5.1 Introduction

This chapter deals with multi-fuzzification of algebraic structures like groups and normal subgroups. Theorems and concepts in multi-fuzzy subgroups are more or less similar to the respective concepts in fuzzy subgroups, but properties of extensions of group homomorphism (or a function) depends on the bridge function. In this chapter we consider order homomorphism as bridge function for the multi-fuzzy extensions of crisp functions. Note that, Order homomorphisms, lattice homomorphisms, arbitrary join preserving maps, complement preserving maps, etc. are some of the useful bridge functions for multi-fuzzy extensions.

\[^{5}\text{Some results of this chapter have appeared in the paper Multi-Fuzzy subgroups, Int. J. Contemp. Math. Sciences, Vol. 6, No. 8 (2011) 365-372.}\]
5.2 Multi-fuzzy Subgroups

We introduce the concepts of multi-fuzzy subgroup and multi-fuzzy subgroupoid. Partial order $\geq$ is the opposite order relation of the partial order $\leq$.

**Definition 5.2.1.** A multi-fuzzy set $A$ of a group $G$ is called a multi-fuzzy subgroup of $G$ if

1. $A(x) \land A(y) \leq A(xy)$, and

2. $A(x^{-1}) = A(x)$, for all $x, y \in G$.

Equivalently, a multi-fuzzy set $A$ is called a multi-fuzzy subgroup of $G$, if

$$A(x) \land A(y) \leq A(xy^{-1}), \forall x, y \in G.$$ 

It follows immediately from this definition that $A(x) \leq A(e)$, for all $x \in G$, where $e$ is the identity element of $G$. If $A$ is a multi-fuzzy subgroup of a group $G$, then

$$A(xy^{-1}) = A(e) \text{ implies } A(x) = A(y), \forall x, y \in G.$$ 

A multi-fuzzy subset $A$ of a group $G$ is called a multi-fuzzy subgroupoid of $G$, if

$$A(x) \land A(y) \leq A(xy), \forall x, y \in G.$$ 

**Theorem 5.2.2.** If $\{A_i : i \in I\}$ is a family of multi-fuzzy subgroups of a group $G$, then $\bigcap A_i$ is a multi-fuzzy subgroup of $G$.

**Proof.** Let $A = \bigcap A_i$. For every $x, y \in G$;

$$A(xy^{-1}) = (\bigcap A_i)(xy^{-1}) = \bigwedge A_i(xy^{-1}) \geq \bigwedge( A_i(x) \land A_i(y))$$

$$= (\bigwedge A_i(x)) \land (\bigwedge A_i(y)) = (\bigcap A_i)(x) \land (\bigcap A_i)(y)$$

$$= A(x) \land A(y).$$
Remark 5.2.3. Union of two multi-fuzzy subgroups of a group \( G \) need not be a multi-fuzzy subgroup of \( G \).

Theorem 5.2.4. Let \( G_1 \) and \( G_2 \) be groups, \( f \) be a group homomorphism from \( G_1 \) into \( G_2 \) and a finite meet preserving order homomorphism \( h : \prod M_i \to \prod L_j \) be the bridge function for the multi-fuzzy extension of \( f \). If \( A \) be a multi-fuzzy subgroup of \( G_1 \), then \( f(A) \) is a multi-fuzzy subgroup of \( G_2 \).

Proof. Let \( x, y \in G_2 \). If either \( f^{-1}(x) \) or \( f^{-1}(y) \) is empty, then

\[
f(A)(x) \land f(A)(y) = 0
\]

and implies

\[
f(A)(xy^{-1}) \geq f(A)(x) \land f(A)(y).
\]

Assume that neither \( f^{-1}(x) \) nor \( f^{-1}(y) \) is empty. Therefore, there exist \( u \in f^{-1}(x) \) and \( v \in f^{-1}(y) \) such that

\[
A(u) = \bigvee \{A(t) : t \in G_1, \ x = f(t)\}
\]

and

\[
A(v) = \bigvee \{A(t) : t \in G_1, \ y = f(t)\}.
\]

We have

\[
xy^{-1} = f(u)(f(v))^{-1} = f(uv^{-1})
\]

and so

\[
uv^{-1} \in f^{-1}(xy^{-1}).
\]
Thus
\[
f(A)(xy^{-1}) = \bigvee \{ h(A(z)) : z \in G_1, xy^{-1} = f(z) \}
\geq h(A(uv^{-1})) \geq h(A(u) \land A(v)) = h(A(u)) \land h(A(v))
= h(\bigvee \{ A(t) : x = f(t) \}) \land h(\bigvee \{ A(t) : y = f(t) \})
= (\bigvee \{ h(A(t)) : x = f(t) \}) \land (\bigvee \{ h(A(t)) : y = f(t) \})
= f(A)(x) \land f(A)(y).
\]

Hence \( f(A) \) is a subgroup of \( G_2 \).

**Theorem 5.2.5.** Let \( G_1 \) and \( G_2 \) be groups, \( f : G_1 \to G_2 \) be a group homomorphism and an order homomorphism \( h : \prod M_i \to \prod L_j \) be the bridge function for the multi-fuzzy extension of \( f \). If \( B \) be a multi-fuzzy subgroup of \( G_2 \), then \( f^{-1}(B) \) is a multi-fuzzy subgroup of \( G_1 \).

**Proof.** Note that \( h^{-1} : \prod L_j \to \prod M_i \) is order preserving and meet preserving (see Note 1.2.14), and \( f^{-1}(B)(y) = h^{-1} \circ B(f(y)) \). For every \( x, y \in G \);

\[
f^{-1}(B)(xy^{-1}) = h^{-1} \circ B(f(x)y^{-1})
= h^{-1} \circ B(f(x)(f(y))^{-1})
\geq h^{-1}(B(f(x)) \land B(f(y)))
= h^{-1}(B(f(x))) \land h^{-1}(B(f(y)))
= (f^{-1}(B)(x)) \land (f^{-1}(B)(y)).
\]

**Theorem 5.2.6.** Let \( f \) be an injective group homomorphism from group \( G_1 \) into group \( G_2 \), an injective meet preserving order homomorphism \( h : \prod M_i \to \prod L_j \) be
5.3 Normal Multi-fuzzy Subgroups

Throughout this chapter we will use order homomorphisms as bridge functions for multi-fuzzy extensions of crisp functions unless otherwise stated.

Definition 5.3.1. A multi-fuzzy subgroup \( A \) of a group \( G \) is called normal, if for each \( x \in G \),

\[
A(x) \leq \bigwedge_{g \in G} A(gxg^{-1})
\]

Theorem 5.3.2. Let \( A \) be a multi-fuzzy subgroup of a group \( G \). Then the following conditions are equivalent for each \( x, g \in G \):

1. \( A \) is a normal multi-fuzzy subgroup of \( G \);
2. \( A(x) \leq A(gxg^{-1}) \);
3. \( A(x) = A(gxg^{-1}) \);
4. \( A(xg) = A(gx) \).

Proof. \( \square \) A(x) \leq \bigwedge_{g \in G} A(gxg^{-1}) \) if and only if \( A(x) \leq A(gxg^{-1}), \forall g \in G \).
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\[ \Rightarrow \quad (2) \quad A(x) \leq A(g x g^{-1}) \leq A(g^{-1}(gxg^{-1})g) = A(x). \]

\[ \Rightarrow \quad (3) \quad \text{is trivial.} \]

\[ \Rightarrow \quad (3) \quad A(xg) = A(g(xg)g^{-1}) = A((gx)(gg^{-1})) = A(gx). \]

\[ \Rightarrow \quad (4) \quad A(g g^{-1}) = A((gx) g^{-1}) = A(g^{-1}(gx)) = A((g^{-1} g)x) = A(x). \]

**Theorem 5.3.3.** If \( A_1 \) and \( A_2 \) are two normal multi-fuzzy subgroups of a group \( G \), then their intersection \( A_1 \cap A_2 \) is a normal multi-fuzzy subgroup of \( G \).

**Proof.** By Theorem 5.2.2, \( A_1 \cap A_2 \) is a multi-fuzzy subgroup of \( G \). For every \( x, g \in g \),

\[ (A_1 \cap A_2)(gxg^{-1}) = A_1(gxg^{-1}) \land A_2(gxg^{-1}) = A_1(x) \land A_2(x) = (A_1 \cap A_2)(x). \]

Hence \( A_1 \cap A_2 \) is a normal multi-fuzzy subgroup of \( G \).

**Theorem 5.3.4.** \( A \) is a normal multi-fuzzy subgroup of a group \( G \) if and only if each level subgroup \( A_{[\alpha]} \) is normal in \( G \), for \( \alpha \in \prod M_i \).

**Proof.** First we prove that, \( A \) is a multi-fuzzy subgroup of \( G \) if and only if each level subset \( A_{[\alpha]} \) is a subgroup of \( G \). Assume that \( A \) is a multi-fuzzy subgroup of \( G \). Let \( \alpha \in \prod M_i \) be arbitrary (fixed), for every \( x, y \in A_{[\alpha]} \) that is,

\[ \alpha \leq A(x) \text{ and } \alpha \leq A(y) \]

and hence

\[ \alpha \leq A(x) \land A(y) \leq A(xy^{-1}). \]

Therefore, \( xy^{-1} \in A_{[\alpha]} \) and implies \( A_{[\alpha]} \) is a subgroup of \( G \).

Conversely assume that \( A_{[\alpha]} \) is a subgroup of \( G \), for each \( \alpha \in \prod M_i \). Hence \( x, y \in A_{[\alpha]} \).
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\( A_{[\alpha]} \) implies \( xy^{-1} \in A_{[\alpha]} \). That is,

\[ \alpha \leq A(x) \leq A(xy^{-1}) \]

and

\[ \alpha \leq A(y) \leq A(xy^{-1}). \]

Therefore,

\[ \alpha \leq A(x) \land A(y) \leq A(xy^{-1}) \]

and hence \( A \) is a multi-fuzzy sub group of \( G \).

Finally, an arbitrary \( \alpha \in \prod M_i \), \( A_{[\alpha]} \) is a normal subgroup of \( G \) if and only if \( x \in A_{[\alpha]} \) implies

\[ gxg^{-1} \in A_{[\alpha]}, \forall g \in G \]

if and only if for every \( x \in X \),

\[ A(x) \leq \bigwedge_{g \in G} A(gxg^{-1}) \]

if and only if \( A \) is a normal multi-fuzzy subgroup of \( G \).

\[ \square \]

**Theorem 5.3.5.** Let \( G_1 \) and \( G_2 \) be groups, \( f \) be a group homomorphism from \( G_1 \) onto \( G_2 \) and a meet preserving order homomorphism \( h : \prod M_i \rightarrow \prod L_j \) be the bridge function for the multi-fuzzy extension of \( f \).

(1) If \( A \) is a normal multi-fuzzy subgroup of \( G_1 \), then \( f(A) \) is a normal multi-fuzzy subgroup;

(2) If \( B \) is a normal multi-fuzzy subgroup of \( G_2 \), then \( f^{-1}(B) \) is a normal multi-fuzzy subgroup.
Proof. (1) For every $x_0 \in G_1$, we have
\[ h(A(x_0)) \leq h(\bigwedge_{g \in G_1} A(gx_0g^{-1})) \leq \bigwedge_{g \in G_1} h(A(gx_0g^{-1})), \]
since
\[ A(x_0) \leq \bigwedge_{g \in G_1} A(gx_0g^{-1}). \]
Let $y = f(x_0)$, then
\[ \bigvee_{x \in f^{-1}(y)} h(A(x)) \leq \bigvee_{x \in f^{-1}(y)} \bigwedge_{g \in G_1} h(A(gxg^{-1})) \leq \bigwedge_{g \in G_1} \bigvee_{x \in f^{-1}(y)} h(A(gxg^{-1})). \]
That is,
\[ f(y) \leq \bigwedge_{g \in G_1} f(A)(f(g)yf(g)^{-1}). \]
Hence $f(A)$ is a normal multi-fuzzy subgroup of $G_2$.

(2) For every $x \in G_1$,
\[ f^{-1}(B)(x) = h^{-1}(B(f(x))) \leq h^{-1}(\bigwedge_{u \in G_2} B(uf(x)u^{-1})). \]
Since $f$ is surjective, there exists a $g \in G_1$ for each $u \in G_2$, such that $u = f(g)$. Hence
\[ f^{-1}(B)(x) \leq h^{-1}(\bigwedge_{g \in G_1} B(f(g)f(x)(f(g))^{-1})) \]
\[ \leq \bigwedge_{g \in G_1} h^{-1}(B(f(g)f(x)(f(g))^{-1})) \]
\[ = \bigwedge_{g \in G_1} h^{-1}(B(f(gxg^{-1}))) = \bigwedge_{g \in G_1} f^{-1}(B)(gxg^{-1}). \]
Hence $f^{-1}(B)$ is a normal multi-fuzzy subgroup of $G_1$. \(\square\)