Introduction

The subject Graph Theory is considered to have its origin in the famous Königsberg bridge problem settled by Euler in 1741. Later such problems were discussed by Poincaré(1810), Listing(1847), Hamilton and the like. Dirac(1952) and Ore (1960) worked on Hamiltonian Graphs. Cayley, von Standt, Kirchhoff, Jordan, Sylvester, Clifford etc. are some other names worth mentioning in the field of Graph Theory. The concepts of Graph Theory are discussed in detail in the works of Harary[31], Clarke and Holton[18], Deo[21], Parthasarathy[50], Bondy and Murty[10] etc.

The first theorem of Graph Theory, i.e., the sum of the degrees of the points of a graph is twice the number of lines, is due to Euler [25]. The operations of composition and cartesian product of graphs were studied by Sabidussi[58,59,60]. The join of two graphs was defined by Zykov[79]. Connectivity of graphs was studied by Whitney[74], Chartrand and Harary[16], Berge[4], Dirac[22], Tutte[71], Menger[41] and the like. Studies on matrices related with a graph include the works of Kirchhoff[35]. Coloring of graphs has been a problem discussed by graph theorists across the world. Some of the important works in this area are by König[37], Heawood [32], Szekeres and Wilf [67], Berge[5], Ore [49], Whitney [73], Hadwiger [30], Birkhoff and Lewis[8] etc. Enumeration theory was pioneered by Cayley, Redfield and Pólya. Almost all graphical enumeration methods in use were anticipated in the paper [55] of Redfield.
One of the important works of early periods is that of Cayley[14]. Pólya’s enumeration theorem[52], Burnside’s lemma[13] are two of the important works in enumeration. Easiest of enumeration problems is that of labeled graphs. The problem of labeling of graphs is one which has attracted a number of researchers. A bibliography of almost all works on labeling is enlisted by Gallian in [28].

The notion of fuzzy sets was introduced by L.A. Zadeh in 1965[77]. It involves the concept of a membership function defined on a universal set. The value of the membership function lies in [0,1]. This concept has found applications in computer science, artificial intelligence, decision analysis, information science, pattern recognition, operations research and robotics. The ideas of fuzzy set theory have been introduced in topology, abstract algebra, geometry, graph theory and analysis. The concept of fuzzy sets, fuzzy logic and its applications are discussed in detail by Klir and Yuan [36], Zimmermann [78] and the like.

Using the concept of fuzzy subsets, the concept of fuzzy graph was introduced by A. Rosenfeld in 1975(cf. [57]). The most general fuzzy graph $G = (V, \mu, \rho)$ is a nonempty set $V$ together with a pair of functions $\mu : V \to [0, 1]$ and $\rho : V \times V \to [0, 1]$ such that for all $x, y$ in $V$, $\rho(x, y) \leq \mu(x) \land \mu(y)$. Other variations are there like those with vertex set crisp and edge set fuzzy. $\mu$ is the fuzzy vertex set and $\rho$ is the fuzzy edge set of $G$. Various concepts on fuzzy graphs are discussed by Mordeson[42], Mordeson and Nair[45], Sunitha and Vijayakumar[64], Bhutani[9], Mordeson and Peng[43], Bhattacharya[6] and the like. Other important works include those of Takeda[68], Yeh and Bang[76], Wu and Chen[75], Cerruti[15], Delgado, Verdegay and Vila[20], McAllister[40], Chen[17], Bhattacharya and Suraweera[7], Luo[38], Arya and Hazarika[2], Craine[19], Mordeson and Nair[44], Tong and Zheng[69], Sunitha and
Vijayakumar[65] etc.

A new concept, namely, graph structure was introduced in [61] by E. Sampathkumar which, in particular, is a generalization of the notions like graphs, signed graphs and edge-coloured graphs with the colourings. According to him, \( G = (V, R_1, R_2, ..., R_k) \) is a graph structure if \( V \) is a nonempty set and \( R_1, R_2, ..., R_k \) are relations on \( V \) which are mutually disjoint such that each \( R_i, i = 1, 2, 3, ..., k, \) is symmetric and irreflexive. The study of graph structures is relevant because it helps to study the various relations and the corresponding edges simultaneously. This is the motivation for our study of fuzzy graph structures.

**Summary of the research work**

**Chapter 1:** Some essential preliminaries in Graph Theory, Fuzzy Graph Theory and Algebra are recalled in chapter 1. Basic concepts of graph structures are also recalled there.

**Chapter 2:** New concepts like \( \rho_i \)-edge, \( \rho_i \)-cycle, \( \rho_i \)-tree, \( \rho_i \)-forest, fuzzy \( \rho_i \)-cycle, fuzzy \( \rho_i \)-tree, fuzzy \( \rho_i \)-forest, \( \rho_i \)-connectedness, \( \rho_i \)-bridge, \( \rho_i \)-cutvertex etc. are introduced and studied in the second chapter.

Also we generalise the above notions and introduce some new concepts like \( \rho_{i_1i_2...i_r} \)-cycle, \( \rho_{i_1i_2...i_r} \)-tree, \( \rho_{i_1i_2...i_r} \)-forest, fuzzy \( \rho_{i_1i_2...i_r} \)-cycle, fuzzy \( \rho_{i_1i_2...i_r} \)-tree, fuzzy \( \rho_{i_1i_2...i_r} \)-forest etc. \( \rho_i \)-regularity and \( \rho_{i_1i_2...i_r} \)-regularity are also discussed in this chapter after introducing \( \rho_i \)-size, \( \rho_i \)-degree, \( \rho_{i_1i_2...i_r} \)-size, \( \rho_{i_1i_2...i_r} \)-degree etc. This has particular significance because it helps to study not only the different fuzzy relations at the same time but also the interconnection between them. \( \rho_i \)-homomorphism and \( \rho_i \)-isomorphism, \( \rho_{i_1i_2...i_r} \)-homomorphism and \( \rho_{i_1i_2...i_r} \)-isomorphism and homomorphism and isomorphism of fuzzy graph structures are also studied in this chapter.
Chapter 3: Operations like composition, lexicographic product etc. of graphs were studied by Sabidussi in [58,59,60]. The operations of fuzzy graphs were studied by Mordeson and Peng [43] and were discussed in Mordeson and Nair[45]. We define the operations of Cartesian product, composition, union and join of graph structure and introduce the above operations on fuzzy graph structures in the third chapter. Some properties are also studied.

We extend the idea of complementation also to fuzzy graph structures. For that, we are using the idea of $\phi$-cyclic complement of a graph structure discussed in [61] and the definitions and results on complement of a fuzzy graph discussed in [65]. $\phi$-self complementary, self complementary, totally self complementary, strong $\phi$-self complementary, strong self complementary and totally strong self complementary fuzzy graph structures are defined analogous to their counterparts in [61]. Some results are obtained. We analyse the results of complementation on union, join and composition of two fuzzy graph structures.

Chapter 4: Labeling of vertices and edges has been a problem discussed a lot in Graph Theory. Many people introduced different types of labelings. A detailed description of various graph labelings is given in [28]. Rosa, A.[56], Graham and Sloane [29] etc. paved the bases for most of the works on labeling of graphs. Graceful, Harmonious, Magic etc. are some of the important types of labelings with each of these types being divided into a number of subtypes. However, Jeurissen [34] has studied another type of labeling. In chapter 4, we introduce $R_t$-labellings, $R_t$-index vectors and admissible $R_t$-index vectors analogous to the labelings, index vectors and admissible index vectors studied in [34] by Jeurissen based on the works of Brouwer[11], Doob [23,24] and Stewart[63]. Jeurissen has proved some results on these using some
operations on incidence matrices. On the same lines, we prove that the collection of all admissible $R_i$-index vectors, the collection of all $R_i$-labellings for the index vector 0 and the collection of all $R_i$-labellings for the $R_i$-index vector $\lambda j$, ($\lambda \in F, F$, a commutative ring, $j$ an all 1-vector), of a graph structure $G = (V, R_1, R_2, ... R_k)$ are $F$-modules. The ranks of the modules are also found in various cases, namely $R_i$-bipartite, char $F = 2$, char $F \neq 2$ etc. Then we introduce labelling matrices and index matrices of a graph structure. We prove similar results for these also. Note that the number of cases is much more in the case of a graph structure.

Chapter 5: If we can establish a link between concepts of Graph Theory with incidence algebra (with its close relation with concepts of number theory), it will open up a vast area for future work in many directions. Ancykutty Joseph studied the relation between directed graphs and incidence algebras in [1]. She used the number of directed paths from one vertex to another for introducing the incidence algebras of directed graphs. In the fifth chapter, we study the incidence algebras of undirected graphs based on the concept of incidence algebra of pre-orders discussed by Foldes and Meletiou [26] and the concept of Ancykutty Joseph[1]. We use the number of paths for introducing the concept of incidence algebras of undirected graphs. We establish another relation between graphs and incidence algebras through the labelings and index vectors of a graph studied by Jeuring[34]. We prove that the collection of all labelings for the collection of all admissible index vectors, the collection of all labelings for 0 and the collection of all labelings for $\lambda j, \lambda \in F, j$, an all 1-vector, are subalgebras of the incidence algebra $I(V, F)$ of $G = (V, E)$.

In a similar way, we prove that the collection of all $R_i$-labellings for the collection of all admissible $R_i$-index vectors, the collection of all $R_i$-labellings for the index...
vector 0 and the collection of all $R_i$-labellings for the $R_i$-index vector $\lambda_i j_i$, ($\lambda_i \in F, F$, a commutative ring, $j_i$ an all 1-vector), of a graph structure $G = (V, R_1, R_2, ..., R_k)$ are subalgebras of the incidence algebra $I(V, F)$ of $G$. We are also proving that the collection of all labelling matrices for the collection of all admissible index matrices, the collection of all labelling matrices for 0 and the collection of all labelling matrices for $\Lambda J$, $\Lambda \in F^{k \times k}$ with $\lambda_i, i = 1, 2, ..., k$ along the diagonal and zeroes elsewhere and $J$ a matrix with column vectors $j_i$ (as mentioned above) along the diagonal and zeroes elsewhere, form subalgebras of the incidence algebra $I(V^k, F^k)$ of $G$.

Chapter 6: In chapter 6, we introduce labellings, index vectors etc. of a fuzzy graph. We also establish relations between a fuzzy algebra of an incidence algebra and these labellings. We also introduce a new concept, namely, fuzzy incidence and extend the labelling of graphs to $(na, ni)$-extended labelling of fuzzy graphs.

Chapter 7: In the seventh chapter we similarly introduce $\rho_i$-labellings, $\rho_i$-index vectors etc. of a fuzzy graph structure and prove similar results. Similar results on fuzzy labelling matrices, fuzzy index matrices etc. are also established. $(na_i, ni_i)$-extended labellings are also introduced.

Chapter 8: Four Color Conjecture and coloring problems have been topics of discussion in Graph Theory for so long. A number of people like König [37], Szekeres and Wilf[67], Berge[5], Ore [49], Dirac[22], Heawood[32], Whitney[73] and a lot of others have worked on problems on coloring. The concept of coloring of fuzzy graph structures are introduced in chapter 8 analogous to the coloring of fuzzy graphs discussed by Muñoz et al.[66], Ramaswami and Poornima[54] and Poornima and Ramaswami[53]. We discuss vertex $-\rho_i$-colouring, $\rho_i$-edge colouring and total $\rho_i$-colouring. We generalise them to get vertex $\rho_{i_1i_2...i_r}$-colouring, $\rho_{i_1i_2...i_r}$-edge colouring
and total $\rho_{i_1i_2\ldots i_r}$-colouring. A new concept, namely, relation colouring of a graph structure and fuzzy graph structure is introduced. Some results on colouring and homomorphism are discussed. We are focussing more on $(d_i, f_i), (d_{i_1i_2\ldots i_r}, f_{i_1i_2\ldots i_r})$ and $(d, f)$-extended colourings.

**Chapter 9:** We go through some applications of fuzzy graph structures in the last chapter. From literature, we have found applications in Environmental Science, Social Science, Linguistics etc. In [12], study of landscape connectivity using Graph Theory is done by Bunn, Urban and Keitt. Therein the authors concentrate only on two focal species of living beings. Using a graph structure, we can improve the study incorporating the path of each species being represented by a relation. Edge thresholding and node removal, two methods used in [12] may be replaced by giving membership grades to $R_i$-edges and vertices. In [3], some basic concepts of network analysis are discussed using analysis of social network by Beránek and Novák. It deals with relations in a group of subjects (individuals/organisations). The relationship can be quite different (for example, friendship, enmity, love etc.). These can be simultaneously studied using a graph structure. The magnitude of relationship being marked by thickness of arrow in a path diagram as is done at present can be replaced by edges with membership grades. Problems in map design from Geography as discussed in [39] by Mackaness and Beard and problems in word association from Linguistics as discussed in [51] by Pigott and in [70] by Torra and Narukawa can be converted to problems of fuzzy graph structures and can more effectively be studied.

Essential preliminaries are given in chapter 1. For concepts from Graph Theory, reference may be made to [31], for Fuzzy Graph Theory, to [45], for graph structures, to [61] and for incidence algebras, to [62].