Chapter 3

Mean square error function of the CRPTE in double sampling with partial information on the auxiliary variable.
3.1 Introduction

The precision or a measure of the closeness of the sample estimates to the census count taken under identical conditions is judged in sampling theory by the variances of the estimators concerned. Here reliance is placed on the fact that with a small variance, the probability of large deviation from the census count will be small. The general principle is to use estimators which will give the highest concentration of the sample estimates (in the sense of probability) around the valued aimed for. With unbiased estimators the method used for judging the degree of concentration is the variance of the estimators.

It may happen sometimes that the degree of concentration of the sample around the valued aimed at is higher for the distribution of a biased estimator than for an unbiased one. In such a situation the biased estimator is preferable to the unbiased one. However in order to compare a biased estimator with an unbiased estimator, or two estimator with different amounts of bias, variance is not a satisfactory criterion, since it measures deviation from the expected value of the estimator, which is not the same as the population value. A useful criterion is the mean square error (MSE) of the estimator, measured from the population value that is being estimated.

3.2 Mean square error function of CRPTE

To obtain the MSE of $t_5$, we proceed as follows;

The proposed CRPTE in double sampling is given by

$$t_5 = \begin{cases} (\bar{y}_u - \rho \bar{x}_u) & \text{if } |\bar{x}_u'| \leq \frac{Z_a}{\sqrt{n_1}} \\ (\bar{y}_u + \rho (\bar{x}_u' - \bar{x}_u)) & \text{if } |\bar{x}_u'| > \frac{Z_a}{\sqrt{n_1}} \end{cases}$$
where \( \bar{y}_h = \sum_i w_i \bar{y}_i \) and \( \bar{x}_h = \sum_i w_i \bar{x}_i \)

and the strata means given by

\[
\bar{x}_h = \frac{\sum_{i=1}^{n_h} x_{ih}}{n_h} \quad \text{and} \quad \bar{y}_h = \frac{\sum_{i=1}^{n_h} y_{ih}}{n_h}
\]

We have assumed that the auxiliary variable \( X \) and the study variable \( Y \) are jointly normally distributed with parameters given by \((\mu_x, \mu_y, \sigma^2_x, \sigma^2_y, \rho)\). The marginal distributions which is the distribution of the study variable \( Y \) and the auxiliary variable \( X \) will also follow normal distribution \( Y \sim N(\mu_y, \sigma^2_y) \) and \( X \sim N(\mu_x, \sigma^2_x) \). The strata population \((x_{ih}, y_{ih})\) being carved out from the parent population are also jointly assumed to follow the bivariate normal distribution with parameters written as \((\mu_{x_i}, \mu_{y_i}, \sigma^2_{x_i}, \sigma^2_{y_i}, \rho)\). The regression estimator depends on weather the covariance matrix is known or not. If known, one may let \( \sigma^2_x = \sigma^2_y = 1 \) without loss of generality (WLOG).

Since the population is assumed to follow normal distribution, the preliminary sample obtained to collect information on the auxiliary variable for the estimation of \( \bar{x}_n' \), is also assumed to follow normal distribution and therefore \( \bar{x}_n' \sim N(\mu_x, \sigma^2_x / n') \) and under the assumption \( \sigma^2_x = \sigma^2_y = 1 \), \( \bar{x}_n' \sim N(\mu_x, 1 / n') \).

Further marginal distributions of \( x_h \) and \( y_h \) are also normal given as \( x_h \sim N(\mu_x, \sigma^2_x) \) and \( y_h \sim N(\mu_y, \sigma^2_y) \).

The derivation of \( MSE(t_i) \) involves conditional expectations, the condition being the acceptance or rejection of the hypothesis considered in the
preliminary test. Further the expectations can be obtained from the integrals involving probability density functions which are assumed to be normal. To obtain the MSE of \( t_s \), we proceed as follows:

\[
MSE(t_s) = \text{var}(t_s) + \{\text{Bias}(t_s)\}^2
\]

\[
= E(t_s^2) - \{E(t_s)\}^2 + \{\text{Bias}(t_s)\}^2
\]

..........(3.1)

Now,

\[
E(t_s^2) = E(t_s^2 \quad \text{if} \quad |x_s| \leq Z_{a/\sqrt{n'}}) + E(t_s^2 \quad \text{if} \quad |x_s| > Z_{a/\sqrt{n'}})
\]

\[
= \{E(y_u - \rho x_u)^2 \quad \text{if} \quad |x_s| \leq Z_{a/\sqrt{n'}} \}
+ \{E(y_u + \rho (x_s - x_u))^2 \quad \text{if} \quad |x_s| > Z_{a/\sqrt{n'}} \}
\]

\[
= \{E(y_u - \rho x_u)^2 \quad \text{if} \quad |x_s| \leq Z_{a/\sqrt{n'}} \}
+ \{E(y_u^2 + 2\rho y_u(x_s - x_u) + \rho^2(x_s - x_u)^2) \quad \text{if} \quad |x_s| > Z_{a/\sqrt{n'}} \}
\]

\[
= E(y_u - \rho x_u)^2
+ E(\rho^2 x_s^2 - 2\rho^2 x_s x_u + 2\rho x_s x_u) \quad \text{if} \quad |x_s| > Z_{a/\sqrt{n'}}
\]

i.e

\[
E(t_s^2) = E(y_u^2) - 2\rho E(x_u y_u) + \rho^2 E(x_u^2)
+ E(\rho^2 x_s^2 - 2\rho^2 x_s x_u + 2\rho x_s x_u) \quad \text{if} \quad |x_s| > Z_{a/\sqrt{n'}}
\]

Thus,

\[
E(t_s^2) = \text{Var}(y_u) + \{E(y_u)\}^2 + \rho^2 \left\{ \text{Var}(x_u) + \{E(x_u)\}^2 \right\}
- 2\rho^2 \sqrt{\text{Var}(x_u)\sqrt{\text{Var}(y_u) - 2\rho E(x_u)E(y_u)}}
+ E(\rho^2 x_s^2 - 2\rho^2 x_s x_u + 2\rho x_s x_u) \quad \text{if} \quad |x_s| > Z_{a/\sqrt{n'}}
\]

..........(3.2)
To evaluate the MSE of \( t_s \), we consider that the joint distribution of \( (x_{n'}, x_{n}, y_{n'}) \) is a multivariate normal with mean \((\mu_x, \mu_x, \mu_y)\) and covariance matrix given by

\[
\Sigma = \begin{pmatrix}
\frac{1}{n'} & \frac{1}{n'} & \rho / n' \\
\frac{1}{n'} & \frac{1}{n} & \rho / n \\
\rho / n' & \rho / n & \frac{1}{n}
\end{pmatrix}
\]

(under the assumptions considered in chapter 2)

3.2.1 Evaluation of \( E\{x_{n'}^2 \text{ if } |x_{n'}| > \frac{Z_{\alpha}}{\sqrt{n'}}\} \)

Let \( I_1 = \{ E(x_{n'}^2) \text{ if } |x_{n'}| > \frac{Z_{\alpha}}{\sqrt{n'}}\} \)

\[
= \{ E(x_{n'}^2) \text{ if } x_{n'} > \frac{Z_{\alpha}}{\sqrt{n'}}\} + \{ E(x_{n'}^2) \text{ if } x_{n'} < -\frac{Z_{\alpha}}{\sqrt{n'}}\}
\]

\[
= \int_{\frac{Z_{\alpha}}{\sqrt{n'}}}^{\infty} x_{n'}^2 f(x_{n'}) dx_{n'} + \int_{-\infty}^{-\frac{Z_{\alpha}}{\sqrt{n'}}} x_{n'}^2 f(x_{n'}) dx_{n'}
\]

Since \( x_{n'} \sim N(\mu_x, 1/\sqrt{n'}) \) we get

\[
f(x_{n'}) = \frac{n'}{2\pi} \exp\left\{ -\frac{1}{2} \left( x_{n'} - \mu_x \right)^2 (1/\sqrt{n'}) \right\}
\]

Putting \( w = (x_{n'} - \mu_x)/(1/\sqrt{n'}) \Rightarrow dw = \sqrt{n'} dx_{n'} \), we have

When \( x_{n'} = \frac{Z_{\alpha}}{\sqrt{n'}} \) then \( w = \sqrt{n'}(\frac{Z_{\alpha}}{\sqrt{n'}} - \mu_x) = Z_{\alpha} - \sqrt{n'}\mu_x = A \) and

When \( x_{n'} = -\frac{Z_{\alpha}}{\sqrt{n'}} \) then \( w = \sqrt{n'}(-\frac{Z_{\alpha}}{\sqrt{n'}} - \mu_x) = -Z_{\alpha} - \sqrt{n'}\mu_x = B \)
Hence $I_i$ becomes

$$
(1/\sqrt{2\pi}) \int_{-\infty}^{\infty} \left[ \mu_x + (w/\sqrt{n'}) \right] \frac{\exp\left[-(1/2)w^2\right]}{\sigma}\,dw + \int_{-\infty}^{b} \left( \mu_x + (w/\sqrt{n'}) \right) \frac{\exp\left[-(1/2)w^2\right]}{\sigma}\,dw
$$

$$
= (\mu_x^2 + 1/n')\{(1 - \Phi(A) + \Phi(B)}
+ (2\mu_x/\sqrt{n'})\{\varphi(A) - \varphi(B)} + (1/n')\{A\varphi(A) - B\varphi(B)}

\cdots\cdots\cdots\cdots(3.3)
$$

where $\Phi(.)$ is the cumulative distribution of $N(0,1)$ and $\varphi(.)$ is the density function.

(Detail derivation is given in Appendix 3)
3.2.2 Evaluation of \( E(\overline{x}, \overline{x'}) \) if \( \lvert \overline{x} \rvert > Z_\alpha / \sqrt{n'} \)

Let
\[
I_2 = \left\{ \frac{\partial}{\partial t_1} \left[ \frac{\partial}{\partial t_2} \left\{ E(e^{t_1 N_0 + t_2 x'}) \right\} \right] \right\} \text{ if } \lvert \overline{x} \rvert > Z_\alpha / \sqrt{n'}
\]

Here \( E(e^{t_1 x + t_2 x'}) \) is the moment generating function of \( (\overline{x}, \overline{x'}) \).

Again let
\[
I_2' = \left\{ E(e^{t_1 x + t_2 x'}) \right\} \text{ if } \lvert \overline{x'} \rvert > Z_\alpha / \sqrt{n'}
\]

\[
= \left[ E(e^{t_1 x + t_2 x'}) \text{ if } \overline{x'} > Z_\alpha / \sqrt{n'} \right] + \left[ E(e^{t_1 x + t_2 x'}) \text{ if } \overline{x'} < -Z_\alpha / \sqrt{n'} \right]
\]
\[
= \int_{\bar{x}_{st} = -\infty}^{\infty} \int_{\bar{x}_{n} = -\infty}^{\infty} e^{t_1 \bar{x}_{st} + t_2 \bar{x}_{n'}} f (\bar{x}_{st}, \bar{x}_{n'}) d \bar{x}_{st} d \bar{x}_{n'}
\]

\[
+ \int_{\bar{x}_{st} = -\infty}^{\infty} \int_{\bar{x}_{n} = -\infty}^{\infty} e^{t_1 \bar{x}_{st} + t_2 \bar{x}_{n'}} f (\bar{x}_{st}, \bar{x}_{n'}) d \bar{x}_{st} d \bar{x}_{n'}
\]

where \( f(\bar{x}_{st}, \bar{x}_{n'}) \) is the bivariate normal probability density function of the pair \((\bar{x}_{st}, \bar{x}_{n'})\) with mean \((\mu_x, \mu_z)\) and the variance covariance matrix given by

\[
\sum^{-1} = \begin{bmatrix} 1/n & 1/n' \\ 1/n' & 1/n' \end{bmatrix}
\]

Under the assumption that \( \sigma_x^2 = \sigma_{x'}^2 = 1 \), \( \sigma_z^2 = \sigma_{z'}^2 = 1 \) (WLOG)

For a bivariate normal density we are given that

\[
f(x, y) = \frac{1}{(2\pi|\Sigma|^{1/2})} \exp \left\{ -1/2 \left( (x - \mu_x, y - \mu_y) \right)^T \Sigma^{-1} \left( (x - \mu_x, y - \mu_y) \right) \right\}
\]

Here

\[
\Sigma^{-1} = n'/(n' - n) \begin{bmatrix} n & -n \\ -n & n' \end{bmatrix}
\]
Thus,

\[
f(x_{st}, x_{st}^{'}) = \frac{1}{2\pi \left\{1/m' - (1/n^{'})^2\right\}^{1/2}} \exp\left[-\frac{1}{2}\left(n(x_{st} - \mu_x)^2 - 2\sqrt{\frac{n}{n'}} \frac{(x_{st} - \mu_x)(x_{st}^{'} - \mu_x)}{1/\sqrt{n'}} + n'(x_{st}^{'})^2\right)\right]
\]

Letting \((x_{st} - \mu_x)/(1/\sqrt{n}) = x_1\) and \((x_{st}^{'})/(1/\sqrt{n'}) = x_2\),

we get \(dx_{st} = dx_1/\sqrt{n}\) and \(dx_{st}^{'} = dx_2/\sqrt{n'}\)

When \(x_{st}^{'}/\sqrt{n'} = Z_a\) then \(x_2 = \sqrt{n'}(Z_a/\sqrt{n'} - \mu_x) = Z_a - \sqrt{n'}\mu_x = A\) and when \(x_{st}^{'}/\sqrt{n'} = -Z_a\) then \(x_2 = \sqrt{n'}(-Z_a/\sqrt{n'} - \mu_x) = -Z_a - \sqrt{n'}\mu_x = B\)

Hence,

\[
I_2 = \int_{-\infty}^{\infty} \int_{x_2 = A}^{\infty} \left\{\sqrt{n'}/(2\pi\sqrt{n' - n}) \right\} \exp\left[t_1 \{\mu_x + (x_1/\sqrt{n})\} + t_2 \{\mu_x + (x_2/\sqrt{n'})\}\right] dx_1 dx_2
\]

\[
+ \left\{\sqrt{n'}/(2\pi\sqrt{n' - n}) \right\} \int_{-\infty}^{\infty} \int_{x_2 = B}^{\infty} \exp\left[t_1 \{\mu_x + (x_1/\sqrt{n})\} + t_2 \{\mu_x + (x_2/\sqrt{n'})\}\right] dx_1 dx_2
\]

\[(\text{Appendix 4})\]
Now we can rewrite
\[
\{x_1^2 - 2\sqrt{n/n'} x_1 x_2 + x_2^2\} - (2(n'-n)/n')(t_1 x_1 / \sqrt{n}) + (t_2 x_2 / \sqrt{n'})
\]
\[
= \left\{ x_1 - (\sqrt{n/n'}) x_2 - (1-n/n') t_1 / \sqrt{n} \right\}^2
\]
\[
+ (1-n/n') \{ x_2 - (\sqrt{n/n'}) t_1 / \sqrt{n} - t_2 / \sqrt{n'} \}^2 - t_1^2 / n - t_2^2 / n' - 2(\sqrt{n/n'}) t_1 t_2 (1 / \sqrt{n'n'})
\]

Substituting
\[
(x_1 - (\sqrt{n/n'}) x_2 - (1-n/n') t_1 / \sqrt{n}) = \{(n'-n)/n'\}^{1/2} u \quad \text{and} \quad x_2 - (\sqrt{n/n'}) t_1 / \sqrt{n} - t_2 / \sqrt{n'} = v
\]

Then in this transformation, the Jacobian is given by
\[
J = \frac{\partial(x_1, x_2)}{\partial(x_1, x_2)} = \sqrt{1 - \left(\frac{n}{n'}\right)} \quad \text{and} \quad dudv = |J| dx_1 dx_2
\]

When
\[
x_2 = A \quad , \quad v = A - (t_1 / \sqrt{n'}) - (t_2 / \sqrt{n'}) = A - \{(t_1 + t_2) / \sqrt{n'}\} = A'
\]
\[
x_2 = B \quad , \quad v = B - (t_1 / \sqrt{n'}) - (t_2 / \sqrt{n'}) = B - \{(t_1 + t_2) / \sqrt{n'}\} = B'
\]
Thus,

\[
I_2' = \left\{ \sqrt{n'/2} \right\} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[\frac{-(n'/2(n' - n))}{(n'/n')u^2} + (1-n/n')\{v^2 - (t_1^2/n) - (t_2^2/n') - (2t_1t_2/n')\}\right] \right\}
\]

\[
\quad + \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[\frac{-(n'/2(n' - n))}{(n'/n')u^2} + (1-n/n')\{v^2 - (t_1^2/n) - (t_2^2/n') - (2t_1t_2/n')\}\right] \right\}
\]

\[
= \exp\left\{ t_1\mu_x + t_2\mu_x + (1/2)\{(t_1^2/n) + (t_2^2/n') + (2t_1t_2/n')\}\right\}
\]

\[
\quad \left\{ \frac{1}{2\pi} \int_{u=\infty}^{\infty} \int_{v=A'}^{\infty} \exp\left[\frac{-(1/2)(u^2 + v^2)}{2}dudv + (1/2\pi) \int_{u=-\infty}^{B'} \int_{v=-\infty}^{\infty} \exp\left[\frac{-(1/2)(u^2 + v^2)}{2}dudv \right] \right\}
\]

\[
= \exp\left\{ t_1\mu_x + t_2\mu_x + (1/2)\{(t_1^2/n) + (t_2^2/n') + (2t_1t_2/n')\}\right\} \left\{ 1 - \Phi(A') + \Phi(B') \right\}
\]

where \( \Phi(.) \) is the cumulative distribution function of \( N(0,1) \).
Now,
\[ I_2 = \mathcal{E}(\bar{x}_n, \bar{x}_m) / |\bar{x}_n| > Z_{\alpha / \sqrt{n}} \]

\[ = \frac{\partial}{\partial t_1} \left[ \frac{\partial}{\partial t_2} \{ \mathcal{E}(e^{r_1 \bar{x}_n + r_2 \bar{x}_m}) / |\bar{x}_n| > Z_{\alpha / \sqrt{n'}} \} \right]_{t_1 = t_2 = 0} \]

\[ = \frac{\partial}{\partial t_1} \left[ \frac{\partial}{\partial t_2} \{ \exp(t_1 \mu_z + t_2 \mu_x + (1/2)(t_1^2 / n) + (t_2^2 / n') + (2t_1t_2 / n') \{ \Phi(A') + \Phi(B') \} \} \right]_{t_1, t_2 = 0} \]

Differentiating under an integral sign is given by the formula

\[ \xi'(y) = \int f(x, y) \, dx , \text{ then} \]

\[ \xi'(y) = \int f(x, y) + h'(y)f(h(y), y) - g'(y)f(g(y), y) \]
Thus,

\[ I_2 = \mu_s^2(1 - \Phi(A) + \Phi(B)) + (\mu_s / \sqrt{n'})(\varphi(A) - \varphi(B)) + (1 / n')(1 - \Phi(A) + \Phi(B)) \]

\[ + (\mu_s / \sqrt{n'})(\varphi(A) - \varphi(B)) + (1 / \sqrt{2n'})(e^{-A^2/2}(A / \sqrt{n'}) - e^{-B^2/2}(B / \sqrt{n'})) \]  

(see Appendix 4)

\[ = \mu_s^2(1 - \Phi(A) + \Phi(B)) + (2 \mu_s / \sqrt{n'})(\varphi(A) - \varphi(B)) \]

\[ + (1 / n')(1 - \Phi(A) + \Phi(B)) + (1 / n')(A \varphi(A) - B \varphi(B)) \]

Hence,

\[ I_2 = E(x_i, x_i' \text{ if } |x_i| > z_s / \sqrt{n'}) \]

\[ = (\mu_s^2 + 1 / n')(1 - \Phi(A) + \Phi(B)) \]

\[ + (2 \mu_s / \sqrt{n'})(\varphi(A) - \varphi(B)) + (1 / n')(A \varphi(A) - B \varphi(B)) \]

\[ \cdots \cdots \cdots (3.4) \]

where \( \Phi(.) \) is the cumulative distribution of \( N(0,1) \) and \( \varphi(.) \) is the density function.

(Detail derivation given Appendix 4)
3.2.3 Evaluation of $E(\vec{y}_{st}, \vec{x}_{n'}$ if $|\vec{x}_{n'}| > Z_{\alpha} / \sqrt{n'}$)

Let 

$$I_3 = \left\{ E(\vec{y}_{st}, \vec{x}_{n'}) \mid |\vec{x}_{n'}| > Z_{\alpha} / \sqrt{n'} \right\}$$

$$= \frac{\partial}{\partial t_1} \left( \frac{\partial}{\partial t_2} \left\{ E(e^{t_1 \vec{y}_{st} + t_2 \vec{x}_{n'}}) \right\} \mid |\vec{x}_{n'}| > Z_{\alpha} / \sqrt{n'} \right) \bigg|_{t_1 = t_2 = 0}$$

Here $E(e^{t_1 \vec{y}_{st} + t_2 \vec{x}_{n'}})$ is the moment generating function $(\vec{y}_{st}, \vec{x}_{n'})$.

Again let 

$$I_3' = \left\{ E(e^{t_1 \vec{y}_{st} + t_2 \vec{x}_{n'}}) \mid |\vec{x}_{n'}| > Z_{\alpha} / \sqrt{n'} \right\}$$

$$= \left[ E(e^{t_1 \vec{y}_{st} + t_2 \vec{x}_{n'}}) \mid \vec{x}_{n'} > Z_{\alpha} / \sqrt{n'} \right] + \left[ E(e^{t_1 \vec{y}_{st} + t_2 \vec{x}_{n'}}) \mid \vec{x}_{n'} < -Z_{\alpha} / \sqrt{n'} \right]$$
\[\begin{align*}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 \bar{y} + t_2 \bar{x}} f(\bar{y}_s, \bar{x}_s) d\bar{y}_s d\bar{x}_s \\
&+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 \bar{y} + t_2 \bar{x}} f(\bar{y}_s, \bar{x}_s) d\bar{y}_s d\bar{x}_s
\end{align*}\]

where \( f(\bar{y}_s, \bar{x}_s) \) is the bivariate normal probability density function of the pair \((\bar{y}_s, \bar{x}_s)\) with mean \((\mu_y, \mu_x)\) and the variance covariance matrix given by

\[
\Sigma_z = \begin{bmatrix}
1/n & \rho/n' \\
\rho/n' & 1/n'
\end{bmatrix}
\]

Under the assumption that \( \sigma_{y}^2 = \sigma_{x}^2 = 1, \sigma_{x}^2 = \sigma_{y}^2 = 1 \) (WLOG)

For a bivariate normal density we are given that

\[
f(x, y) = \frac{1}{(2\pi)^{1/2} |\Sigma|^1/2} \exp\left(\frac{-1}{2}\left(((x - \mu_x)(y - \mu_y)\right)(\Sigma^{-1})\left((x - \mu_x)(y - \mu_y)\right)\right)
\]

Here,

\[
\Sigma_z^{-1} = n'/n' - \rho^2 n \begin{pmatrix}
\frac{n}{n'} & \frac{-n\rho}{n'} \\
\frac{-n\rho}{n'} & n'
\end{pmatrix}
\]
Thus,

$$f(\bar{y}_{st}, \bar{x}^*) = \frac{1}{2\pi \left(1/nn^* - (\rho^2/n^2)^{1/2}\right)} \exp \left[ - \frac{n'}{2(1-n\rho^2)} \left( n(\bar{y}_{st} - \mu_y)^2 - 2\rho \sqrt{\frac{n}{n^*}} \frac{\bar{y}_{st} - \mu_y}{1/\sqrt{n}} + n'(\bar{x}^* - \mu_x)^2 \right) \right]$$

Letting \((\bar{y}_{st} - \mu_y)/(1/\sqrt{n}) = z_1\) and \((\bar{x}^* - \mu_x)/(1/\sqrt{n^*}) = z_2\), we get \(dz_{st} = dz_1 / \sqrt{n}\) and \(d\bar{x}^* = dz_2 / \sqrt{n^*}\).

When \(\bar{x}^* = Z_\alpha / \sqrt{n^*}\) then \(z_2 = \sqrt{n'}(Z_\alpha / \sqrt{n'} - \mu_x) = Z_\alpha - \sqrt{n'}\mu_x = A\) and

when \(-\bar{x}^* = -Z_\alpha / \sqrt{n^*}\) then \(z_2 = \sqrt{n'}(-Z_\alpha / \sqrt{n'} - \mu_x) = -Z_\alpha - \sqrt{n'}\mu_x = B\)

Thus,

$$I_1' = \left[ \frac{\{\sqrt{n'}/(2\pi\sqrt{n'-\rho^2n})\}}{\{\sqrt{n'}/(2\pi\sqrt{n}-\rho^2n)\}} \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ t_1 \{\mu_y + (z_1/\sqrt{n})\} + t_2 \{\mu_x + (z_2/\sqrt{n^*})\} \right]$$

$$\exp \left[ - \{n'/2(n'-\rho^2n)\} \{z_1^2 - 2\rho\sqrt{n'/n'}z_1z_2 + z_2^2\} \right] dz_1 dz_2$$

$$+ \left[ \frac{\{\sqrt{n'}/(2\pi\sqrt{n'-\rho^2n})\}}{\{\sqrt{n'}/(2\pi\sqrt{n}-\rho^2n)\}} \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ t_1 \{\mu_y + (z_1/\sqrt{n})\} + t_2 \{\mu_x + (z_2/\sqrt{n^*})\} \right]$$

$$\exp \left[ - \{n'/2(n'-\rho^2n)\} \{z_1^2 - 2\rho\sqrt{n'/n'}z_1z_2 + z_2^2\} \right] dz_1 dz_2$$
Now we can rewrite
\[
\{z_1^2 - 2\rho(\sqrt{n/n'} z_1 z_2 + z_2^2) - \{2(n' - n\rho^2)/n')(t_1 z_1 / \sqrt{n}) + (t_2 z_2 / \sqrt{n'})\} \\
= \{(z_1 - \rho(\sqrt{n/n'} z_2) - (1 - n\rho^2/n') t_1 / \sqrt{n})^2 \\
+ (1 - n\rho^2/n') [z_2 - \rho t_1 / \sqrt{n'} - t_2 / \sqrt{n'}]^2 - t_1^2 / n - t_2^2 / n' - 2\rho t_1 t_2 / n'\}
\]

Substituting
\[
(z_1 - (\rho(\sqrt{n/n'} z_2) - (1 - n\rho^2/n') t_1 / \sqrt{n}) = \{(n' - n\rho^2)/n'\}^{1/2} u \quad \text{and}
\]
\[
z_2 - \rho(t_1 / \sqrt{n'}) - (t_2 / \sqrt{n'}) = v
\]

Then in this transformation, the Jacobian is given by
\[
J = \frac{\delta(u,v)}{\delta(x_1,x_2)} = \sqrt{1 - (\frac{2\rho}{n'})} \quad \text{and} \quad du dv = \sqrt{J} dx_1 dx_2
\]

When
\[
z_2 = A \quad , \quad v = A - (\rho t_1 / \sqrt{n'}) - (t_2 / \sqrt{n'}) = A - \{(\rho t_1 + t_2) / \sqrt{n'}\} = A'
\]
\[
z_2 = B \quad , \quad v = B - (\rho t_1 / \sqrt{n'}) - (t_2 / \sqrt{n'}) = B - \{(\rho t_1 + t_2) / \sqrt{n'}\} = B'
\]
Hence,

\[
I_i' = \left( \sqrt{\frac{n}{2\pi(\bar{r}' - \rho')}} \right) \text{Exp}(t_1\mu_x + t_2\mu_y) \left[ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \text{Exp}(-n/2(\bar{r}' - \rho')^2) \text{d}u \text{d}v \right. \\
+ \left. \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Exp}(-n/2(\bar{r}' - \rho')^2) \text{d}u \text{d}v \right] \\
+ \left. \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Exp}(-n/2(\bar{r}' - \rho')^2) \text{d}u \text{d}v \right]
\]

\[
= \text{Exp} \left\{ t_1\mu_x + t_2\mu_y + (1/2) \{(t_1^2 / n) + (t_2^2 / n) + (2\rho t_1 t_2 / n') \} \right\} \\
\left\{ \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \text{Exp}(-n/2(\bar{r}' - \rho')^2) \text{d}u \text{d}v \right. \\
+ \left. \left( \frac{1}{2\pi} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Exp}(-n/2(\bar{r}' - \rho')^2) \text{d}u \text{d}v \right]\}
\]

\[
= \text{Exp} \left\{ t_1\mu_x + t_2\mu_y + (1/2) \{(t_1^2 / n) + (t_2^2 / n') + (2\rho t_1 t_2 / n') \} \right\} \left\{ 1 - \Phi(A') + \Phi(B') \right\}
\]

where \( \Phi(.) \) is the cumulative distribution of \( N(0,1) \).
Now,

\[
I_3 = \frac{\partial}{\partial t_1} \left[ \frac{\partial}{\partial t_2} \left\{ E(e^{t_1 y} + t_2 x) \right\} \text{ if } |x| > Z_{\alpha} / \sqrt{n} \right]_{t_1 = t_2 = 0}
\]

\[
= \frac{\partial}{\partial t_1} \left[ \frac{\partial}{\partial t_2} \left\{ \text{Exp}(t_1 \mu_y + t_2 \mu_x + (1/2)(t_1^2/n) + (t_2^2/n') + (2\rho t_1 t_2/n')(1 - \Phi(A') + \Phi(B')) \} \right\} \right]_{t_1 = t_2 = 0}
\]

Differentiation under an integral sign is given by the formula

\[
\xi(y) = \int_{g(y)}^{h(y)} f(x, y) dx, \quad \text{then}
\]

\[
\xi'(y) = \int_{g(y)}^{h(y)} f(x, y) + h'(y) f(h(y), y) - g'(y) f(g(y), y)
\]
Differentiating $I_3$ under an integral sign, we get

$$I_3 = \mu_x \mu_y (1 - \Phi(A) + \Phi(B)) + (\rho \mu_x / \sqrt{n'}) (\varphi(A) - \varphi(B)) + (\rho / n')(1 - \Phi(A) + \Phi(B))$$

$$+ \mu_y (\varphi(A) - \varphi(B)) / \sqrt{n'} + (\rho / \sqrt{2\pi n'}) (e^{-A^2 / 2} \sqrt{n'} - e^{-B^2 / 2} \sqrt{n'})$$

$$= (\mu_x \mu_y + \rho / n')(1 - \Phi(A) + \Phi(B)) + (1 / \sqrt{n'}) (\mu_x + \rho \mu_y) (\varphi(A) - \varphi(B))$$

$$+ (\rho / n')(A \varphi(A) - B \varphi(B))$$

Hence,

$$E(y \mid x_0 > x_a \sqrt{n'})$$

$$= (\mu_x \mu_y + \rho / n')((1 - \Phi(A) + \Phi(B))$$

$$+ (1 / \sqrt{n'}) (\mu_x + \rho \mu_y) (\varphi(A) - \varphi(B)) + (\rho / n')(A \varphi(A) - B \varphi(B))$$

$$\text{.........(3.5)}$$

where $\Phi(.)$ is the cumulative distribution of $\text{N}(0,1)$ and $\varphi(.)$ is the density function.

(Detail derivation given in Appendix 5)
3.2.4 Evaluation of MSE(t_5)

Now, substituting eqn (3.3), (3.4) and (3.5) in eqn (3.2), we get

\[
E (t_5^2) = Var \left( \overline{y}_{st} \right) + \left\{ E \left( \overline{y}_{st} \right) \right\}^2 + \rho^2 \left[ Var \left( \overline{x}_{st} \right) + \left\{ E \left( \overline{x}_{st} \right) \right\}^2 \right] \\
- 2 \rho^2 \sqrt{Var \left( \overline{x}_{st} \right) Var \left( \overline{y}_{st} \right)} - 2 \rho E \left( \overline{x}_{st} \right) E \left( \overline{y}_{st} \right) \\
+ \{(\mu_x^2 + 1/n)(1 - \Phi(A) + \Phi(B)) + (2\mu_x/\sqrt{n'})(\varphi(A) - \varphi(B)) \\
+ (1/n')(A \varphi(A) - B \varphi(B))\} (\rho^2 - 2\rho^2) \\
+ 2 \rho(\mu_x \mu_y + \rho/n')(1 - \Phi(A) + \Phi(B)) \\
+ 2 \rho (1/\sqrt{n'})(\mu_y + \rho \mu_x)(\varphi(A) - \varphi(B)) + 2(\rho^2/n')\{A \varphi(A) - B \varphi(B)\}
\]

\[
= Var \left( \overline{y}_{st} \right) + \left\{ E \left( \overline{y}_{st} \right) \right\}^2 + \rho^2 \left[ Var \left( \overline{x}_{st} \right) + \left\{ E \left( \overline{x}_{st} \right) \right\}^2 \right] \\
- 2 \rho^2 \sqrt{Var \left( \overline{x}_{st} \right) Var \left( \overline{y}_{st} \right)} - 2 \rho E \left( \overline{x}_{st} \right) E \left( \overline{y}_{st} \right) \\
+ \{- (\mu_x^2 + 1/n) \rho^2 + 2 \rho(\mu_x \mu_y + \rho/n')(1 - \Phi(A) + \Phi(B)) \\
+ \{- \rho^2 (2\mu_x/\sqrt{n'}) + (2\rho/n')(A \varphi(A) - B \varphi(B)) \} (\varphi(A) - \varphi(B)) \\
+ \{- \rho^2/n' + (2\rho^2/n')\{A \varphi(A) - B \varphi(B)\}\}
\]
\[
E(t^2_s) = \text{Var}(\overline{y}_{st}) + \left\{ E(\overline{y}_{st}) \right\}^2 + \rho^2 \left[ (\text{Var}(\overline{x}_{st}) + \left\{ E(\overline{x}_{st}) \right\}^2 \right]
- 2 \rho^2 \sqrt{\text{Var}(\overline{x}_{st}) \cdot \text{Var}(\overline{y}_{st})} - 2 \rho \text{E}(\overline{x}_{st}) \text{E}(\overline{y}_{st})
+ \{(2 \mu_x \mu_y - \rho \mu_x^2)\rho + (\rho^2 / n')\}(1 - \Phi(A) + \Phi(B))
+ \{(2 \rho / \sqrt{n'}) \mu_y (\varphi(A) - \varphi(B)) + (\rho^2 / n')(A \varphi(A) - B \varphi(B))
\]

The variance of \( \overline{x}_{st} \) and \( \overline{y}_{st} \) being given by

\[
\text{Var}(\overline{x}_{st}) = \sum W_h^2 \sigma_x^2 / n_h
\]
\[
\text{Var}(\overline{y}_{st}) = \sum W_h^2 \sigma_y^2 / n_h \quad \text{(Cochran 1977)}
\]

which under the stated assumptions becomes

\[
\text{Var}(\overline{x}_{st}) = \sum W_h^2 / n_h
\]
\[
\text{Var}(\overline{y}_{st}) = \sum W_h^2 / n_h \quad \text{(under the assumption } \sigma_x^2 = \sigma_y^2 = 1 \text{ )}
\]

Therefore,

\[
E(t^2_s) = \sum W_h^2 / n_h + \mu_y^2 + \rho^2 \left\{ \sum W_h^2 / n_h + \mu_x^2 \right\} - 2 \rho^2 \sum W_h^2 / n_h - 2 \rho \mu_x \mu_y
+ \{(2 \mu_x \mu_y - \rho \mu_x^2)\rho + (\rho^2 / n')\}(1 - \Phi(A) + \Phi(B))
+ \{(2 \rho / \sqrt{n'}) \mu_y (\varphi(A) - \varphi(B)) + (\rho^2 / n')(A \varphi(A) - B \varphi(B))
\]

when the samples are selected with proportional allocation then the stratum weight is given by \( W_h = (N_h / N) = (n_h / n) \)
Thus \[ \sum_{h} W_{h}^2 / n_{h} = \frac{1}{n} \sum_{h} W_{h} = (1/n) \sum_{h} W_{h} = (1/n) \quad (\text{as } \sum_{h} W_{h} = 1) \]

Hence,

\[
E \left( t_{5}^2 \right) = (1 - \rho^2) / n + \mu_{y}^2 + \rho \mu_{y}^2 - 2 \rho \mu_{x} \mu_{y} + \{(2 \rho \mu_{x} \mu_{y} - \rho^2 \mu_{x}^2 + (\rho^2 / n')) - \{2 \rho \mu_{x} \mu_{y} - \rho^2 \mu_{x}^2 + (\rho^2 / n')\} \{ \Phi (A) - \Phi (B) \} + \{(2 \rho \mu_{y} / \sqrt{n'}) (\varphi (A) - \varphi (B)) + (\rho^2 / n')(A \varphi (A) - B \varphi (B))
\]

\[
= (1 - \rho^2) / n + \mu_{y}^2 + \rho^2 / n' - \{2 \rho \mu_{x} \mu_{y} - \rho^2 \mu_{x}^2 + (\rho^2 / n')\} \{ \Phi (A) - \Phi (B) \} + \{(2 \rho \mu_{y} / \sqrt{n'}) (\varphi (A) - \varphi (B)) + (\rho^2 / n')(A \varphi (A) - B \varphi (B))
\]

\[\text{Mean square error of } t_{5} \text{ is given by}
\]

\[
\text{MSE} (t_{5}) = E(t_{5}^2) - \{E(t_{5})\}^2 + \{E(t_{5}) - \mu_{y}\}^2 \quad \text{from (3.1)}
\]

\[
= E \left( t_{5}^2 \right) - 2 E \left( t_{5} \right) \mu_{y} + \mu_{y}^2 \quad \text{......................(3.7)}
\]

and from Chapter 2

91
\[ E(t_s) = \mu_y - \rho\mu_x \{ \Phi(A) - \Phi(B) \} + \rho (1 / \sqrt{n'}) \{ \phi(A) - \phi(B) \} \] ...

...(3.8)

Therefore, substituting (3.6) and (3.8) in (3.7)

\[
MSE(t_s) = \frac{(1 - \rho^2)}{n} + \mu_y^2 + \rho^2 / n' \\
- \{2 \rho\mu_x\mu_y - \rho^2\mu_x^2 + (\rho^2 / n')\} \{ \Phi(A) - \Phi(B) \} \\
+ \{(2 \rho\mu_y / \sqrt{n'})\{(\phi(A) - \phi(B)) + (\rho^2 / n')(A\phi(A) - B\phi(B))\} \\
- 2\{\mu_y - \rho\mu_x (\Phi(A) - \Phi(B)) + (\rho / \sqrt{n'}) (\phi(A) - \phi(B))\} \mu_y + \mu_y^2
\]

Thus,

\[
MSE(t_s) = \{(1 - \rho^2) / n + \rho^2 / n'\} + (\rho^2 / n')\{A\phi(A) - B\phi(B)\} \\
- \rho^2 (1 / n' - \mu_x^2) \{\Phi(A) - \Phi(B)\} 
\] 

......(3.9)
The Mean square error function of $t_5$ given by equation (3.9) shows that its behavior can be observed for different values of $\mu_x$. As $\text{MSE}(t_5)$ is symmetric about $\mu_x = 0$, hence we need to consider the behavior only for $\mu_x \geq 0$. This above behavior can be studied for selected values of the level of significance $\alpha$ and correlation coefficient $\rho$ and by fixing some hypothetical values for $n$ and $n'$.

### 3.3 Discussion

The above Mean squared error of $t_5$ is given as

$$\text{MSE}(t_5) = \{(1 - \rho^2)/n + \rho^2/n'\} + (\rho^2/n')(A\varphi(A) - B\varphi(B))$$

$$- \rho^2(1/n' - \mu_x^2)(\Phi(A) - \Phi(B))$$

$$= g_1 + h_1,$$

where

$$g_1 = \{(1 - \rho^2)/n + \rho^2/n'\}$$

and

$$h_1 = (\rho^2/n')(A\varphi(A) - B\varphi(B)) - \rho^2(1/n' - \mu_x^2)(\Phi(A) - \Phi(B))$$

We note that $g_1$ is the MSE of $t_4$, the combined linear regression estimator in double sampling (Appendix 6), when information on $\mu_x$ is not known.

The values of $\text{MSE}(t_5)$ can be easily computed for different values of $\mu_x$. In order to get an idea about the behavior of the mean square error function with respect to $\mu_x$, $\text{MSE}(t_5)$ is computed for a set of values of $n$, $n'$, $\alpha$ and $\rho$ which are given in Table 3.1 – 3.2 and Figure 3.1 – 3.2. It is found in general that $\text{MSE}(t_5)$ is minimum at $\mu_x = 0$. As $\mu_x$ is increases, the $\text{MSE}(t_5)$ increases to a maximum and then gradually decreases and then becomes constant. The figures clearly show that when the mean of the auxiliary variable is close to the
hypothetical value, then the $MSE(t_5)$ is minimum. Also as $\mu_x$ moves away from the hypothetical value the $MSE(t_5)$ increases, but after attaining maximum again gradually decreases and then becomes constant. This establishes the utility of the present study that the use of partial information and preliminary test of the auxiliary variable reduces the $MSE(t_5)$ of the proposed estimator.

3.4 Mean square error function of CRPTE computed numerically

It is seen that the analytical method of determining the MSE involves evaluating the mathematical expectation of the random variables like $E(\bar{x}^2)$, $E(\bar{x}_{x_{-}})$ and $E(\bar{x}_{y_{-}})$. The derivation of these expectation is done using moment generating function and also involves the application of single and double integration technique. In the evaluation, using bivariate frequency distributions, a tedious substitution of change of variables is necessary to simplify the integral. The above expectation is finally obtained by differentiating under the integral sign.

The above analytical derivation in the evaluation of MSE is tedious. An alternative method is sought with the help of numerical techniques. The mathematical expectations of the random variables like $E(\bar{x}^2)$, $E(\bar{x}_{x_{-}})$ and $E(\bar{x}_{y_{-}})$ can be evaluated without the use of moment generating function and also avoid the complex substitution. By the use of numerical technique the steps of differentiating under the integral sign can also be avoided. The double integral involves in the evaluation of the above mentioned expectations can be
calculated by using a numerical integration technique for a function in two variables \( f(x, y) \), given by Simpson 1/3\(^{rd}\) rule.

With the advent and rapid development of high speed digital computers and the increasing desire for accurate and faster solution to applied problems, it has become possible to evaluate complex mathematical formulation numerically in a fewer number of steps in a short duration of time. In this study, computers can be used for evaluating the above numerical integrals with the help of programs written in Fortran 77.

By the definition of \( t_5 \)

\[
t_5 = \begin{cases} 
(y_{x'} - \rho \bar{x}_n) & \text{if } |\bar{x}_n| \leq Z_a / \sqrt{n} \\
(y_{x'} + \rho (\bar{x}_n - \bar{x}_x)) & \text{if } |\bar{x}_n| > Z_a / \sqrt{n}
\end{cases}
\]

where \( \bar{y}_{x'} = \sum_h W_h \bar{y}_h \) and \( \bar{x}_n = \sum_h W_h \bar{x}_h \)

and the stratum means given by

\[
\bar{x}_n = \frac{\sum_{i=1}^{n_h} x_{ih}}{n_h} \quad \text{and} \quad \bar{y}_n = \frac{\sum_{i=1}^{n_h} y_{ih}}{n_h}
\]

As before to obtain MSE of \( t_5 \), we notice that

\[
MSE (t_5) = E(t_5^2) - \{E(t_5)\}^2 + \{\text{Bias}(t_5)\}^2
\]

\[
\Rightarrow MSE (t_5) = E(t_5^2) - 2E(t_5)\mu_y + \mu_y^2
\]
Now,
\[ E(t_s^2) = E(t_s^2) \quad \text{if} \quad \left| \bar{x}_n \right| \leq Z_a / \sqrt{n'} \]
\[ + E(t_s^2) \quad \text{if} \quad \left| \bar{x}_n \right| > Z_a / \sqrt{n'} \]

After simplification we get,
\[ E(t_s^2) = Var(y_n') + \left\{ E\left( y_n' \right) \right\}^2 + \rho^2 \left[ Var(x_n') + \left\{ E\left( x_n' \right) \right\}^2 \right] \]
\[ - 2 \rho^2 \sqrt{Var(x_n')} \sqrt{Var(y_n')} - 2 \rho E(x_n') E(y_n') \]
\[ + E\left( \rho^2 x_n'^2 - 2 \rho^2 x_n' y_n' + 2 \rho x_n' y_n' \right) \quad \text{if} \quad \left| \bar{x}_n \right| > Z_a / \sqrt{n'} \]

Now, substituting
\[ I_1 = E(\bar{x}_n^2) \quad \text{if} \quad \left| \bar{x}_n \right| > Z_a / \sqrt{n'} \]
\[ I_2 = E(\bar{x}_n \bar{y}_n') \quad \text{if} \quad \left| \bar{x}_n \right| > Z_a / \sqrt{n'} \]
\[ I_3 = E(\bar{x}_n \bar{y}_n') \quad \text{if} \quad \left| \bar{x}_n \right| > Z_a / \sqrt{n'} \]

\[ MSE(t_s) = (1 / n) + \mu_y^2 + \rho^2 \{ (1 / n) + \mu_y^2 \} - (2 \rho^2 / n) - 2 \rho \mu_x \mu_y \]
\[ + [\rho^2 I_1 - 2 \rho^2 I_2 + 2 \rho I_3] - 2 \rho (I - \mu_x + \mu_y) \mu_y + \mu_y^2 \]

where \( I = E(\bar{x}_n \quad \text{if} \quad \left| \bar{x}_n \right| > Z_a / \sqrt{n'}) \)

Thus,
\[ MSE(t_s) = (1 / n) + \rho^2 \{ (1 / n) + \mu_y^2 \} - (2 \rho^2 / n) \]
\[ + [\rho^2 I_1 - 2 \rho^2 I_2 + 2 \rho I_3] - 2 \rho I \mu_y \]

The mean square error function derived above involves the population parameter \( \mu_y \), the mean of the study variable which is needed to be estimated.

Thus, in order to eliminate this parameter, the analytical result of \( I_1 \) from eqn. (3.5) and \( I \) from section 2.3 are substituted in the above MSE function to obtain the required result given as follows;
\[ \text{MSE} \ (t_s) = \frac{1}{n} + \rho^2 \{(1/n) + \mu_x^2\} - (2 \rho^2/n) + \rho^2 I_1 - 2 \rho^2 I_2 \\
+ 2 \rho \left\{ \mu_x \mu_y + \left( \rho/n \right) (1 - \Phi(A) + \Phi(B)) \right\} \\
+ \left\{ \frac{1}{\sqrt{n'}} \left( \mu_y + \rho \mu_x \right) (\varphi(A) - \varphi(B)) \right\} \\
+ \left( \rho/n' \right) (A \varphi(A) - B \varphi(B)) \\
- 2 \rho \mu_y \left\{ \mu_x (1 - \Phi(A) + \Phi(B)) + (1/\sqrt{n'}) (\varphi(A) - \varphi(B)) \right\} \]

\[ = \frac{1}{n} + \rho^2 \{(1/n) + \mu_x^2\} - (2 \rho^2/n) + \rho^2 I_1 - 2 \rho^2 I_2 \\
+ (2 \rho^2/n') (1 - \Phi(A) + \Phi(B)) \\
+ (2 \rho^2 \mu_x/\sqrt{n'}) (\varphi(A) - \varphi(B)) \\
+ (2 \rho^2/n') (A \varphi(A) - B \varphi(B)) \]

\[ \text{...............(3.10)} \]

In the above MSE function \( \Phi(.) \), the cumulative distribution function of \( N(0,1) \) and \( \varphi(.) \) its density function, are obtained from normal tables.

### 3.4.1 Numerical computation of \( I_1 \)

Now \( I_1 = E(x_{n'}^2 \text{ if } |x_{n'}| > Z_a/\sqrt{n'}) \)

\[ = \{E(x_{n'}^2 \text{ if } x_{n'} > Z_a/\sqrt{n'})\} + \{E(x_{n'}^2 \text{ if } x_{n'} < -Z_a/\sqrt{n'})\} \]

\[ = \int_{Z_a/\sqrt{n'}}^{\infty} x_{n'}^2 f(x_{n'}) dx_{n'} + \int_{-\infty}^{-Z_a/\sqrt{n'}} x_{n'}^2 f(x_{n'}) dx_{n'} \]

Since \( x_{n'} \sim N(\mu_x, 1/n') \) we get

\[ f(x_{n'}) = \frac{n'}{2\pi} \exp \left\{ -\frac{1}{2} (x_{n'} - \mu_x)(1/\sqrt{n'}) \right\} \]
Thus the integrals becomes

\[ I_1 = \left( \frac{\sqrt{n'}}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} x''^2 \exp \left[ -\frac{1}{2} \left( \frac{1}{\sqrt{n'}} \left( x'' - \mu_x \right) \right)^2 \right] d\tilde{x}_{n'} \]

\[ + \left( \frac{\sqrt{n'}}{\sqrt{2\pi}} \right) \int_{-\infty}^{\infty} x''^2 \exp \left[ -\frac{1}{2} \left( \frac{1}{\sqrt{n'}} \left( x'' - \mu_x \right) \right)^2 \right] d\tilde{x}_{n'} \]

Putting \( w = \frac{(x'' - \mu_x)}{1/\sqrt{n'}} \) \( \Rightarrow dw = \sqrt{n'} dx'' \), we have

When \( x'' = \frac{Z_a}{\sqrt{n'}} \) then \( w = \frac{\sqrt{n'}}{\sqrt{2\pi}} \left( Z_a \right) \left( \frac{1}{\sqrt{n'}} - \mu_x \right) = Z_a - \sqrt{n'} \mu_x = A \)

and when \( x'' = -\frac{Z_a}{\sqrt{n'}} \) then \( w = \frac{\sqrt{n'}}{\sqrt{2\pi}} \left( -Z_a \right) \left( \frac{1}{\sqrt{n'}} - \mu_x \right) = -Z_a - \sqrt{n'} \mu_x = B \)

Therefore,

\[ I_1 = \left( \frac{1}{\sqrt{2\pi}} \right) \left[ \int_{-\infty}^{\infty} (\mu_x + (w/\sqrt{n'}))^2 \exp \left[ -\frac{1}{2} w^2 \right] dw \right] + \left. \right|_{-\infty}^{B} \]

\[ \text{or } I_1 = I_{11} + I_{12} \]  \hspace{1cm} \text{(3.11)}

The integrals \( I_{11} \) and \( I_{12} \) are evaluated by the use of Fortran 77 and the output values are given in Table 3.3. (Appendix 7 and 8)
3.4.2 Numerical computation of $I_2$

$$I_2 = \left\{ E(x_n' x_{n''}) / |x_n'| > Z_a / \sqrt{n'} \right\}$$

$$= \left\{ E(x_n' x_{n''}) / x_n' > Z_a / \sqrt{n'} \right\} + \left\{ E(x_n' x_{n''}) / x_n' < -Z_a / \sqrt{n'} \right\}$$

$$= \int_{-\infty}^{\infty} \int_{x_n' = Z_a / \sqrt{n'}}^{\infty} x_n' \cdot f(x_{n''}, x_{n''}) \, d\bar{x}_{n'} \, d\bar{x}_{n''} + \int_{-\infty}^{\infty} \int_{x_n' = -\infty}^{-Z_a / \sqrt{n'}} x_n' \cdot f(x_{n''}, x_{n''}) \, d\bar{x}_{n'} \, d\bar{x}_{n''}$$

where

$$f(x_{n''}, x_{n''}) = \frac{1}{2\pi \left(1/n' - 1/n''^2\right)^{1/2}} \exp \left[ -\frac{1}{2} \left( \frac{n(x_n'' - \mu_n)}{\sqrt{n'}} - 2 \left( \frac{n(x_n'' - \mu_n)}{\sqrt{n'}} \frac{(x_n'' - \mu_n)}{\sqrt{n''}} + n' (x_{n''} - \mu_n)^2 \right) \right]$$

(under the assumptions considered in the present study)

Letting $(x_n'' - \mu_n)/(1/\sqrt{n}) = x_1$ and $(x_n'' - \mu_n)/(1/\sqrt{n'}) = x_2$,

we get $d\bar{x}_{n'} = dx_1 / \sqrt{n}$ and $d\bar{x}_{n''} = dx_2 / \sqrt{n'}$

When $x_n'' = Z_a / \sqrt{n'}$ then $x_2 = \sqrt{n'}(Z_a / \sqrt{n'} - \mu_n) = \sqrt{n'} \mu_n - A$

and when $x_n'' = -Z_a / \sqrt{n'}$, then $x_2 = \sqrt{n'}(-Z_a / \sqrt{n'} - \mu_n) = -\sqrt{n'} \mu_n = B$

Therefore,

$$I_2 = \left[ \left\{ \sqrt{n''}/(2\pi \sqrt{n' - n}) \right\} \int_{-\infty}^{\infty} \int_{x_2 = A}^{\infty} \exp \left[ -\{n'/2(n' - n)\} \{ x_1^2 - 2\sqrt{n/n'} x_1 x_2 + x_2^2 \} \right] dx_1 dx_2 \right]$$

$$+ \left\{ \sqrt{n''}/(2\pi \sqrt{n' - n}) \right\} \int_{-\infty}^{\infty} \int_{x_2 = -\infty}^{B} \exp \left[ -\{n'/2(n' - n)\} \{ x_1^2 - 2\sqrt{n/n'} x_1 x_2 + x_2^2 \} \right] dx_1 dx_2 \right]$$
\[ I_2 = I_{21} + I_{22} \] ..........................(3.12)

The integrals \( I_{21} \) and \( I_{22} \) are evaluated by the use of Fortran 77 and the output values are given in Table 3.4. (Appendix 9 and 10)

3.5 Discussion

For a set of values of \( n, n', \rho \) the integrals \( I_1 \) and \( I_2 \) are computed for different values of the level of significance \( \alpha \). The integrals are evaluated by Simpson’s 1/3rd rule through programs written in Fortran 77 in a Linux operating system and the outputs of the program for the values of \( I_1 \) and \( I_2 \) are given in Tables 3.3 and 3.4. The values of the cumulative function \( \Phi(.) \) and the corresponding density function \( \phi(.) \) are also evaluated for the same set of values of \( n, n', \) and \( \alpha \). All the set of values are compiled in Excel work sheet and finally the \( MSE(t_1) \) is evaluated by substituting all the corresponding set of values of the integrals \( I_1, I_2, \Phi(.) \) and \( \phi(.) \) for various values of level of significant \( \alpha \) and \( \rho \) in (3.10) and are given in Tables 3.5 and 3.6.

The \( MSE(t_2) \) is plotted for different values of level of significance \( \alpha \) (Figure 3.3) and also for different values of correlation coefficient \( \rho \) (Figure 3.4) respectively. It is found in general that \( MSE(t_1) \) is minimum at \( \mu_x = 0 \). As \( \mu_x \) increases, the \( MSE(t_3) \) increases to a maximum and then gradually decreases and thereafter becomes constant with further increase in the value of \( \mu_x \). The figures clearly show that when the mean of the auxiliary variable is close to the
hypothetical value, then the MSE is minimum. Also as \( \mu_x \) moves away from the hypothetical value, the MSE increases, but after attaining maximum again gradually reduces and then becomes constant. Figure 3.5 and 3.6 show that the mean square error obtained by numerical methods depict a pattern similar to that obtained by analytical methods for increasing values of \( \mu_x \). The differences in the values of MSE between analytical and numerical methods of computation are minimal.
Table 3.1 Behavior of MSE(\(t_6\)) computed analytically with respect to \(\mu_x\) for different values of \(\alpha\) and for \(n = 100, \ n' = 200, \ \rho = 0.8\)

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.004</td>
<td>0.01</td>
<td>0.015</td>
<td>0.009</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
</tr>
<tr>
<td>0.05</td>
<td>0.004</td>
<td>0.01</td>
<td>0.01</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
</tr>
<tr>
<td>0.25</td>
<td>0.006</td>
<td>0.008</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
</tr>
</tbody>
</table>

Table 3.2 Behaviour of MSE(\(t_6\)) computed analytically with respect to \(\mu_x\) for different values of \(\rho\) and for \(n = 100, \ n' = 200, \ \alpha = 0.05\)

<table>
<thead>
<tr>
<th>(\rho)</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0.0062</td>
<td>0.0109</td>
<td>0.0161</td>
<td>0.012</td>
<td>0.0095</td>
<td>0.0092</td>
<td>0.0092</td>
<td>0.0092</td>
<td>0.0092</td>
<td>0.0092</td>
<td>0.0092</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0051</td>
<td>0.0112</td>
<td>0.018</td>
<td>0.0126</td>
<td>0.0094</td>
<td>0.0089</td>
<td>0.0089</td>
<td>0.0089</td>
<td>0.0089</td>
<td>0.0089</td>
<td>0.0089</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0038</td>
<td>0.0115</td>
<td>0.0201</td>
<td>0.0133</td>
<td>0.0093</td>
<td>0.0087</td>
<td>0.0086</td>
<td>0.0086</td>
<td>0.0086</td>
<td>0.0086</td>
<td>0.0086</td>
</tr>
</tbody>
</table>
Table 3.3 Numerically computed values of $I_i$ with $n = 100$, $n' = 200$ and $p = 0.8$ for (a) $\alpha = 0.01$ (b) $\alpha = 0.05$ (c) $\alpha = 0.25$.

<table>
<thead>
<tr>
<th></th>
<th>$I_{11}$</th>
<th>$I_{12}$</th>
<th>$I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.000348</td>
<td>0.00348</td>
<td>0.000696</td>
</tr>
<tr>
<td></td>
<td>0.00532</td>
<td>7.93E-06</td>
<td>0.0053279</td>
</tr>
<tr>
<td></td>
<td>0.03154</td>
<td>5.92E-08</td>
<td>0.0315401</td>
</tr>
<tr>
<td></td>
<td>0.08991</td>
<td>1.40E-10</td>
<td>0.08991</td>
</tr>
<tr>
<td></td>
<td>0.16684</td>
<td>1.14E-13</td>
<td>0.16684</td>
</tr>
<tr>
<td></td>
<td>0.25819</td>
<td>2.62E-17</td>
<td>0.25819</td>
</tr>
<tr>
<td></td>
<td>0.36825</td>
<td>1.85E-21</td>
<td>0.36825</td>
</tr>
<tr>
<td></td>
<td>0.49823</td>
<td>4.03E-26</td>
<td>0.49823</td>
</tr>
<tr>
<td></td>
<td>0.648203</td>
<td>2.70E-31</td>
<td>0.648203</td>
</tr>
<tr>
<td></td>
<td>0.81817</td>
<td>5.55E-37</td>
<td>0.81817</td>
</tr>
<tr>
<td></td>
<td>1.0081</td>
<td>3.52E-43</td>
<td>1.0081</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$I_{11}$</th>
<th>$I_{12}$</th>
<th>$I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>0.001162</td>
<td>0.001162</td>
<td>0.002324</td>
</tr>
<tr>
<td></td>
<td>0.010538</td>
<td>4.75E-05</td>
<td>0.010586</td>
</tr>
<tr>
<td></td>
<td>0.042008</td>
<td>6.61E-07</td>
<td>0.042009</td>
</tr>
<tr>
<td></td>
<td>0.096485</td>
<td>2.98E-09</td>
<td>0.096485</td>
</tr>
<tr>
<td></td>
<td>0.15234</td>
<td>4.20E-12</td>
<td>0.168142</td>
</tr>
<tr>
<td></td>
<td>0.258275</td>
<td>1.88E-15</td>
<td>0.258275</td>
</tr>
<tr>
<td></td>
<td>0.368259</td>
<td>2.58E-19</td>
<td>0.368259</td>
</tr>
<tr>
<td></td>
<td>0.498232</td>
<td>1.08E-23</td>
<td>0.498232</td>
</tr>
<tr>
<td></td>
<td>0.648202</td>
<td>1.38E-28</td>
<td>0.648202</td>
</tr>
<tr>
<td></td>
<td>0.818168</td>
<td>5.43E-34</td>
<td>0.818168</td>
</tr>
<tr>
<td></td>
<td>1.0081</td>
<td>6.45E-30</td>
<td>1.0081</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$I_{11}$</th>
<th>$I_{12}$</th>
<th>$I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>0.00299</td>
<td>0.00299</td>
<td>0.00598</td>
</tr>
<tr>
<td></td>
<td>0.016174</td>
<td>2.37E-04</td>
<td>0.016411</td>
</tr>
<tr>
<td></td>
<td>0.047559</td>
<td>6.96E-06</td>
<td>0.047566</td>
</tr>
<tr>
<td></td>
<td>0.098308</td>
<td>6.94E-08</td>
<td>0.098308</td>
</tr>
<tr>
<td></td>
<td>0.16829</td>
<td>2.24E-10</td>
<td>0.16829</td>
</tr>
<tr>
<td></td>
<td>0.25828</td>
<td>2.29E-13</td>
<td>0.25828</td>
</tr>
<tr>
<td></td>
<td>0.368259</td>
<td>7.33E-17</td>
<td>0.368259</td>
</tr>
<tr>
<td></td>
<td>0.498233</td>
<td>7.23E-21</td>
<td>0.498233</td>
</tr>
<tr>
<td></td>
<td>0.648202</td>
<td>2.18E-25</td>
<td>0.648202</td>
</tr>
<tr>
<td></td>
<td>0.818169</td>
<td>2.02E-30</td>
<td>0.818169</td>
</tr>
<tr>
<td></td>
<td>1.0081</td>
<td>5.71E-36</td>
<td>1.0081</td>
</tr>
</tbody>
</table>
Table 3.4 Numerically computed values of $I_2$ with $n = 100$, $n' = 200$ and $\rho = 0.8$ for (a) $\alpha = 0.01$. (b) $\alpha = 0.05$. (c) $\alpha = 0.25$.

<table>
<thead>
<tr>
<th></th>
<th>$I_{21}$</th>
<th>$I_{22}$</th>
<th>$I_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I21</td>
<td>0.000317</td>
<td>0.000317</td>
<td>0.00634</td>
</tr>
<tr>
<td></td>
<td>0.005044</td>
<td>0.000007</td>
<td>0.005051</td>
</tr>
<tr>
<td></td>
<td>0.030573</td>
<td>0</td>
<td>0.030573</td>
</tr>
<tr>
<td></td>
<td>0.088609</td>
<td>0</td>
<td>0.088609</td>
</tr>
<tr>
<td></td>
<td>0.165882</td>
<td>0</td>
<td>0.165882</td>
</tr>
<tr>
<td></td>
<td>0.257301</td>
<td>0</td>
<td>0.257301</td>
</tr>
<tr>
<td></td>
<td>0.367307</td>
<td>0</td>
<td>0.367307</td>
</tr>
<tr>
<td></td>
<td>0.497205</td>
<td>0</td>
<td>0.497205</td>
</tr>
<tr>
<td></td>
<td>0.647086</td>
<td>0</td>
<td>0.647086</td>
</tr>
<tr>
<td></td>
<td>0.816951</td>
<td>0</td>
<td>0.816951</td>
</tr>
<tr>
<td></td>
<td>1.0068</td>
<td>0</td>
<td>1.0068</td>
</tr>
</tbody>
</table>

| (b) |          |          |       |
| I21 | 0.001058 | 0.001058 | 0.002116 |
|     | 0.010032 | 0.000042 | 0.010074 |
|     | 0.040982 | 0.000001 | 0.040983 |
|     | 0.095477 | 0        | 0.095477 |
|     | 0.167265 | 0        | 0.167265 |
|     | 0.25739  | 0        | 0.25739  |
|     | 0.367309 | 0        | 0.367309 |
|     | 0.497205 | 0        | 0.497205 |
|     | 0.647086 | 0        | 0.647086 |
|     | 0.816951 | 0        | 0.816951 |
|     | 1.0068   | 0        | 1.0068   |

| (c) |          |          |       |
| I21 | 0.002727 | 0.002727 | 0.005454 |
|     | 0.015482 | 0.000204 | 0.015686 |
|     | 0.046709 | 0.000017 | 0.046726 |
|     | 0.097393 | 0        | 0.097393 |
|     | 0.167462 | 0        | 0.167462 |
|     | 0.257396 | 0        | 0.257396 |
|     | 0.367309 | 0        | 0.367309 |
|     | 0.497205 | 0        | 0.497205 |
|     | 0.647086 | 0        | 0.647086 |
|     | 0.816951 | 0        | 0.816951 |
|     | 1.0068   | 0        | 1.0068   |
Table 3.5 Behaviour of MSE(tₕ) computed numerically with respect to μₓ for different values of α and for n = 100,  n' = 200,  ρ = 0.8

<table>
<thead>
<tr>
<th>α</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0</td>
<td>0.01</td>
<td>0.012</td>
<td>0.015</td>
<td>0.011</td>
<td>0.007</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>0.05</td>
<td>0</td>
<td>0.01</td>
<td>0.012</td>
<td>0.013</td>
<td>0.008</td>
<td>0.006</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>0.25</td>
<td>0</td>
<td>0</td>
<td>0.011</td>
<td>0.008</td>
<td>0.006</td>
<td>0.006</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 3.6 Behaviour of MSE(tₕ) computed numerically with respect to μₓ for different values of ρ and for n = 100,  n' = 200,  α = 0.05

<table>
<thead>
<tr>
<th>ρ</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>0</td>
<td>0.01</td>
<td>0.011</td>
<td>0.012</td>
<td>0.008</td>
<td>0.007</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>0.8</td>
<td>0</td>
<td>0.01</td>
<td>0.012</td>
<td>0.013</td>
<td>0.008</td>
<td>0.006</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>0.9</td>
<td>0</td>
<td>0.01</td>
<td>0.012</td>
<td>0.014</td>
<td>0.007</td>
<td>0.005</td>
<td>0</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>
Figure 3.1  Behaviour of MSE(t₅) computed analytically with respect to μₓ for different values of α and for n = 100, n' = 200, ρ = 0.8

![Figure 3.1](image1.png)

Figure 3.2  Behaviour of MSE(t₅) computed analytically with respect to μₓ for different values of ρ and for n = 100, n' = 200, α = 0.05

![Figure 3.2](image2.png)
Figure 3.3 Behaviour of the MSE(t₀) computed numerically with respect to μₓ for different values of α and for n = 100, n' = 200, ρ = 0.8

Figure 3.4 Behaviour of the MSE(t₀) computed numerically with respect to μₓ for different values of ρ and for n = 100, n' = 200, α = 0.05
**Figure 3.5** Comparative behaviour of the MSE(t5) with respect to \( \mu_x \) for different values of \( \alpha \) and for \( \rho = 0.8, n = 100, n' = 200 \)

![Graph showing MSE(t5) for different values of \( \alpha \) and \( \rho \).](image1)

**Figure 3.6** Comparative behaviour of the MSE(t6) with respect to \( \mu_x \) for different values of \( \rho \) and for \( \alpha = 0.05, n = 100, n' = 200 \)

![Graph showing MSE(t6) for different values of \( \rho \).](image2)