Chapter 6

Labelling - Extension to Fuzzy Graphs

In this chapter, we extend the concept of labellings and index vectors of graphs discussed in [34] to fuzzy graphs. We also introduce the new concepts of fuzzy incidence and (na,ni)-extended labelling of fuzzy graphs.

6.1 Labellings and index vectors of a fuzzy graph

Convention: Throughout this chapter, $G = (V, V \times V)$ is a finite connected graph without loops and multiple edges.

By a fuzzy algebra of an incidence algebra, we mean a collection of mappings from a fuzzy subset of $V \times V$ to a fuzzy subset of $F$ ($F$ is a commutative ring) which forms a subalgebra of $I(V,F)$. Now we move on to define labellings, index vectors etc. of a fuzzy graph.

Definition 6.1.1. Let $\tilde{G} = (\mu, \rho)$ be a fuzzy graph of $G$ and let $F$ be an integral algebra of Incidence Algebra,Accepted for publication in Int. J. of Fuzzy Math. and Systems

Some results of this chapter are included in the paper Labellings of Fuzzy Graphs as Fuzzy Algebra of Incidence Algebra,Accepted for publication in Int. J. of Fuzzy Math. and Systems
domain. Let \( r : V \to F \) and
\[
x : V \times V \to F.
\]
We have \( x(\rho)(x(u,v)) = \sup_{(e,f) \in x^{-1}(x(u,v))} \rho(e,f) \) and
\[
r(\mu)(r(u)) = \sup_{v \in r^{-1}(r(u))} \mu(v).
\]
Then \( \tilde{r} = (r, r(\mu)) \) is an index vector of \( G \) if \( r \) is so for \( \tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho)) \). \( \tilde{x} = (x, x(\rho)) \) is a labelling of \( G \) if \( x \) is so for \( \tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho)) \).

\textbf{Definition 6.1.2.} For a fuzzy graph \( \tilde{G} = (\mu, \rho) \),
1. \( \tilde{r} = (r, r(\mu)) \) is admissible if \( r \) is so for \( (\text{supp}(\mu), \text{supp}(\rho)) \). Then \( \tilde{x} = (x, x(\rho)) \) is a labelling for \( \tilde{r} \).
2. \( \tilde{r} \) is fuzzy admissible if \( r \) is admissible in \( \tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho)) \) and
\[
r(\mu)(r(v)) \geq \bigwedge_{\rho(v_i, v_j) > 0} x(\rho)(x(v_i, v_j)) \text{ in } \tilde{G}^*.
\]
Then \( \tilde{x} \) is a fuzzy labelling for \( \tilde{r} \).

We now establish some relations between the fuzzy labellings and fuzzy index vectors with a fuzzy algebra of the incidence algebra related to a corresponding graph.

\textbf{Theorem 6.1.1.} Let \( F \) be an integral domain. The set \( FI_{L(A)}(V, F) \) of fuzzy labellings for fuzzy admissible index vectors of a fuzzy graph \( \tilde{G} = (\mu, \rho) \) of \( G \) is a fuzzy algebra of the incidence algebra \( I(\text{supp}(\mu), F) \) of \( \tilde{G}^* = (\text{supp}(\mu), \text{supp}(\rho)) \).

\textbf{Proof.} Let \( \tilde{x}_1, \tilde{x}_2 \) be fuzzy labellings for the fuzzy admissible index vectors \( \tilde{r}_1, \tilde{r}_2 \). Then by definition, \( x_1 \) and \( x_2 \) are labelings for \( r_1 \) and \( r_2 \) in \( \tilde{G}^* \). So from Theorem 5.2.1, \( x_1 + x_2, x_1 x_2 \) and \( f x_1 \) are labellings for \( r_1 + r_2, r_1 r_2 \) and \( f r_1 \) respectively in \( \tilde{G}^* \).
\[
(r_1 + r_2)(\mu)((r_1 + r_2)(v)) = \sup_{u: (r_1 + r_2)(u) = (r_1 + r_2)(v)} \mu(u)
\]
\[
\geq \sup_{u: r_1(u) = r_1(v), r_2(u) = r_2(v)} \mu(u).
\]
But
\[
 r_1(u) = \sum_{(u,v) \in E_u} x_1(u,v), \quad r_1(v_i) = \sum_{(v_i,v_j) \in E_{v_i}} x_1(v_i,v_j)
\]

\[
 r_2(u) = \sum_{(u,v) \in E_u} x_2(u,v), \quad r_2(v_i) = \sum_{(v_i,v_j) \in E_{v_i}} x_2(v_i,v_j)
\]

where \( E_u \) and \( E_{v_i} \) are the sets of edges incident with \( u \) and \( v_i \) respectively in \( \tilde{G}^* \).

Hence

\[
 \sup_{u : r_1(u) = r_1(v_i), r_2(u) = r_2(v_i)} \mu(u) \geq \bigwedge_{\rho(u,v), \rho(v_i,v_j) > 0} \left[ \sup \{ \rho(u,v) \mid (u,v) : \right. \\
\left. \sum_{(u,v) \in E_u} (x_1 + x_2)(u,v) = \sum_{(v_i,v_j) \in E_{v_i}} (x_1 + x_2)(v_i,v_j) \} \right]
\]

Therefore \( \tilde{x}_1 + \tilde{x}_2 \) is a fuzzy labelling for \( \tilde{r}_1 + \tilde{r}_2 \).

\[
 (r_1.r_2)(\mu)((r_1.r_2)(v_i)) = \sup_{u : (r_1.r_2)(u) = (r_1.r_2)(v_i)} \mu(u) \geq \sup_{u : r_1(u) = r_1(v_i), r_2(u) = r_2(v_i)} \mu(u)
\]

since \( (r_1.r_2)(u) = \sum_{s : (u,s) \in supp(\rho)} r_1(u)r_2(s) \)

Hence as in the previous case,

\[
 (r_1.r_2)(\mu)((r_1.r_2)(v_i)) \geq \bigwedge_{\rho(u,v), \rho(v_i,v_j) > 0} \left[ \sup \{ \rho(u,v) \mid (u,v) : \right. \\
\left. \sum_{(u,v) \in E_u} (x_1.x_2)(u,v) = \sum_{(v_i,v_j) \in E_{v_i}} (x_1.x_2)(v_i,v_j) \} \right]
\]

Therefore \( \tilde{x}_1 \cdot \tilde{x}_2 \) is a fuzzy labelling for \( \tilde{r}_1 \cdot \tilde{r}_2 \).

\[
 (fr_1)(\mu)((fr_1)(v_i)) = \sup_{u : (fr_1)(u) = (fr_1)(v_i)} \mu(u) \geq \sup_{u : fr_1(u) = fr_1(v_i)} \mu(u).
\]

As in the previous case,

\[
 \sup_{u : fr_1(u) = fr_1(v_i)} \mu(u) \geq \bigwedge_{\rho(u,v), \rho(v_i,v_j) > 0} \left[ \sup \{ \rho(u,v) \mid (u,v) : \right. \\
\left. \sum_{(u,v) \in E_u} (fx_1)(u,v) = \sum_{(v_i,v_j) \in E_{v_i}} (fx_1)(v_i,v_j) \} \right]
\]

Therefore \( \tilde{f}x_1 \) is a fuzzy labelling for \( \tilde{fr}_1 \).

So \( FI_{L(A)}(V, F) \) is a fuzzy algebra of the incidence algebra \( I(supp(\mu), F) \) of \( \tilde{G}^* \). \( \square \)
Theorem 6.1.2. Let $F$ be an integral domain. The set $FI_{L(0)}(V, F)$ fuzzy labellings for 0 of a fuzzy graph $\tilde{G} = (\mu, \rho)$ of $G$ is a fuzzy algebra of the incidence algebra $I(supp(\mu), F)$ of $\tilde{G}^* = (supp(\mu), supp(\rho))$.

Proof. Let $\tilde{x}_1, \tilde{x}_2$ be fuzzy labellings for the fuzzy index vector $\tilde{0}$. Then by definition, $x_1$ and $x_2$ are labelings for 0 in $\tilde{G}^*$. So from Theorem 5.2.3, $x_1 + x_2, x_1 \cdot x_2$ and $f x_1$ are labelings for 0 in $(supp(\mu), supp(\rho))$.

$$(0 + 0)(\mu)((0 + 0)(v_i)) = \sup_{u: (0 + 0)(u) = (0 + 0)(v_i)} \mu(u) \geq \sup_{u: 0(u) = 0(v_i), 0(u) = 0(v_i)} \mu(u)$$

But

$$0(u) = \sum_{(u,v) \in E_u} x_1(u,v) = \sum_{(u,v) \in E_u} x_2(u,v)$$

$$0(v_i) = \sum_{(v_i,v_j) \in E_{v_i}} x_1(v_i,v_j) = \sum_{(v_i,v_j) \in E_{v_i}} x_2(v_i,v_j)$$

where $E_u$ and $E_{v_i}$ are the sets of edges incident with $u$ and $v_i$ respectively in $\tilde{G}^*$.

Hence

$$\sup_{u: 0(u) = 0(v_i), 0(u) = 0(v_i)} \mu(u) = \bigwedge_{(u,v) \in E_u} \left[ \sup_{\rho(u,v), \rho(v_i,v_j) > 0} \rho(u,v) \right]$$

Therefore $\tilde{x}_1 + \tilde{x}_2$ is a fuzzy labelling for $\tilde{0} + 0$.

$$(0.0)(\mu)((0.0)(v_i)) = \sup_{u: (0.0)(u) = (0.0)(v_i)} \mu(u) \geq \sup_{u: 0(u) = 0(v_i), 0(u) = 0(v_i)} \mu(u)$$

since $0(u) = \sum_{s: (u,s) \in supp(\rho)} 0(u)0(s)$

Hence as in the previous case,

$$(0.0)(\mu)((0.0)(v_i)) \geq \bigwedge_{\rho(u,v), \rho(v_i,v_j) > 0} \left[ \sup_{\rho(u,v), \rho(v_i,v_j) > 0} \rho(u,v) \right]$$
Therefore $\widetilde{x}_1, \widetilde{x}_2$ is a fuzzy labelling for $\widetilde{0}$. 

\[
(f0)(\mu)((f0)(v_i)) = \sup_{u: (f0)(u) = (f0)(v_i)} \mu(u) \\
\geq \sup_{u: f0(u) = f0(v_i)} \mu(u)
\]

As in the previous case,

\[
\sup_{u: f0(u) = f0(v_i)} \mu(u) \geq \bigwedge \{ \sup \{ \rho(u, v) \} | (u, v) : \sum_{(u,v) \in E_u} (fx_1)(u, v) = \sum_{(v_i,v_j) \in E_{v_i}} (fx_1)(v_i, v_j) \} \]

Therefore $\widetilde{fx_1}$ is a fuzzy labelling for $\widetilde{f}0$.

So $FI_{L(0)}(V, F')$ is a fuzzy algebra of $I(supp(\mu), F)$ of $\tilde{G}^*$. \qed

**Theorem 6.1.3.** Let $F$ be an integral domain. The set $FI_{L(\lambda)}(V, F)$ of fuzzy labellings for $\tilde{\lambda}j$ of a fuzzy graph $\tilde{G} = (V, \mu, \rho)$ of $G$ is a fuzzy algebra of the incidence algebra $I(supp(\mu), F')$ of $\tilde{G}^* = (supp(\mu), supp(\rho))$.

**Proof.** Let $\tilde{x}_1, \tilde{x}_2$ be fuzzy labellings for the fuzzy admissible index vectors $\tilde{\lambda}_1, \tilde{\lambda}_2$. Then by definition $x_1$ and $x_2$ are labelings for $\lambda_1$ and $\lambda_2$ in $\tilde{G}^*$. So from Theorem 5.2.2, $x_1 + x_2, x_1.x_2$ and $fx_1$ are labellings for $\lambda_1 + \lambda_2, \lambda_1.\lambda_2$ and $f\lambda_1$ respectively.

\[
(\lambda_1 + \lambda_2)(\mu)((\lambda_1 + \lambda_2)(v_i)) = \sup_{u: (\lambda_1 + \lambda_2)(u) = (\lambda_1 + \lambda_2)(v_i)} \mu(u) \\
\geq \sup_{u: \lambda_1(u) = \lambda_1(v_i), \lambda_2(u) = \lambda_2(v_i)} \mu(u)
\]

But

\[
\lambda_1(u) = \sum_{(u,v) \in E_u} x_1(u, v), \lambda_1(v_i) = \sum_{(v_i,v_j) \in E_{v_i}} x_1(v_i, v_j)
\]

\[
\lambda_2(u) = \sum_{(u,v) \in E_u} x_2(u, v), \lambda_2(v_i) = \sum_{(v_i,v_j) \in E_{v_i}} x_2(v_i, v_j)
\]

where $E_u$ and $E_{v_i}$ are the sets of edges incident with $u$ and $v_i$ respectively in $\tilde{G}^*$.

Hence
\[
\sup_{u: \lambda_1(u) = \lambda_1(v_i), \lambda_2(u) = \lambda_2(v_i)} \mu(u) \geq \bigwedge \left[ \sup_{\rho(u,v)} \rho(u,v) > 0 \left\{ \sum_{(u,v) \in E_u} (x_1 + x_2)(u, v) = \sum_{(v_i,v_j) \in E_{v_i}} (x_1 + x_2)(v_i, v_j) \right\} \right]
\]

Therefore \(\tilde{x}_1 + \tilde{x}_2\) is a fuzzy labelling for \(\tilde{\lambda}_1 + \tilde{\lambda}_2\).

\[
(\lambda_1, \lambda_2)(\mu)((\lambda_1, \lambda_2)(v_i)) = \sup_{u: (\lambda_1, \lambda_2)(u) = (\lambda_1, \lambda_2)(v_i)} \mu(u) \geq \sup_{u: \lambda_1(u) = \lambda_1(v_i), \lambda_2(u) = \lambda_2(v_i)} \mu(u)
\]

since \((\lambda_1, \lambda_2)(u) = \sum_{x: (u, x) \in \text{supp}(\rho)} \lambda_1(u) \lambda_2(s)\)

Hence as in the previous case,

\[
(\lambda_1, \lambda_2)(\mu)((\lambda_1, \lambda_2)(v_i)) \geq \bigwedge \left[ \sup_{\rho(u,v)} \rho(u,v) > 0 \left\{ \sum_{(u,v) \in E_u} (x_1 + x_2)(u, v) = \sum_{(v_i,v_j) \in E_{v_i}} (x_1 + x_2)(v_i, v_j) \right\} \right]
\]

Therefore \(\tilde{x}_1 + \tilde{x}_2\) is a fuzzy labelling for \(\tilde{\lambda}_1 \tilde{\lambda}_2\).

\[
(f \lambda_1)(\mu)((f \lambda_1)(v_i)) = \sup_{u: (f \lambda_1)(u) = (f \lambda_1)(v_i)} \mu(u) \geq \sup_{u: f \lambda_1(u) = f \lambda_1(v_i)} \mu(u)
\]

As in the previous case,

\[
\sup_{u: f \lambda_1(u) = f \lambda_1(v_i)} \mu(u) \geq \bigwedge \left[ \sup_{\rho(u,v)} \rho(u,v) > 0 \left\{ \sum_{(u,v) \in E_u} (f x_1)(u, v) = \sum_{(v_i,v_j) \in E_{v_i}} (f x_1)(v_i, v_j) \right\} \right]
\]

Therefore \(\tilde{f} \tilde{x}_1\) is a fuzzy labelling for \(\tilde{f} \lambda_1\).

So \(FI_{L(\lambda)}(V,F)\) is a fuzzy algebra of the incidence algebra \(I(\text{supp}(\mu), F)\) of \(\tilde{G}^*\).

6.2 Fuzzy Incidence

Now we extend the concept of labelling to fuzzy graphs through another approach similar to the \((d,f)\)-extended coloring of fuzzy graphs introduced by Muñoz et al.[66],
Ramaswamy and Poornima[54] and Poornima and Ramaswamy [53]. First we introduce the term fuzzy incidence of a fuzzy graph as follows.

**Definition 6.2.1.** Let $\tilde{G} = (\mu, \rho)$ be a fuzzy graph of $G$. Let $\psi : V \times (V \times V) \rightarrow [0, 1]$ such that $\psi(v_i, e_j) \leq \mu(v_i) \land \rho(e_j) \forall v_i \in V, e_j \in V \times V$. Then $\psi$ is the fuzzy incidence of $\tilde{G}$.

Denote $\psi(v_i, e_j)$ as $\psi_{ij}$. We can represent $\psi_{ij}$ as a matrix as follows.

**Definition 6.2.2.** Let $\tilde{G} = (\mu, \rho)$ be a fuzzy graph of $G$. Let $\psi : V \times (V \times V) \rightarrow [0, 1]$ such that $\psi(v_i, e_j) \leq \mu(v_i) \land \rho(e_j) \forall v_i \in V, e_j \in V \times V$. Then the matrix $(\psi_{ij})$ is the matrix of fuzzy incidence of $\tilde{G}$ denoted by $FI(\tilde{G})$.

$\begin{align*}
\begin{bmatrix}
  e_1 & e_2 & \ldots & e_q \\
v_0 & \psi_{01} & \psi_{02} & \ldots & \psi_{0q} \\
v_1 & \psi_{11} & \psi_{12} & \ldots & \psi_{1q} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
v_{p-1} & \psi_{(p-1)1} & \psi_{(p-1)2} & \ldots & \psi_{(p-1)q}
\end{bmatrix}
\end{align*}$

**Example 1**

Let $\tilde{G} = (\mu, \rho)$ be a fuzzy graph of the graph $G = (V, V \times V)$. Let $\mu(x_0) = 0.8, \mu(x_1) = 0.9, \mu(x_2) = 0.6, \mu(x_3) = 0.5, \mu(x_4) = 0.6, \mu(x_5) = 0.7,$

$\rho(x_0, x_1) = \rho(e_1) = 0.8, \rho(x_0, x_2) = \rho(e_2) = 0.5, \rho(x_3, x_4) = \rho(e_3) = 0.4, \rho(x_1, x_2) = \rho(e_4) = 0.6, \rho(x_4, x_5) = \rho(e_5) = 0.5, \rho(x_2, x_3) = \rho(e_6) = 0.3, \rho(x_0, x_5) = \rho(e_7) = 0.5.$

Fuzzy incidence is given by

$\begin{align*}
\psi(x_0, e_1) &= 0.7, \psi(x_0, e_2) = 0.5, \psi(x_0, e_3) = 0, \psi(x_0, e_4) = 0, \psi(x_0, e_5) = 0, \psi(x_0, e_6) = 0, \psi(x_0, e_7) = 0.4, \psi(x_1, e_1) = 0.7, \psi(x_1, e_2) = 0, \psi(x_1, e_3) = 0, \psi(x_1, e_4) = 0.6, \psi(x_1, e_5) = \ldots
\end{align*}$
0, \psi(x_1, e_6) = 0, \psi(x_1, e_7) = 0, \psi(x_2, e_1) = 0, \psi(x_2, e_2) = 0.4, \psi(x_2, e_3) = 0, \psi(x_2, e_4) = 0.6, \psi(x_2, e_5) = 0, \psi(x_2, e_6) = 0.2, \psi(x_2, e_7) = 0, \psi(x_3, e_1) = 0, \psi(x_3, e_2) = 0, \psi(x_3, e_3) = 0.2, \psi(x_3, e_4) = 0, \psi(x_3, e_5) = 0, \psi(x_3, e_6) = 0.3, \psi(x_3, e_7) = 0, \psi(x_4, e_1) = 0, \psi(x_4, e_2) = 0, \psi(x_4, e_3) = 0.3, \psi(x_4, e_4) = 0, \psi(x_4, e_5) = 0.5, \psi(x_4, e_6) = 0, \psi(x_4, e_7) = 0, \psi(x_5, e_1) = 0, \psi(x_5, e_2) = 0, \psi(x_5, e_3) = 0, \psi(x_5, e_4) = 0, \psi(x_5, e_5) = 0.3, \psi(x_5, e_6) = 0, \psi(x_5, e_7) = 0.3.

\[
FI(\tilde{G}) = \begin{bmatrix}
0.7 & 0.5 & 0 & 0 & 0 & 0 & 0.4 \\
0.7 & 0 & 0 & 0.6 & 0 & 0 & 0 \\
0 & 0.4 & 0 & 0.6 & 0 & 0 & 0.2 \\
0 & 0 & 0.2 & 0 & 0 & 0.3 & 0 \\
0 & 0.3 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.3 & 0 & 0.3 & 0 
\end{bmatrix}
\]

We introduce a new concept based on this.

**Definition 6.2.3.** Let \( \hat{G} = (\mu, \rho, \psi) \) be a fuzzy graph of \( G \) where \( V = \{v_0, v_1, ..., v_{p-1}\} \), \( \mu : V \rightarrow [0,1] \), \( \rho : V \times V \rightarrow [0,1] \) and \( \psi : V \times (V \times V) \rightarrow [0,1] \) is called a fuzzy incidence graph.

**Remark 6.2.1.** Usual fuzzy graphs are fuzzy incidence graphs with \( \psi(v_i, e_j) = \rho(e_j) \) for each \( v_i \in V, e_j \in V \times V \).

### 6.3 \((na, ni)\)-extended J-labelling

On the lines of \((d, f)\)-extended coloring of fuzzy graph introduced by Muñoz et al.\,[66], Ramaswamy and Poornima \,[54]and Poornima and Ramaswamy \,[53], we define a \((na, ni)\)-extended J-labelling (J-labelling denotes the labeling discussed in \,[34]) of a fuzzy graph.
Let $\tilde{G} = (\mu, \rho)$ be a fuzzy graph of $G$ Let $R = \{r : V \rightarrow F\}$, $X = \{x : V \times V \rightarrow F\}$. Define a distance (non-admissibility) measure $na$ as $na : R \times X \rightarrow [0, \infty)$ with

1. $na(r(v_i), \sum_{e_j \in E_i} x(e_j)) \geq 0 \forall v_i \in V, e_j \in V \times V$

2. $na(r(v_i), \sum_{e_j \in E_i} x(e_j)) = 0 \forall v_i \in V, e_j \in V \times V$ iff $r$ is admissible with labelling $x$.

3. $na(r(v_i), \sum_{e_j \in E_i} x(e_j)) = na(\sum_{e_j \in E_i} x(e_j), r(v_i)) \forall v_i \in V, e_j \in V \times V$.

Define a non-negative non-decreasing real scale function (non-incidence) $ni$ as $ni : I_\psi \rightarrow [0, \infty)$ where $I_\psi = \{\psi_{ij} : v_i \in V, e_j \in V \times V\}$.

**Definition 6.3.1.** Let $\tilde{G} = (\mu, \rho)$ be a fuzzy graph of $G$, $R$ and $X$ be the sets of index vectors and labellings of $G$, $na$ is the non-admissibility measure and $ni$ is a scale function. Let $x_{na,ni} : V \times V \rightarrow F$ and $r_{na,ni} : V \rightarrow F$ with

$$na(r_{na,ni}, \sum_{\rho(e_j) > 0} x_{na,ni}) \leq ni(1 - \psi_{ij}) \forall v_i \in V, e_j \in V \times V$$

where $F$ is an integral domain and $\psi_{ij}$ is the fuzzy incidence of vertex $v_i$ with edge $e_j$ of $\tilde{G}$. Then $x_{na,ni}$ is an $(na, ni)$-extended $J$-labelling of $(na, ni)$-extended index vector $r_{na,ni}$ for $\tilde{G}$.

Note that in a fuzzy graph, $ni(1 - \psi_{ij}) = ni(1 - \rho(e_j))$.

The $(na, ni)$-extension of $J$-labelling to fuzzy graphs is particularly useful like the $(d, f)$-extended coloring of a fuzzy graph. The speciality of $J$-labelling is the admissibility condition $r(v_i) = \sum_{m \in E_i} x(m)$ where $E_i$ is the set of edges incident with vertex $v_i$. This is made use of in $(na, ni)$-extended $J$-labellings. Further it takes into consideration, the fuzzy incidence.