Chapter 4

$I$- and $(I, I)$-Linear Transformations

4.1 Introduction

In [84], Mordeson and Malik discussed weak homomorphism and weak isomorphism of an $L$-subgroup/$L$-subring/$L$-submodule into an $L$-subgroup/$L$-subring/$L$-submodule. Following these notions and restricting $L$ to $I = [0, 1]$, we define weak linear transformation and weak isomorphism of a fuzzy linear space into a fuzzy linear space. Then we introduce the notion of $I$-linear transformation, which generalizes the notion of weak linear transformation. The terms $I$-linear functional and $I$-linear operator are also introduced. Five equivalent conditions for a linear transformation to be an $I$-linear transformation on a fuzzy linear space are obtained. Moreover, some results on images of $\alpha$-cuts of fuzzy linear spaces and of convex fuzzy linear spaces under $I$-linear transformations are given. It is established that if $T : Y \rightarrow Z$ is a surjective linear transformation, then there is one-to-one correspondence between the set of all fuzzy linear spaces $(V, Y)$ such that $T$ is an $I$-linear transformation on $(V, Y)$ and the set of all fuzzy linear spaces

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**Some other results of this chapter have been included in the paper $(I, I)$-Linear transformations (communicated).
4.2 \textit{I-linear transformations}

The algebra of \textit{I}\textdmashtext{-linear transformations} is discussed and it is shown that the set $\text{IL}(V, W)$ of \textit{I}\textdmashtext{-linear transformations}, if it is nonempty, is a linear space over $X_Y$. Furthermore, the set $\text{IL}(V)$ of \textit{I}\textdmashtext{-linear operators} in an algebra with identity over $X_Y$.

The last section of this chapter is devoted to the notions of $(\textit{I}, \textit{I})$-linear transformations, $(\textit{I}, \textit{I})$-linear operators, $(\textit{I}, \textit{I})$-linear functionals, which are the intuitionistic fuzzy analogues of \textit{I}\textdmashtext{-linear transformations}, \textit{I}\textdmashtext{-linear operators}, \textit{I}\textdmashtext{-linear functionals} respectively.

4.2 \textit{I-linear transformations}

At first, following the ideas of weak homomorphism and weak isomorphism in [90], we define weak linear transformation and weak isomorphism of a fuzzy linear space into a fuzzy linear space. Then we define \textit{I}\textdmashtext{-linear transformation} as a generalization of weak linear transformation and and study its properties. Some properties of \textit{I}\textdmashtext{-linear functionals} and \textit{I}\textdmashtext{-linear operators} are also discussed in this section.

Consider fuzzy linear spaces $(V, Y)$ and $(W, Z)$ over $(F, X)$.

\textbf{Definition 4.2.1.} If $T$ is a linear transformation of $Y$ onto $Z$ such that $T(V) \subseteq W$, then $T$ is called a \textit{weak linear transformation} of $(V, Y)$ into $(W, Z)$.

If there exists a weak linear transformation of $(V, Y)$ into $(W, Z)$, then it is expressed by $(V, Y) \sim (W, Z)$.

\textbf{Definition 4.2.2.} If $T$ is an isomorphism of $Y$ onto $Z$ such that $T(V) \subseteq W$, then $T$ is called a \textit{weak isomorphism} of $(V, Y)$ into $(W, Z)$.

If there exists a weak isomorphism of $(V, Y)$ into $(W, Z)$, then $(V, Y)$ is said to be \textit{weakly isomorphic} to $(W, Z)$ and it is expressed by $(V, Y) \simeq (W, Z)$. 
Definition 4.2.3. If $T$ is a linear transformation of $Y$ onto $Z$ such that $T(V) = W$, then $T$ is called a linear transformation of $(V,Y)$ onto $(W,Z)$.

If there exists a linear transformation of $(V,Y)$ onto $(W,Z)$, then it is expressed by $(V,Y) \approx (W,Z)$.

Definition 4.2.4. If $T$ is an isomorphism of $Y$ onto $Z$ such that $T(V) = W$, then $T$ is called an isomorphism of $(V,Y)$ onto $(W,Z)$.

If there exists an isomorphism of $(V,Y)$ onto $(W,Z)$, then $(V,Y)$ is said to be isomorphic to $(W,Z)$ and it is expressed by $(V,Y) \cong (W,Z)$.

Now we define an $I$-linear transformation, $I$-linear functional and $I$-linear operator.

Definition 4.2.5. Let $T : Y \rightarrow Z$ be a linear transformation. If $\mu_w(Ty) \geq \mu_v(y)$ for all $y \in Y$, then $(T, (V,Y), (W,Z))$ is said to be an $I$-linear transformation.

Definition 4.2.6. Let $(F,X)$ be such that $\mu_F(0) = \mu_F(1)$ and $T$ be a linear functional on $Y$. If $\mu_F(Ty) \geq \mu_v(y)$ for all $y \in Y$, then $(T, (V,Y), (F,X))$ is said to be an $I$-linear functional.

Definition 4.2.7. Let $T$ be a linear operator on $Y$. If $\mu_v(Ty) \geq \mu_v(y)$ for all $y \in Y$, then $(T, (V,Y), (V,Y))$ is said to be an $I$-linear operator.

Remark 4.2.8. For any linear transformation $T : Y \rightarrow Z$, $(T, (V,Y), (T(V),Z))$ is an $I$-linear transformation since for all $y \in Y$,

$$\mu_{T(v)}(Ty) = \sup \{ \mu_v(u) : Tu = Ty, u \in Y \} \geq \mu_v(y).$$

Definition 4.2.9. A linear transformation $T : Y \rightarrow Z$ is said to be an $I$-linear transformation on $(V,Y)$ if $\mu_{T(v)}(Ty) = \mu_v(y)$ for all $y \in Y$.

Example 4.2.10. If $T$ is the zero operator on $Y$, then $(T, (V,Y), (V,Y))$ is an $I$-linear operator.
Example 4.2.11. \((I, (V, Y), (V, Y))\), where \(I\) is the identity operator on \(Y\), is an \(I\)-linear operator.

Example 4.2.12. If \((F, \mathbb{R})\) is a fuzzy field of \(\mathbb{R}\) with \(\mu_F(0) = \mu_F(1)\) and \(T\) is the linear operator on \(\mathbb{R}^2\) defined by \(T(a_1, a_2) = (a_2, a_1)\), then \((T, (F \times F, \mathbb{R}^2), (F \times F, \mathbb{R}^2))\) is an \(I\)-linear operator, since for all \((a_1, a_2) \in \mathbb{R}^2\),

\[
\mu_{F \times F}(T(a_1, a_2)) = \mu_{F \times F}(a_2, a_1) = \mu_F(a_2) \land \mu_F(a_1) = \mu_{F \times F}(a_1, a_2).
\]

Example 4.2.13. If \((F, \mathbb{R})\) is a fuzzy field of \(\mathbb{R}\) with \(\mu_F(0) = \mu_F(1)\) and \(T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^2\) are the linear transformations defined by \(T_1(a_1, a_2, a_3) = (a_2, a_3)\), \(T_2(a_1, a_2, a_3) = (a_1 - a_2, a_1 - a_3)\), then \((T_i, (F \times F \times F, \mathbb{R}^3), (F \times F, \mathbb{R}^2))\) (for \(i = 1, 2\)) are \(I\)-linear transformations since

\[
\mu_{F \times F \times F}(T_1(a_1, a_2, a_3)) = \mu_{F \times F \times F}(a_2, a_3) = \mu_F(a_2) \land \mu_F(a_3) \\
\geq \mu_F(a_1) \land \mu_F(a_2) \land \mu_F(a_3) \\
= \mu_{F \times F \times F}(a_1, a_2, a_3)
\]

and

\[
\mu_{F \times F \times F}(T_2(a_1, a_2, a_3)) = \mu_{F \times F \times F}(a_1 - a_2, a_1 - a_3) \\
= \mu_F(a_1 - a_2) \land \mu_F(a_1 - a_3) \\
\geq \mu_F(a_1) \land \mu_F(a_2) \land \mu_F(a_3) \\
= \mu_{F \times F \times F}(a_1, a_2, a_3).
\]

Example 4.2.14. If \((F, \mathbb{R})\) is a fuzzy field of \(\mathbb{R}\) with \(\mu_F(0) = \mu_F(1)\) and \(T\) is the linear operator on \(\mathbb{R}^2\) defined by \(T(a_1, a_2) = (a_1 - a_2, 0)\), then \((T, (F \times F, \mathbb{R}^2), (F \times F, \mathbb{R}^2))\) is an \(I\)-linear operator since

\[
\mu_{F \times F}(T(a_1, a_2)) = \mu_{F \times F}(a_1 - a_2, 0) = \mu_F(a_1 - a_2) \geq \mu_{F \times F}(a_1, a_2).
\]

The following proposition shows that the notion of \(I\)-linear transformation is a generalization of the notion of weak linear transformation.
Proposition 4.2.15. If $T$ is a weak linear transformation of $(V, Y)$ into $(W, Z)$, then $(T, (V, Y), (W, Z))$ is an $I$-linear transformation.

Proof. Since $T$ is a weak linear transformation of $(V, Y)$ into $(W, Z)$, $T(V) \subseteq W$. So $\mu_w(z) \geq \mu_{T(V)}(z)$ for all $z \in Z$. Therefore $\mu_w(Ty) \geq \mu_{T(V)}(Ty)$ for all $y \in Y$. But $\mu_{T(V)}(Ty) \geq \mu_V(y)$ for all $y \in Y$. Therefore $\mu_w(Ty) \geq \mu_V(y)$ for all $y \in Y$. That is, $(T, (V, Y), (W, Z))$ is an $I$-linear transformation.

Proposition 4.2.16. If $(T, (V, Y), (W, Z))$ is an $I$-linear transformation, then $\mu_w(0) \geq \mu_V(y)$ for all $y \in Y$.

Proof. $\mu_w(0) = \mu_w(T0) \geq \mu_V(0) \geq \mu_V(y)$ for all $y \in Y$.

Proposition 4.2.17. For any linear transformation $T : Y \to Z$, $(T, (T^{-1}(W), Y), (W, Z))$ is an $I$-linear transformation.

Proof. For all $y \in Y$, $\mu_w(Ty) = \mu_{T^{-1}(W)}(y)$.

Proposition 4.2.18. Let $T$ be a linear functional on $Y$. If $Ty \in X_V$ for all $y \in Y$, then $(T, (V, Y), (F, X))$ is an $I$-linear functional.

Proof. $Ty \in X_V$ for all $y \in Y \Rightarrow \mu_x(Ty) \geq \mu_V(0) \geq \mu_V(y)$ for all $y \in Y$. Therefore $(T, (V, Y), (F, X))$ is an $I$-linear functional.

Proposition 4.2.19. Let $\{(T_\lambda, (V_\lambda, Y_\lambda), (W_\lambda, Z_\lambda))\}_{\lambda \in J}$ be a family of $I$-linear transformations, $Y = \bigprod_{\lambda \in J} Y_\lambda$, $Z = \bigprod_{\lambda \in J} Z_\lambda$, $V = \bigprod_{\lambda \in J} V_\lambda$ and $W = \bigprod_{\lambda \in J} W_\lambda$. Let $T : Y \to Z$ be the linear transformation defined by $T((y_\lambda)_{\lambda \in J}) = (T_\lambda y_\lambda)_{\lambda \in J}$, where $(y_\lambda)_{\lambda \in J} \in Y$. Then $(T, (V, Y), (W, Z))$ is an $I$-linear transformation.

Proof. For all $y = (y_\lambda)_{\lambda \in J} \in Y$,

$$\mu_w(Ty) = \mu_{\prod_{\lambda \in J} w_\lambda}((T_\lambda y_\lambda)_{\lambda \in J}) = \inf_{\lambda \in J} \mu_{w_\lambda}(T_\lambda y_\lambda) \geq \inf_{\lambda \in J} \mu_{V_\lambda}(y_\lambda) = \mu_V(y).$$

Therefore $(T, (V, Y), (W, Z))$ is an $I$-linear transformation.
Proposition 4.2.20. If \((T_i, (V_i, Y), (W, Z))\) (for \(i = 1, 2, \ldots, n\)) are \(I\)-linear transformations, then \((T, (V_1 + V_2 + \cdots + V_n, Y), (W, Z))\) is an \(I\)-linear transformation.

Proof. Here, for all \(u \in Y\), \(\mu_{w, i}(Tu) \geq \mu_{v_i}(u)\) for \(i = 1, 2, \ldots, n\).

Let \(y \in Y\).

If \(y = y_1 + y_2 + \cdots + y_n\), where \(y_1, y_2, \ldots, y_n \in Y\), then

\[
\mu_{w, i}(Ty) = \mu_{w, i}[T(y_1 + y_2 + \cdots + y_n)] = \mu_{w, i}(Ty_1 + Ty_2 + \cdots + Ty_n) \geq \min \{\mu_{w, i}(Ty_1), \mu_{w, i}(Ty_2), \ldots, \mu_{w, i}(Ty_n)\} \geq \min \{\mu_{v_1}(y_1), \mu_{v_2}(y_2), \ldots, \mu_{v_n}(y_n)\}.
\]

Therefore

\[
\mu_{w, i}(Ty) \geq \sup_{y = y_1 + y_2 + \cdots + y_n} \min \{\mu_{v_1}(y_1), \mu_{v_2}(y_2), \ldots, \mu_{v_n}(y_n)\} = \mu_{v_1 + v_2 + \cdots + v_n}(y).
\]

Hence \((T, (V_1 + V_2 + \cdots + V_n, Y), (W, Z))\) is an \(I\)-linear transformation.

Proposition 4.2.21. Any injective linear transformation on \(Y\) is an \(I\)-linear transformation on \((V, Y)\).

Proof. Let \(T\) be an injective linear transformation on \(Y\).

Let \(y \in Y\).

\[
\mu_{T(V)}(Ty) = \sup \{\mu_{v}(u) : Tu = Ty, u \in Y\} = \mu_{v}(y)\text{ since }T\text{ is injective.}
\]

Therefore \(T\) is an \(I\)-linear transformation on \((V, Y)\).

Proposition 4.2.22. Let \(T : Y \rightarrow Z\) be a linear transformation. If \(T\) is an \(I\)-linear transformation on \((V, Y)\), then \(Z_{T(V)} = T(Y_V)\).

Proof. \(z \in Z_{T(V)} \Rightarrow \mu_{T(V)}(z) = \mu_{T(V)}(0) \Rightarrow \mu_{T(V)}(z) = \mu_{v}(0) > 0 \Rightarrow T^{-1}(z) \neq \phi\) (by Definition 1.2.12) \(\Rightarrow \exists y \in Y\) such that \(Ty = z \Rightarrow \mu_{T(V)}(z) = \mu_{T(V)}(Ty) \Rightarrow \mu_{v}(0) = \mu_{v}(y) \Rightarrow y \in Y_V \Rightarrow z = Ty \in T(Y_V)\). Therefore \(Z_{T(V)} \subseteq T(Y_V)\).
Again, $z \in T(Y_V) \Rightarrow z = Ty$ for some $y \in Y_V \Rightarrow \mu_{T(V)}(z) = \mu_{T(V)}(Ty)$, where $\mu_v(y) = \mu_v(0) \Rightarrow \mu_{T(V)}(z) = \mu_v(y) = \mu_v(0) = \mu_{T(V)}(0) \Rightarrow z \in Z_{T(V)}$. Therefore $T(Y_V) \subseteq Z_{T(V)}$.

Hence $Z_{T(V)} = T(Y_V)$.

\[ \text{Theorem 4.2.23. Let } T : Y \to Z \text{ be a linear transformation. Then the following statements are equivalent.} \]

(i) $T$ is an $I$-linear transformation on $(V,Y)$

(ii) $T^{-1}(T(V)) = V$

(iii) $\mu_v$ is constant on $\ker T$

(iv) $\ker T \subseteq Y_V$

(v) $V$ is $T$-invariant.

\[ \text{Proof. (i) } \Rightarrow (ii). \text{ If } T \text{ is an } I \text{-linear transformation on } (V,Y), \text{ then } \]

$\mu_{T(V)}(Ty) = \mu_v(y)$ for all $y \in Y$. That is, $\mu_{T^{-1}(T(V))}(y) = \mu_v(y)$ for all $y \in Y$.

Therefore $T^{-1}(T(V)) = V$.

(ii) $\Rightarrow$ (iii). $T^{-1}(T(V)) = V \Rightarrow \mu_{T^{-1}(T(V))}(y) = \mu_v(y)$ for all $y \in Y \Rightarrow \mu_{T(V)}(Ty) = \mu_v(y)$ for all $y \in Y \Rightarrow \mu_{T(V)}(0) = \mu_v(y)$ for all $y \in \ker T \Rightarrow \mu_v$ is constant on $\ker T$.

(iii) $\Rightarrow$ (iv) by Proposition 2.3.21.

(iv) $\Rightarrow$ (v). Assume that $\ker T \subseteq Y_V$.

$u, v \in Y$ and $Tu = Tv \Rightarrow u - v \in \ker T \Rightarrow u - v \in Y_V \Rightarrow \mu_v(u) = \mu_v(v)$.

Therefore $V$ is $T$-invariant.

(v) $\Rightarrow$ (i). If $V$ is $T$-invariant, then for all $y \in Y$,

$\mu_{T(V)}(Ty) = \sup \{ \mu_v(u) : Tu = Ty, u \in Y \} = \mu_v(y)$. That is, $T$ is an $I$-linear transformation on $(V,Y)$.

\[ \square \]
**Remark 4.2.24.** Consider fuzzy linear space \((W, Z)\) and a linear transformation \(T : Y \to Z\). By Proposition 4.2.17, \((T, (T^{-1}(W), Y), (W, Z))\) is an \(I\)-linear transformation. In addition to this, \(T\) is an \(I\)-linear transformation on \((T^{-1}(W), Y)\), for, \(\mu_{T^{-1}(W)}(y) = \mu_W(Ty) = \mu_W(0) \forall y \in \ker T\). That is, \(\mu_{T^{-1}(W)}\) is constant on \(\ker T\). Therefore, by Theorem 4.2.23, \(T\) is an \(I\)-linear transformation on \((T^{-1}(W), Y)\).

**Proposition 4.2.25.** Let \(T : Y \to Z\) be a linear transformation. Then there exists a one-to-one correspondence between the set of all fuzzy linear spaces \((V, Y)\) such that \(T\) is an \(I\)-linear transformation on \((V, Y)\) and the set of all fuzzy linear spaces \((W, T(Y))\) (over \((F, X))\).

**Proof.** Consider the map \(f : (V, Y) \mapsto (T(V), T(Y))\).

\[
f(V_1, Y) = f(V_2, Y) \Rightarrow (T(V_1), T(Y)) = (T(V_2), T(Y)) \Rightarrow T(V_1) = T(V_2)
\]

\[
\Rightarrow T^{-1}(T(V_1)) = T^{-1}(T(V_2)) \Rightarrow V_1 = V_2, \text{ by Theorem 4.2.23. Therefore } f \text{ is one-one.}
\]

Given a fuzzy linear space \((W, T(Y))\), consider the fuzzy linear space \((T^{-1}(W), Y)\). \(T\) is an \(I\)-linear transformation on \((T^{-1}(W), Y)\). Also, \(f(T^{-1}(W), Y) = (T(T^{-1}(W)), T(Y)) = (W, T(Y))\), by Note 1.2.16. Therefore \(f\) is onto. 

The following corollary is immediate.

**Corollary 4.2.26.** Let \(T : Y \to Z\) be a linear transformation. If \(T\) is surjective, then there exists a one-to-one correspondence between the set of all fuzzy linear spaces \((V, Y)\) such that \(T\) is an \(I\)-linear transformation on \((V, Y)\) and the set of all fuzzy linear spaces \((W, Z)\) (over \((F, X))\).

Note that the one-to-one correspondence in the above corollary can also be obtained from Proposition 2.3.22, since \(\mu_v\) is constant on \(\ker T\) if and only if \(T\) is an \(I\)-linear transformation on \((V, Y)\), by Theorem 4.2.23.
Proposition 4.2.27. If \((T, (V, Y), (W, Z))\) is an \(I\)-linear transformation, then \(T(V_\alpha) \subseteq W_{\alpha}\) for all \(\alpha \in [0, 1]\).

Proof. If \(y \in V_\alpha\), then \(\mu_\kappa(y) \geq \alpha\), so \(\mu_\kappa(Ty) \geq \mu_\kappa(y) \geq \alpha\). Therefore \(Ty \in W_\alpha\). Therefore \(T(V_\alpha) \subseteq W_\alpha\). \(\square\)

If \(A\) and \(B\) are arbitrary crisp sets, \(f : A \to B\) is an arbitrary crisp function and \(V\) is any fuzzy set in \(A\), then for all \(\alpha \in [0, 1]\), \(f(V_\alpha) \subseteq [f(V)]_\alpha\) and \(f(V_{\alpha+}) = [f(V)]_{\alpha+}\). When \(\alpha = 0\), if \(f\) is surjective, then \(f(V_\alpha) = f(A) = B = [f(V)]_\alpha\). In general, \(f(V_\alpha) \neq [f(V)]_\alpha\). However, equality holds for all \(\alpha \in (0, 1]\) in the case of \(I\)-linear transformations on fuzzy linear spaces.

Proposition 4.2.28. If \(T\) is an \(I\)-linear transformation on \((V, Y)\), then \(T(V_\alpha) = [T(V)]_\alpha\) for all \(\alpha \in (0, 1]\).

Proof. Case (i) \(V_\alpha = \phi\).

Then \(T(V_\alpha) = \phi\), for, if possible, assume that \([T(V)]_\alpha \neq \phi\). Then there exists \(z \in Z\) such that \(z \in [T(V)]_\alpha\). This gives \(\mu_{\tau(V)}(z) \geq \alpha\). That is, \(\sup\{\mu_\kappa(u) : u \in T^{-1}(z)\} \geq \alpha > 0\). Therefore \(T^{-1}(z) \neq \phi\). Let \(u \in T^{-1}(z)\). Then \(\mu_{\tau(V)}(Tu) \geq \alpha\). That is, \(\mu_\kappa(u) \geq \alpha\). So \(u \in V_\alpha\) and hence \(V_\alpha \neq \phi\), a contradiction. Therefore \([T(V)]_\alpha = \phi\). Thus \(T(V_\alpha) = [T(V)]_\alpha\).

Case (ii) \(V_\alpha \neq \phi\).

Then \(T(V_\alpha) \neq \phi\) and so \([T(V)]_\alpha \neq \phi\).

\(z \in [T(V)]_\alpha \Rightarrow T^{-1}(z) \neq \phi\) and \(u \in V_\alpha\) for all \(u \in T^{-1}(z)\) (as in case(i)) \(\Rightarrow z = Tu \in T(V_\alpha)\). Therefore \([T(V)]_\alpha \subseteq T(V_\alpha)\). Hence \(T(V_\alpha) = [T(V)]_\alpha\). \(\square\)

Proposition 4.2.29. Let \(Y\) be a linear space in \(\mathbb{R}^n\) and \(Z\) be a linear space in \(\mathbb{R}^m\). Let \(V\) be a convex fuzzy set in \(Y\) and \((V, Y)\) be a fuzzy linear space over \((F, \mathbb{R})\). If a linear transformation \(T\) of \(Y\) onto \(Z\) is an \(I\)-linear transformation on \((V, Y)\), then \(T(V)\) is a convex fuzzy set in \(Z\).
Proof. Let $0 \leq \lambda \leq 1$ and $Ty_1, Ty_2 \in Z$.

$$
\mu_{T(V)} [\lambda Ty_1 + (1 - \lambda)Ty_2] = \mu_{T(V)} [T(\lambda y_1 + (1 - \lambda)y_2)]
= \mu_v[\lambda y_1 + (1 - \lambda)y_2]
\geq \min \{\mu_v(y_1), \mu_v(y_2)\}
= \min \left\{\mu_{T(V)}(Ty_1), \mu_{T(V)}(Ty_2)\right\}.
$$

So $T(V)$ is a convex fuzzy set in $Z$. \qed

In the next section, we discuss algebra of $I$-linear transformations.

### 4.3 Algebra of $I$-linear transformations

The set $IL(V,W) = \{T : (T, (V,Y), (W,Z)) is an I-linear transformation\}$, if it is nonempty, inherits a natural linear space structure. The set $IL(V) = \{T : (T, (V,Y), (V,Y)) is an I-linear operator\}$ has even more algebraic structure. These ideas are explored in this section.

**Proposition 4.3.1.** If $(T_i, (V,Y), (W,Z))$ (for $i = 1, 2$) are $I$-linear transformations, then $(T_1 \pm T_2, (V,Y), (W,Z))$ are $I$-linear transformations.

**Proof.** For all $y \in Y$,

$$
\mu_w [(T_1 \pm T_2) (y)] \geq \min \{\mu_w(T_1 y), \mu_w(T_2 y)\} \geq \min \{\mu_v(y), \mu_v(y)\} = \mu_v(y). \quad \square
$$

**Proposition 4.3.2.** Let $IL(V,W) = \{T : (T, (V,Y), (W,Z)) is an I-linear transformation\}$. Then the following statements are equivalent.

(i) $IL(V,W)$ is nonempty.

(ii) $O \in IL(V,W)$ where $O$ is the zero linear transformation of $Y$ into $Z$.

(iii) $\mu_w(0_z) \geq \mu_v(0_y)$, where $0_y, 0_z$ are respectively the zero vectors in $Y$ and $Z$. 
Proof. (i) ⇒ (ii) since if $T \in IL(V,W)$, then $O = T - T \in IL(V,W)$.

(ii) ⇒ (iii) since $O \in IL(V,W)$ ⇒ $\mu_w(Oy) \geq \mu_v(y)$ for all $y \in Y$ ⇒ $\mu_w(0_z) \geq \mu_v(0_y)$.

(iii) ⇒ (i) since if $\mu_w(0_z) \geq \mu_v(0_y)$, then $\mu_w(Oy) = \mu_w(0_z) \geq \mu_v(0_y) \geq \mu_v(y)$ for all $y \in Y$ and this gives $O \in IL(V,W)$.

\[\Box\]

**Theorem 4.3.3.** If $IL(V,W)$ is nonempty, then it is a linear space over $X_V$.

**Proof.** Assume that $IL(V,W)$ is nonempty.

In $IL(V,W)$, take the usual sum and scalar multiplication of linear transformations as the operations vector addition and scalar multiplication respectively.

If $T_1, T_2 \in IL(V,W)$, then $T_1 + T_2 \in IL(V,W)$. Since $IL(V,W)$ is nonempty, $O \in IL(V,W)$. If $T \in IL(V,W)$, then $-T = O - T \in IL(V,W)$. Also, for all $T_1, T_2 \in IL(V,W)$, $T_1 + T_2 = T_2 + T_1$. Thus $(IL(V,W), +)$ is an abelian group.

Again, if $a \in X_V$ and $T \in IL(V,W)$, then

$\mu_w[(aT)(y)] = \mu_w(aTy) \geq \mu_f(a) \wedge \mu_w(Ty) \geq \mu_f(a) \wedge \mu_v(y) = \mu_v(y)$ for all $y \in Y$. Therefore $aT \in IL(V,W)$.

All the other linear space axioms are also satisfied by $IL(V,W)$. Hence $IL(V,W)$, if it is nonempty, is a linear space over $X_V$.

**Corollary 4.3.4.** If $(F,X)$ is such that $\mu_f(0) = \mu_f(1)$, then the set $IL(V,F) = \{T : (T, (V,Y), (F,X)) is an I-linear functional\}$ is a linear space over $X_V$.

**Proof.** Consider the zero linear functional $O$ on $Y$.

$\mu_f(Oy) = \mu_f(0) \geq \mu_v(y)$ for all $y \in Y$ and so $O \in IL(V,F)$. This gives $IL(V,F)$ is nonempty. Hence $IL(V,F)$ is a linear space over $X_V$.

**Proposition 4.3.5.** If $(T_1, (V_1,Y_1)), (V_2,Y_2))$ and $(T_2, (V_2,Y_2), (V_3,Y_3))$ are $I$-linear transformations, then $(T_2T_1, (V_1,Y_1), (V_3,Y_3))$ is an $I$-linear transformation, the multiplication being composition of linear transformations.
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Proof. \(\mu_{\nu_3}[(T_2T_1)(y_1)] = \mu_{\nu_3}[T_2(T_1y_1)] \geq \mu_{\nu_2}(T_1y_1) \geq \mu_{\nu_1}(y_1)\) for all \(y_1 \in Y_1\).

\(\therefore (T_2T_1, (V_1, Y_1), (V_3, Y_3))\) is an \(I\)-linear transformation. \(\Box\)

**Theorem 4.3.6.** The set \(IL(V) = \{ T : (T, (V, Y), (V, Y)) \text{ is an } I\text{-linear operator} \}\) is an algebra with identity (over \(X_V\)).

Proof. Consider the zero linear operator \(O\) on \(Y\). \(\mu_{\nu}(Oy) = \mu_{\nu}(0) \geq \mu_{\nu}(y)\) for all \(y \in Y\) and so \(O \in IL(V)\). This gives \(IL(V)\) is nonempty. Hence \(IL(V)\) is a linear space over \(X_V\), by Theorem 4.3.3.

Also, by Proposition 4.3.5, if \((T_i, (V, Y), (V, Y))\) (for \(i = 1, 2\)) are \(I\)-linear operators, then \((T_1T_2, (V, Y), (V, Y))\) is an \(I\)-linear operator, the multiplication being composition of linear operators. This gives if \(T_1, T_2 \in IL(V)\), then \(T_1T_2 \in IL(V)\).

Moreover, the other conditions on vector multiplication in an algebra are satisfied by \(IL(V)\) and the identity operator on \(Y\) is a member of \(IL(V)\).

Thus \(IL(V)\) is an algebra with identity (over \(X_V\)). \(\Box\)

The coming section is to formulate the intuitionistic fuzzy analogues of the notions and some results discussed in the previous sections of this chapter.

4.4 \((I,I)\)-linear transformations

In this section, we discuss some fundamental properties of \((I,I)\)-linear transformations, \((I,I)\)-linear functionals and \((I,I)\)-linear operators, which are intuitionistic fuzzy analogues of \(I\)-linear transformations, \(I\)-linear functionals and \(I\)-linear operators respectively. The algebra of \((I,I)\)-linear transformations is also discussed.
Consider the intuitionistic fuzzy linear spaces \((V,Y)\) and \((W,Z)\) over the intuitionistic fuzzy field \((F,X)\).

**Definition 4.4.1.** Let \(T : Y \rightarrow Z\) be a linear transformation. If \(\mu_{W}(Ty) \geq \mu_{V}(y)\) and \(\nu_{W}(Ty) \leq \nu_{V}(y)\) for all \(y \in Y\), then \((T,(V,Y),(W,Z)))\) is said to be an \((I,I)\)-linear transformation.

**Definition 4.4.2.** Let \((F,X)\) be with \(\mu_{F}(0) = \mu_{F}(1), \nu_{F}(0) = \nu_{F}(1)\) and let \(T\) be a linear functional on \(Y\). If \(\mu_{F}(Ty) \geq \mu_{V}(y)\) and \(\nu_{F}(Ty) \leq \nu_{V}(y)\) for all \(y \in Y\), then \((T,(V,Y),(F,X)))\) is said to be an \((I,I)\)-linear functional.

**Definition 4.4.3.** Let \(T\) be a linear operator on \(Y\). If \(\mu_{V}(Ty) \geq \mu_{V}(y)\) and \(\nu_{V}(Ty) \leq \nu_{V}(y)\) for all \(y \in Y\), then \((T,(V,Y),(V,Y)))\) is said to be an \((I,I)\)-linear operator.

**Remark 4.4.4.** For any linear transformation \(T : Y \rightarrow Z\), \((T,(V,Y),(T(V),Z)))\) is an \((I,I)\)-linear transformation, since
\[
\nu_{T(V)}(Ty) = \inf \{\nu_{V}(u) : Tu = Ty, u \in Y\} \leq \nu_{V}(y) \quad \text{and} \quad \mu_{T(V)}(Ty) \geq \mu_{V}(y) \quad \text{(as in Remark 4.2.8)} \quad \text{for all} \quad y \in Y.
\]

**Definition 4.4.5.** Let \(T : Y \rightarrow Z\) be a linear transformation. Then \(T\) is said to be an \((I,I)\)-linear transformation on \((V,Y)\) if \(\mu_{T(V)}(Ty) = \mu_{V}(y)\) and \(\nu_{T(V)}(Ty) = \nu_{V}(y)\) for all \(y \in Y\).

**Example 4.4.6.** If \(T\) is the zero operator on \(Y\), then \((T,(V,Y),(V,Y)))\) is an \((I,I)\)-linear operator.

**Example 4.4.7.** \((I,(V,Y),(V,Y)))\), where \(I\) is the identity operator on \(Y\), is an \((I,I)\)-linear operator.

**Example 4.4.8.** If \((F,\mathbb{R})\) is an intuitionistic fuzzy field of \(\mathbb{R}\) with \(\mu_{F}(0) = \mu_{F}(1), \nu_{F}(0) = \nu_{F}(1)\) and \(T\) is the linear operator on \(\mathbb{R}^{2}\) defined by \(T(a_{1},a_{2}) = (a_{2},a_{1})\),
then \((T, (F \times F, \mathbb{R}^2), (F \times F, \mathbb{R}^2))\) is an \((I, I)\)-linear operator, since
\[
\nu_{F \times F}(T(a_1, a_2)) = \nu_{F \times F}(a_2, a_1) = \nu_F(a_2) \lor \nu_F(a_1) = \nu_{F \times F}(a_1, a_2)
\]
and analogously
\[
\mu_{F \times F}(T(a_1, a_2)) = \mu_{F \times F}(a_1, a_2).
\]

**Example 4.4.9.** Let \((F, \mathbb{R})\) be an intuitionistic fuzzy field of \(\mathbb{R}\) with \(\mu_{F}(0) = \mu_{F}(1)\) and \(\nu_{F}(0) = \nu_{F}(1)\). If \(T_1, T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2\) are the linear transformations defined by \(T_1(a_1, a_2, a_3) = (a_2, a_3)\), \(T_2(a_1, a_2, a_3) = (a_1 - a_2, a_1 - a_3)\), then \((T_i, (F \times F \times F, \mathbb{R}^3), (F \times F, \mathbb{R}^2), (F, \mathbb{R}))\) (for \(i = 1, 2\)) are \((I, I)\)-linear transformations, since
\[
\nu_{F \times F}(T_1(a_1, a_2, a_3)) = \max\{\nu_F(a_1), \nu_F(a_2), \nu_F(a_3)\} \leq \max\{\nu_F(a_1), \nu_F(a_2), \nu_F(a_3)\} = \nu_{F \times F \times F}(a_1, a_2, a_3)
\]
and analogously \(\mu_{F \times F}(T_1(a_1, a_2, a_3)) \geq \mu_{F \times F \times F}(a_1, a_2, a_3)\);
\[
\nu_{F \times F}(T_2(a_1, a_2, a_3)) = \nu_{F \times F}(a_1 - a_2, a_1 - a_3) \leq \max\{\nu_F(a_1), \nu_F(a_2), \nu_F(a_3)\} = \nu_{F \times F \times F}(a_1, a_2, a_3)
\]
and analogously \(\mu_{F \times F}(T_2(a_1, a_2, a_3)) \geq \mu_{F \times F \times F}(a_1, a_2, a_3)\).

**Example 4.4.10.** Let \((F, \mathbb{R})\) be an intuitionistic fuzzy field of \(\mathbb{R}\) with \(\mu_{F}(0) = \mu_{F}(1)\) and \(\nu_{F}(0) = \nu_{F}(1)\). If \(T\) is the linear operator on \(\mathbb{R}^2\) defined by \(T(a_1, a_2) = (a_1 - a_2, 0)\), then \((T, (F \times F, \mathbb{R}^2), (F \times F, \mathbb{R}^2))\) is an \((I, I)\)-linear operator, since \(\nu_{F \times F}(T(a_1, a_2)) = \nu_{F \times F}(a_1 - a_2, 0) = \nu_F(a_1 - a_2) \lor \nu_F(0) = \nu_F(a_1 - a_2) \leq \nu_F(a_1) \lor \nu_F(a_2) = \nu_{F \times F}(a_1, a_2)\) and analogously \(\mu_{F \times F}(T(a_1, a_2)) \geq \mu_{F \times F}(a_1, a_2)\).

**Proposition 4.4.11.** If \((T, (V, Y), (W, Z))\) is an \((I, I)\)-linear transformation, then \(\mu_{W}(0) \geq \mu_{V}(y)\) and \(\nu_{W}(0) \leq \nu_{V}(y)\) for all \(y \in Y\).

**Proof.** \(\nu_{W}(0) = \nu_{W}(T0) \leq \nu_{V}(0) \leq \nu_{V}(y)\) and analogously \(\mu_{W}(0) \geq \mu_{V}(y)\) for all \(y \in Y\). \(\square\)
**Proposition 4.4.12.** Let \( T : Y \rightarrow Z \) be a linear transformation. Then 
\((T, (T^{-1}(W), Y), (W, Z))\) is an \((I, I)\)-linear transformation.

**Proof.** For all \( y \in Y \), \( \mu_w(Ty) = \mu_{T^{-1}(W)}(y) \) and \( \nu_w(Ty) = \nu_{T^{-1}(W)}(y) \).

**Proposition 4.4.13.** Let \( \{(T_\lambda, (V_\lambda, Y_\lambda), (W_\lambda, Z_\lambda))\}_{\lambda \in J} \) be a family of \((I, I)\)-linear transformations, \( Y = \prod_{\lambda \in J} Y_\lambda \), \( Z = \prod_{\lambda \in J} Z_\lambda \), \( V = \prod_{\lambda \in J} V_\lambda \) and \( W = \prod_{\lambda \in J} W_\lambda \). Let 
\( T : Y \rightarrow Z \) be the linear transformation defined by \( T((y_\lambda)_{\lambda \in J}) = (T_\lambda y_\lambda)_{\lambda \in J} \), where \((y_\lambda)_{\lambda \in J} \in Y\). Then \((T, (V, Y), (W, Z))\) is an \((I, I)\)-linear transformation.

**Proof.** For all \( y = (y_\lambda)_{\lambda \in J} \in Y \),
\[
\nu_w(Ty) = \nu_{\prod_{\lambda \in J} W_\lambda}((T_\lambda y_\lambda)_{\lambda \in J}) = \sup_{\lambda \in J} \nu_{\lambda W_\lambda}(T_\lambda y_\lambda) \leq \sup_{\lambda \in J} \nu_{\lambda W_\lambda}(y_\lambda) = \nu_y(y)
\]
and as in Proposition 4.2.19, \( \mu_w(Ty) \geq \mu_y(y) \).

Therefore \((T, (V, Y), (W, Z))\) is an \((I, I)\)-linear transformation.

**Proposition 4.4.14.** If \((T, (V, Y), (W, Z))\) is an \((I, I)\)-linear transformation and \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta \leq 1 \), then \( T(N_{\alpha, \beta}(V)) \subseteq N_{\alpha, \beta}(W) \).

**Proof.** Let \( y \in N_{\alpha, \beta}(V) \). Then \( \mu_y(y) \geq \alpha \) and \( \nu_y(y) \leq \beta \). So 
\( \mu_w(Ty) \geq \mu_y(y) \geq \alpha \) and \( \nu_w(Ty) \leq \nu_y(y) \leq \beta \).

Therefore \( Ty \in N_{\alpha, \beta}(W) \). This gives \( T(N_{\alpha, \beta}(V)) \subseteq N_{\alpha, \beta}(W) \).

**Proposition 4.4.15.** Let \( T : Y \rightarrow Z \) be a linear transformation. If \( T \) is injective, then \( T \) is an \((I, I)\)-linear transformation on \((V, Y)\).

**Proof.** \( \nu_{T(V)}(Ty) = \inf\{\nu_y(u) : Tu = Ty, u \in Y\} = \nu_y(y) \) and analogously 
\( \mu_{T(V)}(Ty) = \mu_y(y) \) for all \( y \in Y \).

**Proposition 4.4.16.** Let \( Y \) be a linear space in \( \mathbb{R}^n \), \( Z \) be a linear space in \( \mathbb{R}^m \), \( T \) be a linear transformation of \( Y \) onto \( Z \) and \((V, Y)\) be an intuitionistic fuzzy linear space over \((F, \mathbb{R})\). If \( V \) is convex in \( Y \) and \( T \) is an \((I, I)\)-linear transformation on \((V, Y)\), then \( T(V) \) is a convex intuitionistic fuzzy set in \( Z \).
Proof. Let \( \lambda \in [0, 1] \) and \( Ty_1, Ty_2 \in Z \). Then
\[
\nu_{T(V)}(\lambda Ty_1 + (1 - \lambda)Ty_2) = \nu_v(\lambda y_1 + (1 - \lambda)y_2)
\]
\[
\leq \max \{ \nu_v(y_1), \nu_v(y_2) \}
\]
\[
= \max \left\{ \nu_{T(V)}(Ty_1), \nu_{T(V)}(Ty_2) \right\}
\]
and \( \mu_{T(V)}(\lambda Ty_1 + (1 - \lambda)Ty_2) \geq \min \left\{ \mu_{T(V)}(Ty_1), \mu_{T(V)}(Ty_2) \right\} \), as in Proposition 4.2.29. Therefore \( T(V) \) is a convex intuitionistic fuzzy set in \( Z \).

Theorem 4.4.17. Let \( T : Y \rightarrow Z \) be a linear transformation. Then the following statements are equivalent.

(i) \( T \) is an \((I,I)\)-linear transformation on \((V,Y)\)

(ii) \( T^{-1}(T(V)) = V \)

(iii) \( \mu_v \) and \( \nu_v \) are constant on \( \ker T \)

(iv) \( \ker T \subseteq Y_V \)

(v) \( V \) is \( T \)-invariant.

Proof. (i) \( \Rightarrow \) (ii). If \( T \) is an \((I,I)\)-linear transformation on \((V,Y)\), then
\[
\mu_{T(V)}(Ty) = \mu_v(y) \quad \text{and} \quad \nu_{T(V)}(Ty) = \nu_v(y)
\]
for all \( y \in Y \). That is,
\[
\mu_{T^{-1}(T(V))}(y) = \mu_v(y) \quad \text{and} \quad \nu_{T^{-1}(T(V))}(y) = \nu_v(y)
\]
for all \( y \in Y \). Therefore \( T^{-1}(T(V)) = V \).

(ii) \( \Rightarrow \) (iii). \( T^{-1}(T(V)) = V \Rightarrow \mu_{T^{-1}(T(V))}(y) = \mu_v(y) \) and \( \nu_{T^{-1}(T(V))}(y) = \nu_v(y) \)
for all \( y \in Y \) \( \Rightarrow \) \( \mu_{T(V)}(Ty) = \mu_v(y) \) and \( \nu_{T(V)}(Ty) = \nu_v(y) \)
for all \( y \in Y \) \( \Rightarrow \) \( \mu_{T(V)}(0) = \mu_v(y) \) and \( \nu_{T(V)}(0) = \nu_v(y) \)
for all \( y \in \ker T \) \( \Rightarrow \) \( \mu_v \) and \( \nu_v \) are constant on \( \ker T \).

(iii) \( \Rightarrow \) (iv) \( \mu_v \) and \( \nu_v \) are constant on \( \ker T \) \( \Rightarrow \) \( \mu_v(y) = \mu_v(0) \) and \( \nu_v(y) = \nu_v(0) \)
for all \( y \in \ker T \) \( \Rightarrow \) \( y \in Y_V \) for all \( y \in \ker T \) \( \Rightarrow \) \( \ker T \subseteq Y_V \).
(iv) ⇒ (v). Assume that \( \ker T \subseteq Y \).

Let \( u, v \in Y \) and \( Tu = Tv \) ⇒ \( u - v \in \ker T \Rightarrow u - v \in Y \Rightarrow \mu_v(u) = \mu_v(v) \) and \( \nu_v(u) = \nu_v(v) \). Therefore \( V \) is \( T \)-invariant.

(v) ⇒ (i). If \( V \) is \( T \)-invariant, then for all \( y \in Y \),

\[
\nu_W[(T_1 \pm T_2)(y)] \leq \max\{\nu_{W}(T_1 y), \nu_{W}(T_2 y)\} \\
\leq \max\{\nu_v(y), \nu_v(y)\} = \nu_v(y)
\]

That is, \( T \) is an \((I,I)\)-linear transformation on \((V,Y)\)

\[\square\]

\textbf{Algebra of \((I,I)\)-linear transformations}

\textbf{Proposition 4.4.18.} If \((T_i, (V,Y), (W,Z)) \) (for \( i = 1, 2 \)) are \((I,I)\)-linear transformations, then \((T_1 \pm T_2, (V,Y), (W,Z)) \) are \((I,I)\)-linear transformations.

\textit{Proof.} For all \( y \in Y \),

\[
\nu_W[(T_1 \pm T_2)(y)] \leq \max\{\nu_{W}(T_1 y), \nu_{W}(T_2 y)\} \\
\leq \max\{\nu_v(y), \nu_v(y)\} = \nu_v(y)
\]

and as in Proposition 4.3.1, \( \mu_W[(T_1 \pm T_2)(y)] \geq \mu_v(y) \).

Therefore \((T_1 \pm T_2, (V,Y), (W,Z)) \) are \((I,I)\)-linear transformations.

\[\square\]

\textbf{Proposition 4.4.19.} If \((I,I)L(V,W) = \{ T : (T, (V,Y), (W,Z)) \) is an \((I,I)\)-linear transformation\}, then the following statements are equivalent.

(i) \((I,I)L(V,W)\) is nonempty.

(ii) \( O \in (I,I)L(V,W), \) where \( O \) is the zero linear transformation of \( Y \) into \( Z \).

(iii) \( \mu_W(0_z) \geq \mu_v(0_y) \) and \( \nu_w(0_z) \leq \nu_v(0_y), \) where \( 0_y, 0_z \) are respectively the zero vectors in \( Y \) and \( Z \).
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Proof. (i) ⇒ (ii) since if \( T \in (I, I)L(V, W) \), then \( O = T - T \in (I, I)L(V, W) \).

(ii) ⇒ (iii) since \( O \in (I, I)L(V, W) \) ⇒ \( \mu_w(Oy) \geq \mu_v(y) \) and \( \nu_w(Oy) \leq \nu_v(y) \) for all \( y \in Y \).

(iii) ⇒ (i) since if \( \mu_w(0z) \geq \mu_v(0y) \) and \( \nu_w(0z) \leq \nu_v(0y) \), then \( \mu_w(Oy) = \mu_w(0z) \geq \mu_v(0y) \) and analogously \( \nu_w(Oy) \leq \nu_v(y) \) for all \( y \in Y \). So \( O \in (I, I)L(V, W) \).

Theorem 4.4.20. If \( (I, I)L(V, W) \) is nonempty, then it is a linear space over \( X_V \).

Proof. Assume that \( (I, I)L(V, W) \) is nonempty.

In \( (I, I)L(V, W) \), take the usual sum and scalar multiplication of linear transformations as the operations vector addition and scalar multiplication respectively.

If \( T_1, T_2 \in (I, I)L(V, W) \), then \( T_1 + T_2 \in (I, I)L(V, W) \). Since \( (I, I)L(V, W) \) is nonempty, \( O \in (I, I)L(V, W) \). If \( T \in (I, I)L(V, W) \), then \( -T = O - T \in (I, I)L(V, W) \). Also, \( T_1 + T_2 = T_2 + T_1 \) for all \( T_1, T_2 \in (I, I)L(V, W) \). Thus \( ((I, I)L(V, W), +) \) is an abelian group.

Now, if \( a \in X_V \) and \( T \in (I, I)L(V, W) \), then \( \nu_w[(aT)(y)] = \nu_w(aTy) \leq \nu_r(a) \vee \nu_w(Ty) \leq \nu_r(a) \vee \nu_v(y) = \nu_v(y) \) and analogously \( \mu_w[(aT)(y)] \geq \mu_v(y) \) for all \( y \in Y \). Therefore \( aT \in (I, I)L(V, W) \).

All the other linear space axioms are also satisfied by \( (I, I)L(V, W) \). Hence \( (I, I)L(V, W) \), if it is nonempty, is a linear space over \( X_V \).

Corollary 4.4.21. If \( (F, X) \) is such that \( \mu_r(0) = \mu_r(1) \) and \( \nu_r(0) = \nu_r(1) \), then the set \( (I, I)L(V, F) = \{ T : (T, (V, Y), (F, X)) \} \) is an \( (I, I) \)-linear functional} is a linear space over \( X_V \).
Proof. Consider the zero linear functional $O$ on $Y$. $\mu_\nu(Oy) = \mu_\nu(0) \geq \nu_\nu(y)$ and $\nu_\nu(Oy) = \nu_\nu(0) \leq \nu_\nu(y)$ for all $y \in Y$. So $O \in (I, I)L(V, F)$. This gives $(I, I)L(V, F)$ is nonempty. Hence $(I, I)L(V, F)$ is a linear space over $X_V$. \hfill \Box

Proposition 4.4.22. If $(T_1, (V_1, Y_1), (V_2, Y_2))$ and $(T_2, (V_2, Y_2), (V_3, Y_3))$ are $(I, I)$-linear transformations, then $(T_2T_1, (V_1, Y_1), (V_3, Y_3))$ is an $(I, I)$-linear transformation, the multiplication being composition of linear transformations.

Proof. For all $y_1 \in Y_1$, $\nu_{v_2}[(T_2T_1)(y_1)] = \nu_{v_2}[T_2(T_1y_1)] \leq \nu_{v_2}(T_1y_1) \leq \nu_{v_1}(y_1)$ and analogously $\mu_{v_3}[(T_2T_1)(y_1)] \geq \mu_{v_3}(y_1)$. Therefore $(T_2T_1, (V_1, Y_1), (V_3, Y_3))$ is an $(I, I)$-linear transformation. \hfill \Box

Theorem 4.4.23. The set $(I, I)L(V) = \{T : (T, (V, Y), (V, Y))\}$ is an $(I, I)$-linear operator} is an algebra with identity (over $X_V$).

Proof. Consider the zero linear operator $O$ on $Y$. For all $y \in Y$, $\mu_\nu(Oy) = \mu_\nu(0) \geq \nu_\nu(y)$ and $\nu_\nu(Oy) = \nu_\nu(0) \leq \nu_\nu(y)$. So $O \in (I, I)L(V)$. Therefore $(I, I)L(V)$ is nonempty and hence it is a linear space over $X_V$, by Theorem 4.4.20.

Also, by Proposition 4.4.22, if $(T_i, (V, Y), (V, Y))$ (for $i = 1, 2$) are $(I, I)$-linear operators, then $(T_1T_2, (V, Y), (V, Y))$ is an $(I, I)$-linear operator, the multiplication being composition of linear operators. This gives if $T_1, T_2 \in (I, I)L(V)$, then $T_1T_2 \in (I, I)L(V)$.

Moreover, the other conditions on vector multiplication in an algebra are satisfied by $(I, I)L(V)$ and the identity operator on $Y$ is a member of $(I, I)L(V)$.

Thus $(I, I)L(V)$ is an algebra with identity (over $X_V$). \hfill \Box