Chapter 3

Graphical Realization of Finite Topologies

A topology $\tau$, on a non-empty ground set $X$ is said to be graphically realizable if there exists a connected graph $G = (V, E)$ and a set-indexer $f : V(G) \rightarrow 2^X$ such that $f(V(G)) \cup f(\Theta(E(G))) = \tau$; $G$ is called the graphical realization of $\tau$. In this chapter, we analyze the non isomorphic graphical realizations of all topologies of order up to 8.

3.1 Introduction

Doignon and Falmagne give different aspects of knowledge space in [3]. We envisage a field of knowledge that can be parsed into a set of questions each of which has a correct response. We shall consider a basic set of such questions, called the domain, that is large enough to give a fine-grained, representative coverage of the field. The knowledge state
of an individual is represented by the set of questions in the domain that the individual is capable of answering in ideal conditions.

Definition 3.1.1. Let \( Q \) be a non-empty set and \( K \) be a family of subsets of \( Q \) containing at least \( Q \) and the empty set, \( \emptyset \). The set \( Q \) is called the domain of the knowledge structure. Its elements are referred to as questions or items and the subsets in the family \( K \) are labeled (knowledge) states. We shall say that \( K \) is a knowledge structure on a set \( Q \) to mean that \((Q, K)\) is a knowledge structure.

Definition 3.1.2. When the family \( K \) of knowledge structure \((Q, K)\) is closed under union—that is when \( \bigcup F \in K \) whenever \( F \subset K \)—we shall say that \((Q, K)\) is a (knowledge) space, or equivalently, that \( K \) is a (knowledge) space on \( Q \).

Let \( Q = \{a, b, c, d, e\} \) be a domain and \( K = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}, Q\} \) be a knowledge structure on \( Q \). The graphical representation of the knowledge structure is given in Figure 3.1. This knowledge structure contains 11 states. The domain \( Q \) and empty set \( \emptyset \), the latter symbolizing complete ignorance are among them. The edges of the graph represent the covering
relation of set inclusion: an edge linking a state $K$ and $K'$ located to its right in the graph means that $K \subset K'$ and that there is no state $K''$ such that $K \subset K'' \subset K'$. This graphic representation is often used.

When scanned from left to right this graphic representation suggests a learning process: at first a subject knows nothing about the field and thus in state $\emptyset$, which is represented by the empty box on the left of the figure. The individual then gradually progress from state to state, following one of the paths in Figure 3.1 until a complete mastery of the topic is achieved in state $Q$.

Figure 3.1 illustrate a vertex assignment where $Q$ is the underlying set.

![Figure 3.1](image-url)
Thus, this problem intimately related to the concept of set-valuation of a graph. Moreover, the path chosen by the individual, whichever it may be, to master the field is always a set-indexed path. As the graphical representation of knowledge structure implies the different possibilities of the learning process arrangements, it is particularly important to investigate the possible graphical representations of topologies of different cardinalities on finite sets. This lead us to the concept of graphical realization of topologies.

### 3.2 Graphical realization of finite topologies

Let $X$ be any non-empty set, $\mathcal{S}(X)$ denote the set of all topologies on $X$ and let $\tau \in \mathcal{S}(X)$. We shall say that $\tau$ is ‘graphical’ if there exists a graph $G = (V, E)$ and a set-labeling $f : V(G) \to 2^X$ of $G$ such that $f(V(G)) \cup f^\oplus(E(G)) =: \tau$. Construct a graph $G = (V, E)$ with vertex set $V$ such that $f : V(G) \longrightarrow \tau$ is a bijection and edge set $E = \{\{A, B\} : A, B \in \tau \text{ and } A \cap B = \emptyset\}$. Then, by the definition of a topology on $X$, $uv \in E(G) \iff f(u) \cap f(v) = \emptyset \iff f^\oplus(uv) = f(u) \bigoplus f(v) = (f(u) \cup f(v)) - (f(u) \cap f(v)) = f(u) \cup f(v) \in \tau$ since $f$ is a bijection, and hence $f(V(G)) \cup f^\oplus(E(G)) = \tau$. Thus, we have
Proposition 3.2.1. Every topology on a non-empty set $X$ is graphical.

Definition 3.2.1. Let $X$ be any non-empty set. A topology $\tau$ on $X$ is said to be graphically realizable if there exists a connected graph $G = (V, E)$ and a set-indexer $f : V(G) \rightarrow 2^X$ such that \( f(V(G)) \cup f(\bigoplus E(G)) = \tau \); and $G$ is called the graphical realization of $\tau$.

Example 3.2.2. Consider the topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ on $X = \{1, 2, 3\}$. Figure 3.2 gives a graphical realization of $\tau$.

Theorem 3.2.3. Every topology is graphically realizable.

Proof. Let $\tau$ be a topology of cardinality $n$ on a non-empty set $X$. Then, $\tau = \{\emptyset, A_1, A_2, \ldots, A_{n-1}\}$ where $A_{n-1} = X$ and $A_i \in 2^X$ for
Then, $K_{1,n-1}$ on $n$ vertices, with the central vertex assigned with the label $\emptyset$ and the other vertices are labeled with the non-empty subsets of $X$, namely $A_1, A_2, \ldots, A_{n-1}$ gives a graphical realization of $\tau$.

Figure 3.3 illustrates the Theorem 3.2.3 for the topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$.

Remark 3.2.1. Figure 3.2 and Figure 3.3 give the graphical realizations of the same topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. Thus, the graphical realization of a topology is not unique.
As we have noted, a topology can have more than one graphical realization. Hence, it will be interesting to find out all the possible non-isomorphic graphical realizations of topologies of different cardinality. Since the structure and number of topologies of different cardinality is an open problem we analyze topologies of cardinality up to 8 and establish their non-isomorphic graphical realizations.

In [2], it is shown that the topology of cardinality 4 has 2 structures, the topology of cardinality 5 has 3 structures, the topology of cardinality 6 has 5 structures, the topology of cardinality 7 has 8 structures and the topology of cardinality 8 has 15 structures. Based on these facts we initiate a study on the graphical realizations of different structures of topologies with cardinality $n$, where $2 \leq n \leq 8$.

**Remark 3.2.2.** Let $\tau = \{\emptyset, A_1, A_2, \ldots, A_{n-1}\}$ where $A_{n-1} = X$ be a topology on a non-empty set $X$. Let $G = (V, E)$ be the graphical realization of $\tau$. Let $f(V(G))$ and $f\oplus(E(G))$ denote the sets of vertex assignments and edge assignments respectively. Then,

1. $|V(G)| \leq |\tau|$ and $|E(G)| \leq |\tau| - 1$.
2. $uv \in E(G)$ if and only if $f(u) \oplus f(v) \in \tau$ for every $u, v \in V(G)$.
3. For any $A_i \in \tau$, if $A_i \neq A_i \oplus A_j$ for any non-empty $A_i, A_j$ in $\tau$
Then, \( A_l \in f(V(G)) \).

4. Since \( G \) is connected, \( vv_k \in E(G) \) where the vertex \( v \) is such that 
\( f(v) = \emptyset \) and the vertices \( v_k \)’s are such that \( f(v_k) = A_k \) where 
\( A_k \neq A_i \oplus A_j \) for all non-empty \( A_i, A_j \) in \( \tau \).

5. Let \( S_1, S_2, \ldots, S_r \) be subsets of \( \tau \) such that the symmetric difference 
of any two elements in \( S_i \) is again in \( S_i \). Let \( |S_i| = k_i \) and 
\( \emptyset \notin S_i \). Let \( S_0 = \tau - \bigcup_i S_i - \emptyset \).

   (a) If none of the \( S_i \) has any non-empty elements in common then,
   \[
   \sum_{i=1}^{r} \left\lceil \frac{k_i}{2} \right\rceil + |S_0| + 1 \leq |V(G)|.
   \]

   (b) Let there be at least two pairs \( S_i, S_j \) such that \( S_i \cap S_j \neq \emptyset \) 
   and \( |S_i \cap S_j| = l \) for some \( i, j \). Then, \( \left\lceil \frac{k_i}{2} \right\rceil + \left\lceil \frac{k_j}{2} \right\rceil \) in the above 
   inequality can be replaced by \( \left\lceil \frac{k_i}{2} \right\rceil + \left\lceil \frac{k_j}{2} \right\rceil - l \). This result can be 
   extended to any number of such pairs.

**Definition 3.2.2.** [5] A family \( \mathcal{F} \) of subsets of a set \( X \) is said to have 
the finite intersection property (f.i.p.) if for any \( n \in \mathbb{N} \) and \( F_1, F_2, \ldots, F_n \) 
\( \in \mathcal{F} \), the intersection \( \bigcap_{i=1}^{n} F_i \) is non-empty.

Let \( \tau = \{ \emptyset, A_1, A_2, \ldots, A_{n-1} \} \); \( A_{n-1} = X \), be a topology on \( X \) such 
that the non-empty elements of \( \tau \) has the finite intersection property.
Then, there is at least one non-empty element, $A_i \in \tau$ for some $i$ such that $A_i \subset A_j$ for every $j \in \{1, 2, \ldots, n - 1\}$. Then, $A_i \bigoplus A_m$ does not contain $A_i$ for any $l, m \in \{1, 2, \ldots, n - 1\}$ and hence, does not belong to $\tau$. Then, invoking Remark 3.2.2 (2) and Remark 3.2.2 (4), the graphical realization of $\tau$ is isomorphic to $K_{1,n-1}$. Thus, we have

**Theorem 3.2.4.** The graphical realization of a topology $\tau$ of cardinality $n$ whose non-empty elements has the finite intersection property is isomorphic to $K_{1,n-1}$.

**Definition 3.2.3.** The topology $\tau = \{\emptyset, A_1, A_2, \ldots, A_{n-1}\}; A_{n-1} = X$ on a non-empty set $X$ is called a chain topology if $A_1 \subset A_2 \subset \cdots \subset A_{n-1}$.

The following result is an immediate consequence of Theorem 3.2.4

**Corollary 3.2.5.** The graphical realization of chain topology of cardinality $n$ is isomorphic to $K_{1,n-1}$.

### 3.3 Enumeration of graphical realizations

**Theorem 3.3.1.** The graphical realization of any topology of cardinality 2, is isomorphic to $P_2$. 
Proof. Let $\tau$ be any topology of cardinality 2 with the underlying non-empty set $X$. Then, $\tau$ is necessarily be $\{\emptyset, X\}$. By assigning the elements of $\tau$ to the vertices and joining them will give $P_2$, the unique graphical realization of $\tau$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.4.png}
\caption{Figure 3.4:}
\end{figure}

Corollary 3.3.2. The graphical realization of indiscrete topology is isomorphic to $P_2$.

Theorem 3.3.3. The graphical realization of any topology of cardinality 3 is isomorphic to $K_{1,2}$.

Proof. The topology of cardinality 3 is of the form $\tau = \{\emptyset, A, X\}$ where $A \subset X$ and $X$ is the non-empty ground set. That is, $\tau$ is a chain topology, $\emptyset \subset A \subset X$ of cardinality 3. Hence, invoking Corollary 3.2.5, the graphical realization of $\tau$ is isomorphic to $K_{1,2}$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.4.png}
\caption{Figure 3.4:}
Theorem 3.3.4. The graphical realization of any topology of cardinality 4 is isomorphic to $K_{1,2}$ or $K_{1,3}$ or $K_3$.

Proof. Let $\tau$ be a topology of cardinality 4. Then, $\tau = \{\emptyset, A, B, X\}$ where $X$ is the ground set. The two structures and the corresponding graphical realization are given below.

Case 1: $\tau$ is such that $\emptyset \subset A \subset B \subset X$, the chain topology. Then, invoking Corollary 3.2.5, the graphical realization is isomorphic to $K_{1,3}$.

Case 2: Here, $\tau$ is such that $A \cap B = \emptyset$, $A \cup B = X$. By Remark 3.2.2 (3), empty set, $\emptyset$, must be a vertex assignment. Since $A \bigoplus B = X$, at least two of the sets $A, B, X$ must be vertex assignments. Hence, the graphical realization can have at least three vertices and at most four vertices; at least 2 edges and at most 3 edges. Hence, the graphical realization is isomorphic to any one of the graphs given in Figure 3.4.

Let $\tau = \{\emptyset, A, B, C, X\}$ be a topology of cardinality 5. Then, the elements of $\tau$ are such that:

Case 1: $\emptyset \subset A \subset B \subset C \subset X$

Case 2: $A \cap B = \emptyset$, $A \cup B = C$, $C \subset X$

Case 3: $\emptyset \subset A$, $A \subset B$, $A \subset C$, $B \cup C = X$
Case 1: Invoking Corollary 3.2.5, the graphical realization is isomorphic to $K_{1,4}$.

Case 2: By Remark 3.2.2 (3), $\emptyset, X$ must be vertex assignments. Since $A \oplus B = C$ at least two of $A, B, C$ must be vertex assignments. Hence, the graphical realization can have at least four vertices and at most five vertices. Moreover, the graphical realization can have at least three edges and at most four edges. The non-isomorphic graphical realizations are given in Figure 3.5.

Case 3: In this case, by Theorem 3.2.4, the graphical realization is isomorphic to $K_{1,4}$. 
Thus we have

**Theorem 3.3.5.** The graphical realization of any topology of cardinality 5 is isomorphic to any one of the graphs given in Figure 3.5.

Now, let \( \tau = \{\emptyset, A, B, C, D, X\} \) be the topology of cardinality 6. Then, the elements of \( \tau \) are such that:

**Case 1:** \( \emptyset \subset A \subset B \subset C \subset D \subset X \)

**Case 2:** \( \emptyset \subset A, A \subset B, A \subset C, B \cup C = D, D \subset X \)

**Case 3:** \( \emptyset \subset A \subset B, B \subset C, B \subset D, C \cup D = X \)

**Case 4:** \( A \cap B = \emptyset, A \cup B = C, C \subset D \subset X \)
Case 5: \( A \cap B = \emptyset, A \cup B = C, B \subset D, C \cup D = X \)

Case 1: Invoking Corollary 3.2.5, the graphical realization in this case is isomorphic to \( K_{1,5} \).

Case 2: Note that \( A \) is contained in every non-empty elements of \( \tau \). Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,5} \).

Case 3: In this case also, \( A \) is contained in every non-empty elements of \( \tau \). Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,5} \).

Case 4: By Remark 3.2.2 (3), \( \emptyset, D, X \) must be vertex assignments and since \( A \oplus B = C \) at least two of \( A, B, C \) must also be vertex assignments. Hence, the graphical realization can have at least five vertices and at most six vertices; at least four edges and at most five edges. The possible non-isomorphic realizations are given in Figure 3.6.

Case 5: In this case, \( A \oplus B = C \) and \( A \oplus D = X \). Thus, by assigning at least three of \( A, B, C, D, X \) and \( \emptyset \) to the vertices we can construct the graphical realizations. Hence, the graphical realization can have at least four vertices and at most six vertices; at least three edges and at most five edges. The possible non-isomorphic graphical realizations are given in Figure 3.7.
Then, we have

**Theorem 3.3.6.** The graphical realization of any topology of cardinality 6 is isomorphic to any one of the graphs given in Figure 3.6 and Figure 3.7.

Consider a topology of cardinality 7. Then, \( \tau = \{\emptyset, A, B, C, D, E, X\} \) where \( A, B, C, D, E \) are subsets of the ground set \( X \). Then, the elements of \( \tau \) are such that:

**Case 1:** \( \emptyset \subset A \subset B \subset C \subset D \subset E \subset X \)
Case 2: $\emptyset \subset A, A \subset B, A \subset C, B \cup C = D, D \subset E \subset X$

Case 3: $\emptyset \subset A \subset B, B \subset C, B \subset D, C \cup D = E, E \subset X$

Case 4: $\emptyset \subset A \subset B \subset C, C \subset D, C \subset E, D \cup E = X$

Case 5: $\emptyset \subset A, A \subset B, A \subset C, B \cup C = D, B \subset E, D \cup E = X$

Case 6: $A \cap B = \emptyset, A \cup B = C, C \subset D \subset E \subset X$

Case 7: $A \cap B = \emptyset, A \cup B = C, A \subset D, C \cup D = E, E \subset X$

Case 8: $A \cap B = \emptyset, A \cup B = C, C \subset D, C \subset E, D \cup E = X$
Case 1: Invoking Corollary 3.2.5, the graphical realization in this case is isomorphic to $K_{1,6}$.

Case 2: In this case, $A$ is contained in every non-empty elements of $\tau$. Hence, by Theorem 3.2.4, the graphical realization is isomorphic to $K_{1,6}$.

Case 3: In this case, $A$ is contained in every non-empty elements of $\tau$. Hence, by Theorem 3.2.4, the graphical realization is isomorphic to $K_{1,6}$.

Case 4: In this case, $A$ is contained in every non-empty elements of $\tau$. Hence, by Theorem 3.2.4, the graphical realization is isomorphic to $K_{1,6}$.

Case 5: In this case, $A$ is contained in every non-empty elements of $\tau$. Hence, by Theorem 3.2.4, the graphical realization is isomorphic to $K_{1,6}$.

Case 6: In this case, by Remark 3.2.2 (3), $\emptyset, D, E, X$ must be vertex assignments. Since $A \bigoplus B = C$ at least two of the sets $A, B, C$ must be vertex assignments. Hence, the graphical realization can have at least six vertices and at most seven vertices; at least five edges and at most six edges. The possible non-isomorphic graphical realizations are given in Figure 3.8.
Case 7: By Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), $\emptyset, X$ and any three of the sets $A, B, C, D, E$ must be assigned to the vertices. Then, the graphical realization can have at least five vertices and at most seven vertices; at least four edges and at most six edges. The possible non-isomorphic graphical realizations are given in Figure 3.9 and Figure 3.10.
Case 8: Here, $A \oplus B = C$. Then, by Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), $\emptyset, D, E, X$ and at least two of the sets $A, B, C$ must be vertex assignments. Hence, the graphical realization can have at least six vertices and at most seven vertices; at most five edges and at most six edges. The possible non-isomorphic graphical realizations in this case are same as that of Case 6, which are depicted in Figure 3.8.
**Theorem 3.3.7.** The graphical realization of any topology of cardinality 7 is isomorphic to any one of the graphs given in Figure 3.8, Figure 3.9 and Figure 3.10.

Now, consider \( \tau = \{\emptyset, A, B, C, D, E, F, X\} \), a topology of cardinality 8.

Then, the elements of \( \tau \) are such that:

**Case 1:** \( \emptyset \subset A \subset B \subset C \subset D \subset E \subset F \subset X \)

**Case 2:** \( \emptyset \subset A, A \subset B, A \subset C, B \cup C = D, D \subset E \subset F \subset X \)

**Case 3:** \( A \cap B = \emptyset, A \cup B = C, C \subset D \subset E \subset F \subset X \)

**Case 4:** \( \emptyset \subset A \subset B, B \subset C, B \subset D, C \cup D = E, E \subset F \subset X \)
Case 5: $\emptyset \subset A \subset B \subset C, C \subset D, C \subset E, D \cup E = F, F \subset X$

Case 6: $\emptyset \subset A \subset B \subset C \subset D, D \subset E, D \subset F, E \cup F = X$

Case 7: $A \cap B = \emptyset, A \cup B = C, A \subset D, D \cup C = E, E \subset F \subset X$

Case 8: $\emptyset \subset A, A \subset B, A \subset C, B \cup C = D, B \subset E, E \cup D = F, F \subset X$

Case 9: $\emptyset \subset A \subset B, B \subset C, B \subset D, C \cup D = E, C \subset F, E \cup F = X$

Case 10: $A \cap B = \emptyset, A \cup B = C, C \subset D, C \subset E, D \cup E = F, F \subset X$

Case 11: $\emptyset \subset A, A \subset B, A \subset C, B \cup C = D, D \subset E, D \subset F,$

$E \cup F = X$

Case 12: $A \cap B = \emptyset, A \cup B = C, C \subset D, D \subset E, D \subset F, E \cup F = X$
Case 13: \( A \cap B = \emptyset, A \cup B = C, A \subset D, C \cup D = E, C \subset F, E \cup F = X \)

Case 14: \( A \cap B = \emptyset, A \cup B = C, B \subset D, C \cup D = E, D \subset F, E \cup F = X \)

Case 15: \( A \cap B = \emptyset, A \cap C = \emptyset, B \cap C = \emptyset, A \cup B = D, B \cup C = E, A \cup C = F, D \cup E = F \cup E = D \cup F = X \)

Case 1: Invoking Corollary 3.2.5, the graphical realization in this case is isomorphic to \( K_{1,7} \).

Case 2: In this case, \( A \) is contained in every non-empty elements of \( \tau \). Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,7} \).

Case 3: Here, \( A \bigoplus B = C \). Then, by Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), \( \emptyset, D, E, F, X \) and at least two of the sets \( A, B, C \) must be vertex assignments. Hence, the graphical realization can have at least seven vertices and at most eight vertices; at least six edges and at most seven edges. The possible non-isomorphic graphical realizations are given in Figure 3.11.

Case 4: In this case, \( A \) is contained in every non-empty elements of \( \tau \). Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,7} \).

Case 5: In this case, \( A \) is contained in every non-empty elements of
Figure 3.13:

\[ \tau \]. Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,7} \).

**Case 6:** In this case, \( A \) is contained in every non-empty elements of \( \tau \). Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,7} \).

**Case 7:** In this case, \( A \oplus B = C, \; B \oplus D = E \). Then, by Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), \( \emptyset, F, X \) and any two of the sets \( A, B, C \) and any one of the sets \( D, E \) must be vertex assignments. Hence, the graphical realization can have at least six vertices and at most eight vertices; at least five edges and at most seven edges. The possible non-isomorphic graphical realizations in this case are given in
Figure 3.14:

Case 8: In this case, \( A \) is contained in every non-empty elements of \( \tau \). Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,7} \).

Case 9: In this case, \( A \) is contained in every non-empty elements of \( \tau \). Hence, by Theorem 3.2.4, the graphical realization is isomorphic to \( K_{1,7} \).

Case 10: In this case, \( A \oplus B = C \). Then, by Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), \( \emptyset, D, E, F, X \) and any two of the sets \( A, B, C \) must be vertex assignments. Hence, the graphical realization can have at least seven vertices and at most eight vertices; at least six edges and at
most seven edges. The possible non-isomorphic graphical realizations in this case are same as that of Case 3, which are depicted in Figure 3.11.

**Case 11:** In this case, $A$ is contained in every non-empty elements of $\tau$. Hence, by Theorem 3.2.4, the graphical realization is isomorphic to $K_{1,7}$.

**Case 12:** In this case, $A \bigoplus B = C$. Then, by Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), $\emptyset, D, E, F, X$ and any two of the sets $A, B, C$ must be vertex assignments. Hence, the graphical realization can have at least seven vertices and at most eight vertices; at least six edges and at most seven edges. The possible non-isomorphic graphical realizations in this case are same as that of Case 3 which are depicted in Figure 3.11.
Case 13: In this case $A \bigoplus B = C$. Then, by Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), $\emptyset, D, E, F, X$ and any two of the sets $A, B, C$ must be vertex assignments. Hence, the graphical realization can have at least seven vertices and at most eight vertices; at least six edges and at most seven edges. The possible non-isomorphic graphical realizations in this case are same as that of Case 3, which are depicted in Figure 3.11.

Case 14: In this case, $A \bigoplus B = C$. Then, by Remark 3.2.2 (3) and Remark 3.2.2 (5(a)), $\emptyset, D, E, F, X$ and any two of the sets $A, B, C$ must be vertex assignments. Hence, the graphical realization can have at least seven vertices and at most eight vertices; at least six edges and at most seven edges. The possible non-isomorphic graphical realizations
in this case are same as that of Case 3 which are depicted in Figure 3.11.

**Case 15:** In this case any non-empty elements can be expressed as the symmetric difference of any two non-empty elements of \( \tau \). Hence, \( \emptyset \) and any four of the sets \( A, B, C, D, E, F, X \) must be the vertex assignments. Hence, the graphical realization can have at least five vertices and at most eight vertices; at least four edges and at most seven edges. The possible non-isomorphic graphical realizations in this case are given in Figure 3.14, Figure 3.15, Figure 3.16, Figure 3.17 and Figure 3.18.

**Theorem 3.3.8.** The graphical realization of any topology of cardinality 8 is isomorphic to any one of the graphs given in Figure 3.11.
Figure 3.18:

**Problem 1:** Characterize the graphical realization of the discrete topology.

**Problem 2:** Graphical realization of different structures of topology with cardinality \( n \geq 9 \) is a long term problem to investigate.

As a consequence of Theorem 3.3.6, Theorem 3.3.7 and Theorem 3.3.8 we have

**Corollary 3.3.9.** \( C_5 \), the cycle on 5 vertices is not topogenic.
Proof. If possible, suppose that $C_5$ is topogenic. Then, by Theorem 2.2.11 the topogenic strength of $C_5$ is such that $6 \leq |\tau| \leq 8$. In other words, if $C_5$ is topogenic and $\tau$ is the corresponding topology then, either $|\tau| = 6$ or $|\tau| = 7$ or $|\tau| = 8$. Invoking Theorem 3.3.6, Theorem 3.3.7 and Theorem 3.3.8 none of the graphical realizations are isomorphic to $C_5$. Hence, $C_5$ is not topogenic.

In Theorem 2.2.14 we have proved that $K_4$, is not topogenic. A smaller proof to prove that $K_4$ is not topogenic can be reduced from Theorem 3.3.7.

**Corollary 3.3.10.** $K_4$ the complete graph on 4 vertices is not topogenic.

Proof. Let $K_4$ be topogenic. Then, by Theorem 2.2.13 the topogenic strength of $K_4$ is 7. But by Theorem 3.3.7 none of the graphical realizations are isomorphic to $K_4$.

The following problem is worth for further investigation.

**Problem 3:** Given a topology $\tau$, determine the structure of the topological space $(X, \tau)$ whose graphical realization is isomorphic to (i) complete graph and (ii) trees.
3.4 References


