Chapter 8

Colouring - Extension to Fuzzy Graph Structures

In this chapter, we extend the concepts of vertex coloring, edge coloring and total coloring of fuzzy graphs discussed in [66], [54] and [53] to fuzzy graph structures. We also introduce a new concept, namely, relation colouring of graph structures as well as fuzzy graph structures.

8.1 Vertex colouring of a fuzzy graph structure

First we define the \( R_i \)-colouring and \( R_i \)-chromatic number of a graph structure. Our definition is different from the edge coloring concept of [61].

Definition 8.1.1. An \( R_i \)-vertex colouring of a graph structure \( G = (V, R_1, R_2, \ldots, R_k) \) is an assignment of colours to its points so that no two points which are adjacent by an \( R_i \)-edge have the same colour.

Definition 8.1.2. An \( n-R_i \)-colouring of a graph structure \( G = (V, R_1, R_2, \ldots, R_k) \) is such an \( R_i \)-vertex colouring which uses \( n \) colours.

\(^1\)Some results of this chapter are included in the paper Colouring of Fuzzy Graph Structures, Communicated.
**Definition 8.1.3.** The $R_i$-chromatic number $\chi_i(G)$ is the minimum $n$ for which $G$ has an $n$-$R_i$-colouring.

We define vertex colouring for a fuzzy graph structure on the lines of Muñoz et al.[66]. For this, we take $\tilde{G}^* = (\text{supp}(\mu), \rho_1, \rho_2, \ldots, \rho_k)$.

$\tilde{G}_{i}^{\alpha} = (\text{supp}(\mu), \rho_1, \rho_2, \ldots, \rho_i^\alpha, \ldots, \rho_k)$, where $\alpha \in I_i$.

$\rho_i^\alpha = \{(r, s)|\rho_i(r, s) \geq \alpha, \alpha \in I\}$. Let $\chi_i^\alpha$ be the $\rho_i$-chromatic number of $\tilde{G}_{i}^{\alpha}$.

**Convention:** Throughout this section, unless otherwise specified, we take $\tilde{G}$ as a fuzzy graph structure $(\mu, \rho_1, \rho_2, \ldots, \rho_k)$ with $\text{supp}(\mu) = V$ of a graph structure $G = (V, R_1, R_2, \ldots, R_k)$.

**Definition 8.1.4.** Given a fuzzy graph structure $\tilde{G}$, the $\rho_i$-chromatic number of $\tilde{G}$ is $\chi_i(\tilde{G}) = \{(x, \nu_i(x))|x \in X\}$ where $X = \{1, 2, \ldots, |V|\}$,

$\nu_i(x) = \sup \{\alpha \in I|x \in A_i^\alpha\} \forall x \in X, A_i^\alpha = \{1, 2, \ldots, \chi_i^\alpha\}\forall \alpha \in I_i$.

**Definition 8.1.5.** Given a fuzzy graph structure $\tilde{G}$, the chromatic number of $\tilde{G}$ is $\chi(\tilde{G}) = \{((x, \nu_1(x)), (x, \nu_2(x)), \ldots, (x, \nu_k(x)))|x \in X\}$, where $X = \{1, 2, \ldots, |V|\}$,

$\nu_i(x) = \sup \{\alpha_i \in I|x \in A_i^{\alpha_i}\} \forall x \in X, A_i^{\alpha_i} = \{1, 2, \ldots, \chi_i^{\alpha_i}\}\forall \alpha_i \in I_i$,

$i = 1, 2, \ldots, k$.

Now we move on to $(d_i, f_i)$-extended colouring.

**§ (d_i, f_i)-extended $\rho_i$-colouring**

Let $S_i$ be the available $\rho_i$-colour set. $d_i$ be the $\rho_i$-dissimilarity measure defined by $d_i : S_i \times S_i \rightarrow [0, \infty)$ with

i. $d_i(r, s) \geq 0 \forall r, s \in S_i$
ii. \( d_i(r, s) = 0 \iff r = s \forall r, s \in S_i \)

iii. \( d_i(r, s) = d_i(s, r) \forall r, s \in S_i \)

\( I_i \) be the image of membership function of fuzzy graph structure.

\( f_i : I_i \rightarrow [0, \infty) \) be non negative, non-decreasing and real scale function.

ie., \( f_i(a) \leq f_i(a') \forall a, a' \in I_i, a < a' \).

**Definition 8.1.6.** Given a fuzzy graph structure \( \tilde{G} \) of \( G \), a colour set \( S_i \), a \( \rho_i \)-dissimilarity measure \( d_i \) defined on \( S_i \) and a \( \rho_i \)-scale function \( f_i \), a \( (d_i, f_i) \)-extended \( \rho_i \)-coloring function \( \tilde{C}_i \) is a mapping \( \tilde{C}_i : V \rightarrow S_i \) with the property

\[ d_i(\tilde{C}_i(r), \tilde{C}_i(s)) \geq f_i(\rho_i(r, s)) \forall r, s \in V \text{ such that } r \neq s. \]

A \( (d_i, f_i) \)-extended \( p_i \)-\( \rho_i \)-coloring \( C_{p_i}^i \) is a \( (d_i, f_i) \)-extended colouring function with no more than \( p_i \) different colors, \( C_{p_i}^i : V \rightarrow S_i \) where \( S_i = \{1, 2, \ldots, p_i\} \).

**Definition 8.1.7.** Given a fuzzy graph structure \( \tilde{G} \), a \( \rho_i \)-dissimilarity measure \( d_i \) and a \( \rho_i \)-scale function \( f_i \), the minimum value \( p_i \) for which a \( (d_i, f_i) \)-extended \( \rho_i \)-colouring of \( \tilde{G} \) exists is the \( (d_i, f_i) \)-\( \rho_i \)-chromatic number of \( \tilde{G} \) denoted by \( \chi_i^{(d_i, f_i)}(\tilde{G}) \).

\( \text{§}(d, f) \)-extended colouring

Now we generalise the above concepts to get a \( (d, f) \)-extended colouring of a fuzzy graph structure as follows.

\( (d, f) \)-extended colouring problem consists of determining the \( (d_i, f_i) \)-\( \rho_i \)-chromatic numbers of the fuzzy graph structure and the associated \( (d_i, f_i) \)-extended \( \rho_i \)-colouring functions \( C_i \) for \( i = 1, 2, \ldots, k \).

Thus we have \( \chi = (\chi_1, \chi_2, \ldots, \chi_k); C = (C_1, C_2, \ldots, C_k); f = (f_1, f_2, \ldots, f_k). \) Here

\( C : V^k \rightarrow S_1 \times S_2 \times \ldots \times S_k \)

\( f : I_1 \times I_2 \times \ldots \times I_k \rightarrow [0, \infty)^k. \) We say that \( C \) is a \( (d, f) \)-extended colouring function if

\[ d_i(C_i(r), C_i(s)) \geq f_i(\rho_i(r, s)) \forall r, s \in V, r \neq s, i = 1, 2, \ldots, k. \]

A \( (d, f) \)-extended
\[ p = (p_1, p_2, \ldots, p_k) \text{-colouring} \]

\[ C^p \text{ is a } (d, f) \text{-extended colouring function with no more than } p \text{ different colours, i.e., } C^p : V^k \rightarrow S_1 \times S_2 \times \cdots \times S_k \text{ where } S_i = \{1, 2, \ldots, p_i\}. \]

**Definition 8.1.8.**

Given a fuzzy graph structure \( \tilde{G} \), a \( \rho_{i_1i_2\ldots i_r} \)-colour set \( \bigcup_{i=1}^{i_r} S_i \), a \( \rho_{i_1i_2\ldots i_r} \)-dissimilarity measure \( d_{i_1i_2\ldots i_r} \) defined on \( \bigcup_{i=1}^{i_r} S_i \) and a \( \rho_{i_1i_2\ldots i_r} \)-scale function \( f_{i_1i_2\ldots i_r} \), a \( (d_{i_1i_2\ldots i_r}, f_{i_1i_2\ldots i_r}) \)-extended \( \rho_{i_1i_2\ldots i_r} \)-coloring function of \( \tilde{G} \) denoted as \( C_i \) is a mapping \( C_i : V \rightarrow \bigcup_{i=1}^{i_r} S_i \) with the property

\[ d_{i_1i_2\ldots i_r}(C_i(r), C_i(s)) \geq f_{i_1i_2\ldots i_r}(\rho_i(r, s)) \forall r, s \in V \text{ such that } r \neq s \text{ where } (r, s) \text{ is a } \rho_i \text{-edge, } i_1 \leq i \leq i_r. \]

A \( (d_{i_1i_2\ldots i_r}, f_{i_1i_2\ldots i_r}) \)-extended \( \rho_{i_1i_2\ldots i_r} \)-coloring \( C^{pi_1i_2\ldots i_r} \) is a \( (d_{i_1i_2\ldots i_r}, f_{i_1i_2\ldots i_r}) \)-extended \( \rho_{i_1i_2\ldots i_r} \)-colouring function with no more than \( p_1 + p_2 + \cdots + p_r \).
different colors, i.e., \( C_{i}^{p_{i}} : V \to \bigcup_{i=1}^{i_{r}} S_{i} \) where 
\( S_{i} = \{1, 2, ..., p_{i}\} \).

**Definition 8.1.9.** Given a fuzzy graph structure \( G \), a \( \rho_{i_{1}i_{2}...i_{r}} \)-dissimilarity measure \( d_{i_{1}i_{2}...i_{r}} \) and a \( \rho_{i_{1}i_{2}...i_{r}} \)-scale function \( f_{i_{1}i_{2}...i_{r}} \), the minimum value \( \sum_{i=1}^{i_{r}} p_{i} \) for which a \( \rho_{i_{1}i_{2}...i_{r}} \)-extended \( d_{i_{1}i_{2}...i_{r}}, f_{i_{1}i_{2}...i_{r}} \)-colouring of \( \tilde{G} \) exists is the \( \rho_{i_{1}i_{2}...i_{r}} \)-chromatic number of \( \tilde{G} \) denoted by \( \chi_{i_{1}i_{2}...i_{r}}^{(d_{i_{1}i_{2}...i_{r}}, f_{i_{1}i_{2}...i_{r}})}(\tilde{G}) \).

Note that we can select \( r \) among \( \rho_{1}, \rho_{2}, ..., \rho_{k} \) in \( k(k-1)(k-2)...(k-r+1) \) ways.

So the following result is obvious.

**Result 1**

There exist \( k(k-1)(k-2)...(k-r+1) \) different vertex \( \rho_{i_{1}i_{2}...i_{r}} \)-colourings for a fuzzy graph structure \( \tilde{G} \).

### 8.2 Some results on vertex colouring of a fuzzy graph structure

Now we recall the following result from [31] and extend it to fuzzy graph structures.

**Convention:** Throughout this section, unless otherwise specified, we take \( \tilde{G} \) as a fuzzy graph structure \( (\mu, \rho_{1}, \rho_{2}, ..., \rho_{k}) \) with \( \text{supp}(\mu) = V \) of a graph structure \( G = (V, R_{1}, R_{2}, ..., R_{k}) \).

**Theorem 8.2.1.** [31] For any graph \( G \) and any elementary homomorphism \( \phi \) of \( G \),
\( \chi(G) \leq \chi(\phi G) \) where \( \chi(G) \) is the chromatic number of \( G \).

**Theorem 8.2.2.** Let \( \tilde{G} \) be a fuzzy graph structure and \( \epsilon_{i} \) a \( \rho_{i} \)-homomorphism on \( \tilde{G} \).
Let \( \chi_{i} \) be the \( \rho_{i} \)-chromatic number of \( \tilde{G} \). Then \( \chi_{i}(\tilde{G}) \leq \chi_{i}(\epsilon_{i}\tilde{G}) \).
Proof. Let $\chi_i(\epsilon, \tilde{G}) = r$.

Then $d_i(C^k_i(\epsilon, u), C^k_i(\epsilon, v)) \geq f_i(\rho_i(\epsilon, u, \epsilon, v))$. If we use the same $\rho_i$-colouring for $\tilde{G}$, we have

$d_i(C^k_i(u), C^k_i(v)) = d_i(C^k_i(\epsilon, u), C^k_i(\epsilon, v)) \geq f_i(\rho_i(\epsilon, u, \epsilon, v)) \geq f_i(\rho_i(u, v))$ since $\epsilon$ is a $\rho_i$-homomorphism.

Therefore $\chi_i(\tilde{G}) \leq \chi_i(\epsilon, \tilde{G})$. \qed

**Theorem 8.2.3.** Let $\tilde{G}$ be a fuzzy graph structure and $\epsilon_{i_{1_{1}2_{...i}}}i_r$ be a $\rho_{i_{1_{1}2_{...i}}}i_r$-homomorphism on $\tilde{G}$. Let $\chi_{i_{1_{1}2_{...i}}}i_r$ be the $\rho_{i_{1_{1}2_{...i}}}i_r$-chromatic number of $\tilde{G}$.

Then $\chi_{i_{1_{1}2_{...i}}}i_r(\tilde{G}) \leq \chi_{i_{1_{1}2_{...i}}}i_r(\epsilon_{i_{1_{1}2_{...i}}}i_r, \tilde{G})$.

Proof. Let $\chi_{i_{1_{1}2_{...i}}}i_r(\epsilon_{i_{1_{1}2_{...i}}}i_r, \tilde{G}) = s$.

Then

$d_{i_{1_{1}2_{...i}}}i_r(C^k_i(\epsilon_{i_{1_{1}2_{...i}}}i_r, u), C^k_i(\epsilon_{i_{1_{1}2_{...i}}}i_r, v)) \geq f_{i_{1_{1}2_{...i}}}i_r(\rho_{i_{1_{1}2_{...i}}}i_r(\epsilon_{i_{1_{1}2_{...i}}}i_r, u, \epsilon_{i_{1_{1}2_{...i}}}i_r, v)), i \in \{i_1, i_2, ..., i_r\}, 1 < r < k$.

If we use the same $\rho_{i_{1_{1}2_{...i}}}i_r$-colouring for $\tilde{G}$, we have

$d_{i_{1_{1}2_{...i}}}i_r(C^k_i(u), C^k_i(v)) = d_{i_{1_{1}2_{...i}}}i_r(C^k_i(\epsilon_{i_{1_{1}2_{...i}}}i_r, u), C^k_i(\epsilon_{i_{1_{1}2_{...i}}}i_r, v))$

$\geq f_{i_{1_{1}2_{...i}}}i_r(\rho_{i_{1_{1}2_{...i}}}i_r(\epsilon_{i_{1_{1}2_{...i}}}i_r, u, \epsilon_{i_{1_{1}2_{...i}}}i_r, v)), i_1 \leq i \leq i_r, 1 \leq r \leq k$,

$\geq f_{i_{1_{1}2_{...i}}}i_r(\rho_i(u, v))$ since $\epsilon_{i_{1_{1}2_{...i}}}i_r$ is a $\rho_{i_{1_{1}2_{...i}}}i_r$-homomorphism.

Therefore $\chi_{i_{1_{1}2_{...i}}}i_r(\tilde{G}) \leq \chi_{i_{1_{1}2_{...i}}}i_r(\epsilon_{i_{1_{1}2_{...i}}}i_r, \tilde{G})$. \qed

**Theorem 8.2.4.** Let $\tilde{G}$ be a fuzzy graph structure and $\epsilon$ a homomorphism on $\tilde{G}$. Let $\chi_i$ be the $\rho_i$-chromatic number of $\tilde{G}$ for $i = 1, 2, ..., k$. Then $\chi_i(\tilde{G}) \leq \chi_i(\epsilon, \tilde{G})$ for $i = 1, 2, ..., k$.

Proof. $\epsilon$ is a homomorphism on $\tilde{G}$. Hence it is a $\rho_i$-homomorphism on $\tilde{G}$ for $i = 1, 2, ..., k$. Hence the result is obvious from Theorem 8.2.2. \qed
8.3 \((d_i, f_i)\)-extended \(\rho_i\)-edge colouring of a fuzzy graph structure

Convention: Throughout this section, unless otherwise specified, we take \(\tilde{G}\) as a fuzzy graph structure \((\mu, \rho_1, \rho_2, ..., \rho_k)\) of a graph structure \(G = (V, R_1, R_2, ..., R_k)\) with \(\mu(v) = 1 \forall v \in V\).

\(\mathcal{G}(d_i, f_i)\)-extended \(\rho_i\)-edge colouring

We define \((d_i, f_i)\)-extended \(\rho_i\)-edge colouring of a fuzzy graph structure in the following way similar to the \((d, f)\)-extended edge coloring of a fuzzy graph defined in [54].

Let \(S_i\) be a colour set. A \(\rho_i\)-dissimilarity measure \(d_i\) defined on \(S_i\) is a function \(d_i : S_i \times S_i \to [0, \infty)\) which satisfies

i. \(d_i(r, s) = 0 \iff r = s \forall (r, s) \in S_i\)

ii. \(d_i(r, s) = d_i(s, r) \forall (r, s) \in S_i\)

Let \(I_i\) be the set of all \(\rho_i\)-membership grades assigned to \(\rho_i\)-edges. Let \(f_i : I_i \to [0, \infty)\) be non decreasing,i.e., \(f_i(a) \leq f_i(a')\forall a, a' \in I_i, a < a'\). \(f_i\) is a \(\rho_i\)-scale function.

Definition 8.3.1. Consider a fuzzy graph structure \(\tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k, S_1, S_2, ..., S_k, d_1, d_2, ..., d_k, f_1, f_2, ..., f_k)\). A \((d_i, f_i)\)-extended \(\rho_i\)-colouring function of \(\tilde{G}\), \(C_i^{(d_i, f_i)}\) or \(C_i\) is a mapping \(C_i : R_i \to S_i\) with the properties

i. \(d_i(C_i(r, s), C_i(r, l)) \geq \wedge\{f_i(\rho_i(r, s)), f_i(\rho_i(r, l))\}\) for all \(\rho_i\)-edges \((r, s), (r, l)\)

ii. \(d_i(C_i(r, s), C_i(l, s)) \geq \wedge\{f_i(\rho_i(r, s)), f_i(\rho_i(l, s))\}\) for all \(\rho_i\)-edges \((r, s), (l, s)\).

A \((d_i, f_i)\)-extended \(p_i - \rho_i\)-colouring \(C_i^{p_i}\) is a \((d_i, f_i)\)-extended \(\rho_i\)-colouring function
which takes maximum $p_i$ different colours,

ie., $C^p_i : R_i \rightarrow S_i$ where $S_i = \{1, 2, ..., p_i\}$ which satisfies

i. $d_i(C^p_i(r,s), C^p_i(r,l)) \geq \wedge \{f_i(\rho_i(r,s)), f_i(\rho_i(r,l))\}$ for all $\rho_i$-edges $(r,s), (r,l)$

ii. $d_i(C^p_i(r,s), C^p_i(l,s)) \geq \wedge \{f_i(\rho_i(r,s)), f_i(\rho_i(l,s))\}$ for all $\rho_i$-edges $(r,s), (l,s)$

**Definition 8.3.2.** For a given fuzzy graph structure

$\tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k, S_1, S_2, ..., S_k, d_1, d_2, ..., d_k, f_1, f_2, ..., f_k)$, the minimum value of $p_i$ for which a $(d_i, f_i)$-extended $p_i$-$\rho_i$-coloring exists is called the $(d_i, f_i)$-$\rho_i$-edge chromatic number of $\tilde{G}$ and is denoted by $\chi^{d_i,f_i}_i(\tilde{G})$.

$(d_i, f_i)$-extended edge colouring problem consists of determining the $(d_i, f_i)$-$\rho_i$-edge chromatic numbers of the fuzzy graph structure and associated $(d_i, f_i)$-extended $\rho_i$-edge colouring functions for $i = 1, 2, ..., k$.

$\S(d,f)$-extended edge colouring

Let $\chi^{(d,f)}(\tilde{G}) = (\chi^{(d_1,f_1)}_1, \chi^{(d_2,f_2)}_2, ..., \chi^{(d_k,f_k)}_k)$, $C = (C_1, C_2, ..., C_k)$,

$f = (f_1, f_2, ..., f_k)$. Here $C : V^k \rightarrow S_1 \times S_2 \times ... \times S_k$,

$f : I_1 \times I_2 \times ... \times I_k \rightarrow [0, \infty)^k$. We say that $C$ is a $(d,f)$-extended edge colouring if

i. $d_i(C_i(r), C_i(s)) \geq \wedge \{f_i(\rho_i(r,s)), f_i(\rho_i(r,l))\}$ for all $\rho_i$-edges $(r,s), (r,l)$,

and ii. $d_i(C_i(r), C_i(s)) \geq \wedge \{f_i(\rho_i(r,s)), f_i(\rho_i(l,s))\}$ for all $\rho_i$-edges $(r,s), (l,s)$

for $i = 1, 2, ..., k$.

A $(d,f)$-extended $p$-edge colouring $C^p$ is a $(d,f)$-extended edge colouring function with no more than $p$ different colours.

ie., $C^p : R_1 \times R_2 \times ... \times R_k \rightarrow S_1 \times S_2 \times ... \times S_k$ where $S_i = \{1, 2, ..., p_i\}$.
\( \xi(d_{i_1i_2...i_r}, f_{i_1i_2...i_r}) \)-extended \( \rho_{i_1i_2...i_r} \)-edge colouring

Let \( \{S_1, S_2, \ldots, S_k\} \) be the available colour set with \( S_i \cap S_j = \phi \) for \( i_1 \leq i, j \leq i_r, 1 \leq r \leq k \). A \( \rho_{i_1i_2...i_r} \)-dissimilarity measure \( d_{i_1i_2...i_r} \) defined on \( \bigcup_{i=i_1}^{i_r} S_i \) is a function

\[
d_{i_1i_2...i_r} : \bigcup_{i=i_1}^{i_r} S_i \times \bigcup_{i=i_1}^{i_r} S_i \to [0, \infty)
\]

i. \( d_{i_1i_2...i_r}(r, s) = 0 \iff r = s \)

ii. \( d_{i_1i_2...i_r}(r, s) = d_{i_1i_2...i_r}(s, r) \)

Let \( I_{i_1i_2...i_r} = \bigcup_{i=i_1}^{i_r} I_i \) be the set of all \( \rho_i \)-membership grades, assigned to \( \rho_r \)-edges, \( i = i_1, i_2, \ldots, i_r \). Let \( f_{i_1i_2...i_r} : I_{i_1i_2...i_r} \to [0, \infty) \) be non decreasing.

ie., \( f_{i_1i_2...i_r}(a) \leq f_{i_1i_2...i_r}(a') \forall a, a' \in I_{i_1i_2...i_r} \) such that \( a < a' \). \( f_{i_1i_2...i_r} \) is a \( \rho_{i_1i_2...i_r} \)-scale function.

**Definition 8.3.3.** Consider a fuzzy graph structure

\( \tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_k, S_1, S_2, \ldots, S_k, d_1, d_2, \ldots, d_k, f_1, f_2, \ldots, f_k) \).

A \( (d_{i_1i_2...i_r}, f_{i_1i_2...i_r}) \)-extended \( \rho_{i_1i_2...i_r} \)-colouring function of \( \tilde{G} \), \( C_i \) is a mapping

\( C_i : \bigcup_{i=i_1}^{i_r} R_i \to \bigcup_{i=i_1}^{i_r} S_i, i_1 \leq i \leq i_r \) with the properties

i. \( d_{i_1i_2...i_r}(C_i(r, s), C_i(r, l)) \geq \bigwedge \{f_{i_1i_2...i_r}(\rho_i(r, s)), f_{i_1i_2...i_r}(\rho_i(r, l))\} \) for all \( \rho_i \)-edges, \( (r, s), (r, l), i_1 \leq i \leq i_r \)

ii. \( d_{i_1i_2...i_r}(C_i(r, s), C_i(l, s)) \geq \bigwedge \{f_{i_1i_2...i_r}(\rho_i(r, s)), f_{i_1i_2...i_r}(\rho_i(l, s))\} \) for all \( \rho_i \)-edges, \( (r, s), (l, s), i_1 \leq i \leq i_r \).

A \( (d_{i_1i_2...i_r}, f_{i_1i_2...i_r}) \)-extended \( \rho_{i_1i_2...i_r} \)-colouring \( C_{i_1i_2...i_r} \) is a \( (d_{i_1i_2...i_r}, f_{i_1i_2...i_r}) \)-extended \( \rho_{i_1i_2...i_r} \)-colouring function which takes maximum \( p_{i_1i_2...i_r} \) different colours,
ie., $C_{\rho_1,\ldots,\rho_r}^{\rho_{i_1} \ldots \rho_{i_r}} : \bigcup_{i=i_1}^{i_r} R_i \rightarrow \bigcup_{i=i_1}^{i_r} S_i$

where $S_i = \{1, 2, \ldots, p_i\}$ which satisfies

i. $d_{i_1i_2 \ldots i_r}(C_{\rho_1 \ldots \rho_r}^{\rho_{i_1} \ldots \rho_{i_r}}(r,s), C_{\rho_1 \ldots \rho_r}^{\rho_{i_1} \ldots \rho_{i_r}}(r,l)) \geq \land \{f_{i_1i_2 \ldots i_r}(\rho_i(r,s)), f_{i_1i_2 \ldots i_r}(\rho_i(r,l))\}$ for all $\rho_i$-edges, $(r,s), (r,l), i_1 \leq i \leq i_r$

ii. $d_{i_1i_2 \ldots i_r}(C_{\rho_1 \ldots \rho_r}^{\rho_{i_1} \ldots \rho_{i_r}}(r,s), C_{\rho_1 \ldots \rho_r}^{\rho_{i_1} \ldots \rho_{i_r}}(l,s)) \geq \land \{f_{i_1i_2 \ldots i_r}(\rho_i(r,s)), f_{i_1i_2 \ldots i_r}(\rho_i(l,s))\}$ for all $\rho_i$-edges, $(r,s), (l,s), i_1 \leq i \leq i_r$

Note that we can select $r$ among $\rho_1, \rho_2, \ldots, \rho_k$ in $k(k-1)(k-2)\ldots(k-r+1)$ ways.

So the following result is obvious.

**Result 2**

There exist $k(k-1)(k-2)\ldots(k-r+1)$ different edge $\rho_{i_1i_2 \ldots i_r}$-colourings for the fuzzy graph structure $\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_k)$.

### 8.4 Relation colouring of a graph structure and fuzzy graph structure

We introduce a new type of colouring in a graph structure which is quite different from the edge colouring given by Sampathkumar[61].

**Definition 8.4.1.** Let $G = (V, R_1, R_2, \ldots, R_k)$ be a graph structure. Then $C = \{C_1, C_2, \ldots, C_k\}$ is a relation colouring of $G$ if every $(u,v) \in R_i$ is coloured with colour $C_i$ and $C_i \neq C_j$ if $i \neq j$.

We give the $(d, f)$-extended relation colouring of a fuzzy graph structure as follows.

Let $S$ be the available colour set of $k$ colours. $d$ is the dissimilarity measure $d : S \times S : [0, \infty)$ with
\[ d(r, s) = 0 \text{ iff } r = s \forall r, s \in S \]
\[ d(r, s) = d(s, r) \forall r, s \in S. \]

Let \( I \) be the set of all membership grades of \( \tilde{G} \). Let \( f : I \to [0, \infty) \) be non decreasing. This is the scale function.

**Definition 8.4.2.** A \((d, f)\)-extended relation colouring of a fuzzy graph structure \( \tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k) \) of \( G = (V, R_1, R_2, ..., R_k) \) denoted by \( C_R \) is a mapping \( C_R : \{R_1, R_2, ..., R_k\} \to S \) satisfying
\[
\begin{align*}
&i. \ d(C_R(\rho_i)), C_R(\rho_j) \geq \land\{f(\rho_i(r, s)), f(\rho_j(r, l)))\} \forall (r, s) \in \text{supp}(\rho_i), (r, l) \in \text{supp}(\rho_j) \\
&ii. \ d(C_R(\rho_i), C_R(\rho_j)) \geq \land\{f(\rho_i(r, s)), f(\rho_j(l, s)))\} \forall (r, s) \in \text{supp}(\rho_i), (l, s) \in \text{supp}(\rho_j) \\
\end{align*}
\]

**8.5 Total colouring of a fuzzy graph structure**

**Convention:** Throughout this section, unless otherwise specified, we take \( \tilde{G} \) as a fuzzy graph structure \( (\mu, \rho_1, \rho_2, ..., \rho_k) \) of \( G = (V, R_1, R_2, ..., R_k) \) with \( \mu(v) = 1 \forall v \in V \) of a graph structure \( G = (V, R_1, R_2, ..., R_k) \).

\[ (d_i, f_i)\)-extended total \( \rho_i \)-colouring of a fuzzy graph structure

We define \( \rho_i \)-total colouring as follows.

**Definition 8.5.1.** Let \( \tilde{G} = (\mu, \rho_1, \rho_2, ..., \rho_k, S_1, S_2, ..., S_k, d_1, d_2, ..., d_k, f_1, f_2, ..., f_k) \) be a fuzzy graph structure. A \((d_i, f_i)\)-extended total \( \rho_i \)-colouring of \( \tilde{G} \) denoted by \( C_T^i \) is a mapping from \( V \cup R_i \) to \( S_i \) satisfying
\[
\begin{align*}
&i. \ d_i(C_T^i(r), C_T^i(s)) \geq f_i(\rho_i(r, s)) \forall r, s \in V \\
&ii. \ d_i(C_T^i(r, s), C_T^i(r, l)) \geq \land\{f_i(\rho_i(r, s)), f_i(\rho_i(r, l))\} \\
\text{and } d_i(C_T^i(r, s), C_T^i(l, s)) \geq \land\{f_i(\rho_i(r, s)), f_i(\rho_i(l, s))\} \forall \rho_i \text{-edges } (r, s), (r, l), (l, s)
\end{align*}
\]
iii. \( d_i(C^T_i(r), C^T_i(r, s)) \geq f_i(\rho_i(r, s)) \)
and \( d_i(C^T_i(s), C^T_i(r, s)) \geq f_i(\rho_i(r, s)) \forall \rho_i\)-edges \((r, s)\).

A \((d_i, f_i)\)-extended total \(\rho_i\)-coloring \(C^{TP}_{i \rho_i}\) is a \((d_i, f_i)\)-extended total \(\rho_i\)-coloring function which takes maximum \(p_i\) different colours.

The minimum \(p_i\) for which a \((d_i, f_i)\)-total \(\rho_i\)-coloring exists for a fuzzy graph structure is the \((d_i, f_i)\)-total \(\rho_i\)-chromatic number \(\chi^{T(d_i, f_i)}_{i \rho_i}\).

§ \((d, f)\)-extended total colouring of a fuzzy graph structure

Definition 8.5.2. Let \(\chi^{T(d, f)}_{\tilde{G}} = (\chi^{T(d_1, f_1)}_{1}, \chi^{T(d_2, f_2)}_{2}, ..., \chi^{T(d_k, f_k)}_{k}),\)
\(C^T = (C^T_1, C^T_2, ..., C^T_k), f = (f_1, f_2, ..., f_k).\) Here \(C^T : V^k \rightarrow S_1 \times S_2 \times ... \times S_k,\)
\(f : I_1 \times I_2 \times ... \times I_k \rightarrow [0, \infty)^k.\) \(C^T\) is a \((d, f)\)-extended total colouring if

i. \( d_i(C^T_i(r), C^T_i(s)) \geq f_i(\rho_i(r, s)) \forall r, s \in V \)

and \( d_i(C^T_i(r, s), C^T_i(r, l)) \geq \land \{f_i(\rho_i(r, s)), f_i(\rho_i(r, l))\} \)

ii. \( d_i(C^T_i(r, s), C^T_i(r, l)) \geq \land \{f_i(\rho_i(r, s)), f_i(\rho_i(r, l))\} \forall \rho_i\)-edges \((r, s), (r, l), (l, s)\)

and \( d_i(C^T_i(r, s), C^T_i(r, l)) \geq \land \{f_i(\rho_i(r, s)), f_i(\rho_i(r, l))\} \forall \rho_i\)-edges \((r, s)\)

and \( d_i(C^T_i(r, s), C^T_i(r, l)) \geq \land \{f_i(\rho_i(r, s)), f_i(\rho_i(r, l))\} \forall \rho_i\)-edges \((r, s)\)

for \(i = 1, 2, ..., k.\)

A \((d, f)\)-extended \(p\)-total colouring \(C^p\) of \(\tilde{G}\) is a \((d, f)\)-extended total colouring function which takes maximum \(p\) different colours.

§ \((d_1i_12...i_r, f_1i_12...i_r)\)-extended total \(\rho_{1i_12...i_r}\)-colouring of a fuzzy graph structure

We define \((d_1i_12...i_r, f_1i_12...i_r)\)-total \(\rho_{1i_12...i_r}\)-colouring of a fuzzy graph structure as follows.
Definition 8.5.3. Let $\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_k, S_1, S_2, \ldots, S_k, d_1, d_2, \ldots, d_k, f_1, f_2, \ldots, f_k)$ be a fuzzy graph structure. A $(d_{i_1 i_2 \ldots i_r}, f_{i_1 i_2 \ldots i_r})$-extended total $\rho_{i_1 i_2 \ldots i_r}$-colouring of $\tilde{G}$ denoted by $C^T_i$ is a mapping from $V \cup \bigcup_{i=i_1}^{i_r} R_i$ to $\bigcup_{i=i_1}^{i_r} S_i, S_i \cap S_j = \emptyset; i_1 \leq i, j \leq i_r$ satisfying

i. $d_{i_1 i_2 \ldots i_r} (C^T_i(r), C^T_i(s)) \geq f_{i_1 i_2 \ldots i_r}(\rho_i(r, s)) \forall r, s \in V, (\rho_i > 0)$

ii. $d_{i_1 i_2 \ldots i_r}(C^T_i(r, s), C^T_i(r, l)) \geq \wedge \{f_{i_1 i_2 \ldots i_r}(\rho_i(r, s)), f_{i_1 i_2 \ldots i_r}(\rho_i(r, l))\}$

and $d_{i_1 i_2 \ldots i_r}(C^T_i(r, s), C^T_i(l, s)) \geq \wedge \{f_{i_1 i_2 \ldots i_r}(\rho_i(r, s)), f_{i_1 i_2 \ldots i_r}(\rho_i(l, s))\}$

$\forall \rho_i$-edges $(\rho_i > 0), (r, s), (r, l), (l, s), i_1 \leq i \leq i_r, 1 < r < k$

iii. $d_{i_1 i_2 \ldots i_r}(C^T_i(r), C^T_i(r, s)) \geq f_{i_1 i_2 \ldots i_r}(\rho_i(r, s))$

and $d_{i_1 i_2 \ldots i_r}(C^T_i(s), C^T_i(r, s)) \geq f_{i_1 i_2 \ldots i_r}(\rho_i(r, s)) \forall \rho_i$-edges $(\rho_i > 0), (r, s) \in \bigcup_{i=i_1}^{i_r} R_i$.

A $(d_{i_1 i_2 \ldots i_r}, f_{i_1 i_2 \ldots i_r})$-extended total $\rho_{i_1 i_2 \ldots i_r}$-coloring exists for a fuzzy graph structure $\tilde{G}$ if and only if the $(d_{i_1 i_2 \ldots i_r}, f_{i_1 i_2 \ldots i_r})$-total $\rho_{i_1 i_2 \ldots i_r}$-chromatic number $\chi^{T(d_{i_1 i_2 \ldots i_r}, f_{i_1 i_2 \ldots i_r})}$.

Note that we can select $r$ among $\rho_1, \rho_2, \ldots, \rho_k$ in $k(k-1)(k-2)\ldots(k-r+1)$ ways. So the following result is obvious.

Result 3

There exist $k(k-1)(k-2)\ldots(k-r+1)$ different total $\rho_{i_1 i_2 \ldots i_r}$-colourings for a fuzzy graph structure $\tilde{G} = (\mu, \rho_1, \rho_2, \ldots, \rho_k)$ of $G = (V, R_1, R_2, \ldots, R_k)$. 
8.6 Some results on total colouring of a fuzzy graph structure

In this section, we extend the results in vertex colouring to total colouring also.

Convention: Throughout this section, unless otherwise specified, we take \( \tilde{G} \) as a fuzzy graph structure \((\mu, \rho_1, \rho_2, ..., \rho_k)\) with \( \mu(v) = 1 \forall v \in V \) of a graph structure \( G = (V, R_1, R_2, ..., R_k) \).

**Theorem 8.6.1.** Let \( \epsilon_i \) be a \( \rho_i \)-homomorphism on \( \tilde{G} \). Let \( \chi^T_i \) be the \((d_i, f_i)\)-extended total \( \rho_i \)-chromatic number of \( \tilde{G} \). Then \( \chi^T_i(\tilde{G}) \leq \chi^T_i(\epsilon_i, \tilde{G}) \).

**Proof.** Let \( \chi^T_i(\epsilon_i, \tilde{G}) = n \).

Then

i. \( d_i(C^T_i(r), C^T_i(s)) \geq f_i(\rho_i(r, s)) \forall r, s \in V \)

ii. \( d_i(C^T_i(r, s), C^T_i(r, l)) \geq \land \{f_i(\rho_i(r, s)), f_i(\rho_i(r, l))\} \)

and \( d_i(C^T_i(r, s), C^T_i(l, s)) \geq \land \{f_i(\rho_i(r, s)), f_i(\rho_i(l, s))\} \forall \rho_i\)-edges, \((r, s), (r, l), (l, s)\)

iii. \( d_i(C^T_i(r), C^T_i(r, s)) \geq f_i(\rho_i(r, s)) \)

and \( d_i(C^T_i(s), C^T_i(r, s)) \geq f_i(\rho_i(r, s)) \forall \rho_i\)-edges \((r, s)\).

If we use the same \( \rho_i \)-colouring for \( \tilde{G} \), we have

i. \( d_i(C^T_i(u), C^T_i(v)) = d_i(C^T_i(\epsilon_i u), C^T_i(\epsilon_i v)) \geq f_i(\rho_i(\epsilon_i u, \epsilon_i v)) \geq f_i(\rho_i(u, v)) \) since \( \epsilon_i \) is a \( \rho_i \)-homomorphism.

ii. \( d_i(C^T_i(u), C^T_i(u, v)) = d_i(C^T_i(\epsilon u), C^T_i(\epsilon u, \epsilon v)) \geq f_i(\rho_i(\epsilon u, \epsilon v)) \geq f_i(\rho_i(u, v)) \)

and \( d_i(C^T_i(v), C^T_i(u, v)) = d_i(C^T_i(\epsilon v), C^T_i(\epsilon u, \epsilon v)) \)
Let $\epsilon_i$ be a $\rho_i$-homomorphism.

Therefore $\chi_i^T(\tilde{G}) \leq \chi_i^T(\epsilon_i \tilde{G})$.

\begin{flushright}
$\Box$
\end{flushright}

**Theorem 8.6.2.** Let $\epsilon_{i_1i_2...i_r}$ a $\rho_{i_1i_2...i_r}$-homomorphism on $\tilde{G}$. Let $\chi_{i_1i_2...i_r}$ be the $\rho_{i_1i_2...i_r}$-chromatic number of $\tilde{G}$. Then $\chi_{i_1i_2...i_r}(\tilde{G}) \leq \chi_{i_1i_2...i_r}(\epsilon_{i_1i_2...i_r} \tilde{G})$

**Proof.** Let $\chi_{i_1i_2...i_r}(\epsilon_{i_1i_2...i_r} \tilde{G}) = n$.

Then

1. $d_{i_1i_2...i_r}(C_i^{Tn}(\epsilon_{i_1i_2...i_r} u), C_i^{Tn}(\epsilon_{i_1i_2...i_r} v)) \geq f_{i_1i_2...i_r}(\rho_i(\epsilon_{i_1i_2...i_r} u, \epsilon_{i_1i_2...i_r} v))$

2. $d_{i_1i_2...i_r}(C_i^{Tn}(u), C_i^{Tn}(v)) \geq \land \{f_{i_1i_2...i_r}(\rho_i(u, v)), d_{i_1i_2...i_r}(\rho_i(l, v)))

3. $d_{i_1i_2...i_r}(C_i^{Tn}(u), C_i^{Tn}(v)) \geq f_{i_1i_2...i_r}(\rho_i(u, v))$

If we use the same $\rho_{i_1i_2...i_r}$-colouring for $\tilde{G}$, we have

1. $d_{i_1i_2...i_r}(C_i^{Tn}(u), C_i^{Tn}(v)) = d_{i_1i_2...i_r}(C_i^{Tn}(\epsilon_{i_1i_2...i_r} u), C_i^{Tn}(\epsilon_{i_1i_2...i_r} v))$

2. $d_{i_1i_2...i_r}(C_i^{T(n+1)}(u), C_i^{T(n+1)}(v))$

Since $\epsilon_{i_1i_2...i_r}$ is a $\rho_{i_1i_2...i_r}$-homomorphism.
for $i \in \{i_1, i_2, ..., i_r\}, 1 < r < k$.

Therefore $\chi_{i_1i_2...i_r}(\tilde{G}) \leq \chi_{i_1i_2...i_r}(\epsilon_{i_1i_2...i_r} \tilde{G})$.

\begin{theorem}
Let $\epsilon$ be a homomorphism on $\tilde{G}$. Let $\chi_i^T$ be the $(d, f)$-extended total $\rho_i$-chromatic number of $\tilde{G}$ for $i = 1, 2, ..., k$. Then $\chi_i^T(\tilde{G}) \leq \chi_i^T(\epsilon \tilde{G})$ for $i = 1, 2, ..., k$.
\end{theorem}

\begin{proof}
$\epsilon$ is a homomorphism on $\tilde{G}$. Hence it is a $\rho_i$-homomorphism on $\tilde{G}$ for $i = 1, 2, ..., k$. Hence the result is obvious from Theorem 8.6.1.
\end{proof}