (4) implies (1) under the assumption that \(_R M_\) is finitely presented.

It is well-known that when \(_R M_\) is finitely presented for any left \(_R\)-module \(M'\) the \(\_R^{-1}C_\)-modules \(\_R^{-1}\text{Hom}_R(M, M')\) and \(\text{Hom}_R(\_R^{-1}M, \_R^{-1}M')\) are canonically isomorphic. (See, e.g., Proposition 2.13", Chapter I of [L].) Now let \(m \in M\) and \(v \in \text{Max}(C)\). Identifying \(\text{Hom}_R(M, R)_v\) and \(\text{Hom}_R(M_v, R_v)\) as above, we have

(\text{using standard properties of localization}):

\[
(W(m))_v = (\text{C}_v m/1 + m/\text{Hom}_{R_v}(M_v, R_v)m/1)/(m/\text{Hom}_{R_v}(M_v, R_v)m/1)
\]

Since \(M_v\) is a regular \(_R_v\)-module by assumption, this yields \(W(m)_v = 0\) for each \(v \in \text{Max}(C)\).

It follows (by an application of an analogue of Proposition 3.8 of [AM] for non-commutative rings) that \(W(m) = 0\). Therefore \(m\) is a regular element of \(M\) by 1.63. #
1.65. Remark. As pointed out in §1.1 E, there is a definition of regular modules ("f-regular modules") due to Fieldhouse. For these modules he has proved (Theorem 11.2 of [F:69]): Let $R$ be a commutative ring and $M$ an $R$-module. Then $\hat{R}^M$ is f-regular $\Rightarrow_R M$ is f-regular for each maximal ideal $v$ of $R$. 