Chapter 1
Preliminaries

1.1 Introduction

Graph Theory began in 1736 when Leonard Euler (1707 – 1783) solved the well known Königsberg bridge problem. This problem asked for a circular walk through the town of Königsberg is such a way as to cross over each of the seven bridges spanning the river Pregel once, and only once. Euler realized that the precise shape of the island and the other three territories involved are not important; the solvability depends only on their connection properties. He represented the four territories by points and the bridges by curves joining the respective points as shown in Figure 1.1. He proved that it is impossible to solve the problem. Moreover, he gave a necessary and sufficient condition for an arbitrary graph to admit a circular tour of this kind.

There are several reasons for the acceleration of interest in graph theory. Graph Theory has a lot of applications in various subjects like physical sciences [19], chemical sciences [11], [24], [25], social sciences [20],
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Figure 1.1: The graph of Königsgberg bridge problem

[21], biological sciences [17], computer science [22] and engineering [10]. The theory is also intimately related to many branches of mathematics, including group theory [9], matrix theory [6], topology [16] and combinatorics [7]. In fact, graph theory serves as a mathematical model for any system involving a binary relation and its abstract concepts can be pictured by concrete spatial relations.

1.2 Graphs

A graph $G$ is a pair $G = (V, E)$, consisting of a non-empty set $V$ and a set $E$ of two element subsets of $V$. The elements of $V$ are called vertices. An element $e = \{a, b\}$ of $E$ is called an edge with end vertices $a$ and $b$. We say that $a$ and $b$ are incident with $e$ and that $a$ and $b$ are adjacent of each other. When $a = b$, $e$ is a loop. If two edges $e_1, e_2$ have the same end vertices then $e_1, e_2$ are called parallel edges (multiple edges). A graph in which, both loops and multiple edges are allowed
is called a *pseudograph* and a graph in which no loop is allowed is a *multigraph*. A graph with no loops and multiple edges is called a *simple graph*. The *order* of $G$ is the number of vertices in $G$ and the *size* of $G$ is the number of edges in $G$. Thus, if $G$ has $p$ vertices and $q$ edges, then we say that $G$ is a $(p, q)$-graph.

A *subgraph* of $G$, is a graph having all of its vertices and edges in $G$. A *spanning subgraph* of $G$ is a subgraph containing all the vertices of $G$. For any set $S \subset V$, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of $G$ with vertex set in $S$. Thus, two vertices of $S$ are adjacent in $\langle S \rangle$ if and only if they are adjacent in $G$.

A *walk* of a graph $G$ is an alternating sequence of vertices and edges $v_0x_1v_1x_2\ldots v_{n-1}x_nv_n$ beginning and ending with vertices, in which, each edge $x_i$ is incident with the two vertices immediately preceding and following the edge $x_i$. This walk joins $v_0$ and $v_n$ and may also be denoted $v_0v_1\ldots v_n$ and is called a $v_0$-$v_n$ walk. It is *closed* if $v_0 = v_n$ and is *open* otherwise. The walk is called a *trail* if all the edges are distinct and called a *path* if all the vertices are distinct. A closed path is called a *cycle*. The *length of a walk* is the number of occurrences of edges in it.

In the graph $G$ of Figure 1.2 $v_1v_2v_5v_2v_3$ is a walk, which is not a trail and $v_1v_2v_5v_4v_2v_3$ is a trail, which is not a path and $v_1v_2v_5v_4$ is a path and $v_2v_4v_5v_2$ is a cycle. A path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. 
A graph $G$ is *connected* if every pair of vertices are joined by a path. A maximal connected subgraph of $G$ is called a *connected component* or simply a *component* of $G$. The *girth* of a graph $G$, denoted $g(G)$, is the length of the shortest cycle in $G$; the *circumference* $c(g)$ is the length of the largest cycle.

The *distance* $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of the shortest path joining them; otherwise $d(u, v) = \infty$. A shortest $u$-$v$ path is called a *geodesic*. The *diameter* $d(G)$ of a connected graph $G$ is the length of the longest geodesic. The graph of Figure 1.2 has girth $g(G) = 3$, circumference $c(G) = 4$ and diameter $d(G) = 2$. The *degree* of a vertex $v$ in $G$ denoted by $deg v$ is the number of edges incident with $v$. $\min_v\{deg v : v \in V(G)\}$ is denoted by $\delta(G)$, while $\max_v\{deg v : v \in V(G)\}$ is denoted by $\Delta(G)$. If $\delta(G) = \Delta(G) = r$, then $G$ is called an *$r$-regular* graph. A vertex $v$ of a graph $G$ is called *even*, if its degree is even and *odd*, if its degree is odd. Also, if $deg v = 0$, $v$
is called an isolated vertex, and if $\text{deg } v = 1$, it is called an end vertex. Also, if $e = uv$ is an edge of a graph $G$ such that either $\text{deg } u = 1$ or $\text{deg } v = 1$, then $e$ is called a pendant edge of $G$.

The complement $\overline{G}$ of a graph has $V(G)$ as the vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. A complete graph $K_n$ has every pair of its $n$ vertices adjacent. The graphs $\overline{K_n}$ are totally disconnected.

A bipartite graph $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ has one end in $V_1$ and the other end in $V_2$. If $G$ contains every edge joining $V_1$ and $V_2$, then $G$ is a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then $K_{m,n}$ denotes a complete bipartite graph. A star is the complete bipartite graph $K_{1,n}$. A $k$-partite graph is one in which there is a partitioning of the vertices into $k$ subsets so that all edges have one end in $V_i$ and other end in $V_j$, where $i \neq j$. The notion of a complete bipartite graph can also be extended to a complete $k$–partite graph. A graph is a complete $k$–partite graph if all the pairs of vertices belonging to different subsets are adjacent.

Two graphs $G$ and $H$ are said to be equal if $V(G) = V(H)$ and $E(G) = E(H)$. Two graphs $G$ and $H$ are said to be isomorphic (written as $G \cong H$) if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. If two graphs $G$ and $H$ are not isomorphic, then they are called non-isomorphic graphs and is written as $G \not\cong H$. 
1.3 Operations on graphs

Let \( G_1 = (V_1, E_1) \), \( G_2 = (V_2, E_2) \) be two graphs such that \( V_1 \cap V_2 = \emptyset \) and \( E_1 \cap E_2 = \emptyset \).

The **union** \( G = G_1 \cup G_2 \) has \( V = V_1 \cup V_2 \) and \( E = E_1 \cup E_2 \).

The **join** of \( G_1 \) and \( G_2 \), denoted as \( G_1 + G_2 \) is such that \( V(G_1 + G_2) = V_1 \cup V_2 \) and the edge set consists of \( E(G_1) \cup E(G_2) \) and all edges joining the vertices in \( V_1 \) with the vertices in \( V_2 \).

The **product** of two graphs \( G_1 \) and \( G_2 \) is defined as the graph, with vertex set \( V(G_1 \times G_2) = V_1 \times V_2 \) and two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are adjacent in \( G_1 \times G_2 \) if and only if \( u_1 = v_1 \) and \( u_2 \) adjacent to \( v_2 \) or \( u_1 \) adjacent to \( v_1 \) and \( u_2 = v_2 \).

The **composition** \( G = G_1[G_2] \) has \( V(G_1[G_2]) = V_1 \times V_2 \) as its vertex set and \( u = (u_1, u_2) \) is adjacent to \( v = (v_1, v_2) \) whenever \( u_1 \) is adjacent to \( v_1 \) or \( u_1 = v_1 \) and \( u_2 \) adjacent to \( v_2 \).
If $G$ and $H$ are any two graphs then, $G\bullet H$ is obtained by the identification of any vertex of $G$ with an arbitrary vertex of $H$, which results in a unique graph (up to isomorphism).

### 1.4 Directed graphs

A directed graph (or just digraph) $D$ consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs. Then $V(D)$ is called the vertex set and $A(D)$ the arc set and the corresponding digraph $D$ is denoted by $D = (V, A)$. For an arc $(u, v)$, the first vertex $u$ is its tail and the second vertex $v$ is its head. The head and tail of an arc are its end points. For an arc $(u, v)$, $u$ is said to be adjacent to $v$ and $v$ is said to be adjacent from $u$. The vertices $u$ and $v$ are said to be incident with the arc $(u, v)$. The number of vertices to which a vertex $v$ is adjacent is
the outdegree of $v$ and is denoted by $od\ v$. The number of vertices from which a vertex $v$ is adjacent is the indegree of $v$ and is denoted by $id\ v$.

In a digraph $D$, the sequence $W : u = u_0u_1 \ldots u_k = v$ of vertices of $D$ such that $u_i$ is adjacent to $u_{i+1}$ for all $i$, $(1 \leq i \leq k - 1)$ is called a directed $u$-$v$ walk in $D$. The number of occurrences of arcs on a walk is the length of the walk. A walk in which no arc is repeated is a directed trail and a walk in which no vertex is repeated is a directed path. A closed directed path is a directed cycle.

The underlying graph of a digraph $D$ is the graph $G$ obtained by removing all directions from the arcs of $D$ and replacing any resulting pair of parallel edges by a single edge. A digraph $D$ is connected if the underlying graph of $D$ is connected. A digraph $D$ is strong (or strongly connected) if $D$ contains both a $u$-$v$ path and a $v$-$u$ path for every pair $u, v$ of distinct vertices of $D$.

### 1.5 Graph labeling

In the mathematical discipline of graph theory, a graph labeling is the assignment of labels, traditionally represented by integers, to the edges, or vertices, or both, of a graph [26]. Formally, given a graph $G$, a vertex labeling is a function mapping vertices of $G$ to a set of labels. A graph with such a function defined is called a vertex-labeled graph. Likewise, an edge labeling is a function mapping edges of $G$ to a set of ‘labels’.
In this case, G is called an *edge-labeled graph*. When the edge labels are members of an ordered set (such as the real numbers), it may be called a *weighted graph*. When used without qualification, the term labeled graph generally refers to a vertex-labeled graph with all labels distinct. Such a graph may equivalently be labeled by the consecutive integers $1, 2, \ldots, n$, where $n$ is the number of vertices in the graph [26]. For many applications, the edges or vertices are given labels, that are meaningful in the associated domain. For example, the edges may be assigned weights representing the ‘cost’ of traversing between the incident vertices [8]. In the above definition a graph is understood to be a finite undirected simple graph. However, the notion of labeling may be applied to all extensions and generalizations of graphs. For example, in automata theory and formal language theory it is convenient to consider labeled pseudographs [12].

Labeled graphs are becoming an increasingly useful family of mathematical models for a wide range of applications such as coding, X-ray, crystallography, radar tracking, remote control, radio-astronomy, communication in networks, network flows, etc.

### 1.5.1 Number labeling

Most graph labelings trace their origin to the labeling defined by Alexander Rosa [23]. In his 1967 paper, Rosa [23] identified three types of labelings, which he called $\alpha$-, $\beta$-, and $\rho$-labelings. $\beta$-labelings were later renamed *graceful* by Golomb [14] and the name has been popular since.
Many other deviations of labeled graphs such as, harmonious graphs, elegant graphs, additive graphs, felicitous graph, etc. have been studied in literature [see Gallian [13]].

1.5.2 Set labeling

Acharya [1, 2] introduced the set-labeling of graphs as an analogue of the number labeling of graphs. Acharya [1, 2] defined a set-indexer of a graph $G = (V, E)$ to be an injective set-assignment $f : V(G) \rightarrow 2^X$ such that the induced set-assignment $f^\oplus : E(G) \rightarrow 2^X$ on the edges of $G$ is also injective, when $f^\oplus(uv) = f(u) \bigoplus f(v)$ for every $uv \in E(G)$. Acharya [1, 2] called a graph $G = (V, E)$ to be set-graceful if there exists a non-empty set $X$ and a set-indexer $f : V(G) \rightarrow 2^X$ such that $f^\oplus(E(G)) = 2^X - \emptyset$, such an indexer being called a set-graceful labeling of $G$. He [1, 3] also defined a graph $G = (V, E)$ to be set-sequential if it admits a set-sequential labeling which is a bijection $f : V(G) \cup E(G) \rightarrow 2^X - \emptyset$ such that $f(uv) = f(u) \bigoplus f(v)$ for every $uv \in E(G)$. A set-indexer $f$ is said to be a topological set-indexer (or a $T$-set-indexer) of a graph $G$ if $f(V(G))$ is a topology on $X$. The topological number (or, $T$-number) of $G$ denoted by $t(G)$ is the smallest cardinality of a set $X$ with respect to which $G$ has a T-set-indexer. A graph $G$ is said to be topologically set-graceful if it admits a topologically set-graceful set-indexer, which is a set-indexer $f$ such that $f^\oplus(E(G)) = 2^X - \emptyset$ and $f(V(G))$ is a topology on $X$. In the thesis, we analyze yet another variation of set-valuation.
1.6 Topological space

A topological space is a pair \((X, \tau)\) where \(X\) is a set and \(\tau\) is a family of subsets of \(X\) satisfying:

(i) \(\emptyset \in \tau\) and \(X \in \tau\),
(ii) \(\tau\) is closed under arbitrary unions and
(iii) \(\tau\) is closed under finite intersections.

The family \(\tau\) is said to be a topology on the set \(X\). Members of \(\tau\) are said to be open in \(X\) or open subsets of \(X\).

It may happen that the topology \(\tau\) on the set \(X\) consists only of empty set, \(\emptyset\) and \(X\). Then, \(\tau\) is called the indiscrete topology on \(X\). The other extreme is the discrete topology on \(X\), in which every set is open; in other words the topology coincides with the power set \(\mathcal{P}(X)\).

1.7 Outline of the thesis

Chapter 1: Chapter 1 is the introductory chapter of the thesis.

Chapter 2: This chapter contains some foundational results on topogenic set-indexers of graphs, which is defined as follows: A graph \(G\), not necessarily finite, is called topogenic if there exists a non-empty ground set \(X\) and an injective ‘set-assignment’ \(f : V(G) \rightarrow 2^X\), such that (i) the induced edge function \(f:\bigoplus : E(G) \rightarrow 2^X - \emptyset\) defined by \(f:\bigoplus(uv) = f(u) \bigoplus f(v)\), \(uv \in E(G)\), where ‘\(\bigoplus\)’ denotes the binary operation of taking symmetric difference of the subsets of \(X\), is also injective, and (ii) \(f(V(G)) \cup f:\bigoplus(E(G)) = \tau_f\) is a topology on \(X\); such
a set-assignment $f$, if it exists, is called a *topogenic set-indexer* of $G$. In particular, if $f(V(G)) \cup f(\oplus(E(G)) = 2^X$, then $f$ is called a graceful topogenic set-indexer. We also establish a necessary condition for a graph to be topogenic and analyze the relationship between topogenic graphs and existing various categories of set-valued graphs. We also establish the existence of non-topogenic graphs. In this chapter, we also identify certain classes of graphs that admit topogenic set-indexers and discuss about graceful topogenic set-indexer of graphs. We establish a necessary condition for a graph to admit graceful topogenic set-indexer with respect to a non-empty set $X$ and identify certain classes of graphs that always admit graceful topogenic set-indexer.

**Chapter 3**: In this chapter, we discuss about the graphical realization of topologies of different cardinalities on finite sets. A topology $\tau$, on a ground set $X$ is said to be *graphically realizable* if there exists a connected graph $G = (V, E)$ and a set-indexer $f : V(G) \rightarrow 2^X$ such that $f(V(G)) \cup f(\oplus(E(G)) = \tau$. Then, $G$ is called the *graphical realization* of $\tau$. We analyzes the non-isomorphic graphical realizations of all topologies of order up to 8 and find the graphical realizations of certain topologies like chain topology and topologies admitting finite intersection property.

**Chapter 4**: In this chapter, we define yet another variation of set-valued graphs namely sequential topogenic graphs which is defined as follows: A set-indexer $f$ of $G = (V, E)$ is called a *segregation* of $X$ on
G if the sets \( f(V(G)) = \{f(u) : u \in V(G)\} \) and \( f\oplus(E(G)) = \{f\oplus(e) : e \in E(G)\} \) are disjoint. A segregation \( f \) is called a sequential topogenic set-indexer, if \( f(V(G)) \cup f\oplus(E(G)) = \tau - \emptyset \) for some topology \( \tau \) on \( X \). A graph is called sequential topogenic if it admits a sequential topogenic labeling with respect to some non-empty set \( X \). We analyze the properties of sequential topogenic graphs and establish the existence of sequential topogenic graphs and existence of non-sequential topogenic graphs.

**Chapter 5**: In this chapter, we show that every graph \( G \) has a topogenic host graph \( H \), in the sense that \( G \) is contained as an induced subgraph in \( H \) which is a topogenic graph, implying thereby that there is no ‘forbidden subgraph characterization’ of topogenic graphs. We also analyze the relation between certain graph parameters like independence number, clique number, domination number, chromatic number, vertex covering number, edge covering number and cycle rank of \( G \) with the host graph \( H \). We also show that every graph has a graceful topogenic host graph and sequential topogenic host graph and hence implies that none of these variations of set-valuation has a ‘forbidden subgraph characterization’.

**Chapter 6**: In this chapter, we discuss about \( k \)-transitive digraphs which is defined as follows: A digraph \( D \), is said to be \( k \)-transitive if for the dipaths \( P_i : a_1a_2a_3 \ldots a_{k-1}a_k \) there exist an arc \((a_1, a_i) \in A(D)\), for \( 3 \leq i \leq k \). A \( k \)-transitive digraph \( D \) is said to be totally \( k \)-transitive
if there exists an arc from the origin to the terminus of all the longest paths. A totally \( k \)-transitive digraph \( D \) is said to be totally maximal path \( k \)-transitive (tmp-\( k \)-transitive) if \((a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k) \in A(D) \implies (a_1, a_i) \in A(D), 3 \leq i \leq k\), for every dipath \( P_i, 3 \leq i \leq k \).

In this chapter, we initiate a study on \( k \)-transitive digraph, totally \( k \)-transitive digraph and totally maximal path \( k \)-transitive digraph. We also study the tmp-directed transitive lattice digraph which is the directed graph of the lattice of open sets determined by the topology of \( D \). We also establish a one-to-one correspondence between the set of all tmp-\( k \)-transitive digraphs with exactly one source and the set of all topologies having the finite intersection property with \(| \bigcap_i T_i | = 1\).

**Chapter 7**: In this chapter, we discuss about the set-indexers of a directed graph. Given a simple digraph \( D = (V, A) \) and a set-valuation \( f : V(G) \to 2^X \), to each arc \((u, v)\) in \( D \) we assign the set \( f(u) - f(v) \). A set-valuation \( f \) of a given digraph \( D = (V, A) \) is a set-indexer of \( D \) if both \( f \) and its ‘arc-induced function’ \( g_f \), defined by letting \( g_f(u, v) = f(u) - f(v) \), for each arc \((u, v)\) of \( D \), are injective. Further, \( f \) is arc-bounded if \(|g_f(u, v)| < |f(v)|\), for each \((u, v) \in A\).

**Chapter 8**: This chapter is on new directions for research where, we enumerate some conjectures and open problems which are worth of further investigation.

For the notions in graph and digraph theory, we would respectively
follow [18] and [21].

1.8 References


10. Deo, Narsingh, **Graph theory with Applications to Engineering and Computer Science**, PHI Pvt. Ltd., India, 1974.


