CHAPTER 3

GORENSTEIN $n$-FLAT MODULES AND THEIR COVERS

3.1 Introduction

Enochs et al. [18] first introduced and studied Gorenstein flat modules over Gorenstein rings (that is, noetherian rings with finite self-injective dimension). One of the most interesting results is that over a right coherent ring the class of Gorenstein flat left modules and its right orthogonal class form a complete hereditary cotorsion pair, which is due to Enochs, Jenda and López-Ramos [17]. Since the class of Gorenstein flat modules over a right coherent ring is closed under direct limits, one can get further that all left modules over a right coherent ring have Gorenstein flat covers. The existence of Gorenstein flat covers was first proved for modules over Gorenstein rings in [21].

Recently, in [24] Gang et al. proved all modules over a left $GF$-closed ring have Gorenstein flat covers. We proved all modules have $n$-flat covers.
in Chapter 2. These motivate us to introduce the notion of Gorenstein \( n \)-flat module and its cover. In this chapter, we show that all left \( R \)-modules over right \( n \)-coherent ring have Gorenstein \( n \)-flat covers. This chapter is organized as follows. In Section 3.3, we prove that over a right \( n \)-coherent ring, every direct limit of Gorenstein \( n \)-flat modules is again Gorenstein \( n \)-flat module. Also, we introduce the notion of Gorenstein \( n \)-absolutely pure right \( R \)-modules and study the relation between them and we show that over a right \( n \)-coherent ring, any pure submodule of a Gorenstein \( n \)-flat module is Gorenstein \( n \)-flat. In Section 3.4, we prove given a ring \( R \), the class of all Gorenstein \( n \)-flat left \( R \)-modules is a Kaplansky class and also we prove that all modules over a right \( n \)-coherent ring have Gorenstein \( n \)-flat covers and show that over a right \( n \)-coherent ring, Gorenstein \( n \)-flat cover of \( M \) is an \( n \)-flat cover of \( M \).

3.2 Gorenstein \( n \)-flat modules

First, we introduce the definition of Gorenstein \( n \)-flat module as follows:

**Definition 3.2.1.** A left \( R \)-module \( M \) is said to be Gorenstein \( n \)-flat, if there exists an exact sequence of \( n \)-flat left \( R \)-modules,

\[
\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots
\]

such that \( M \cong \text{Im}(F_0 \to F^0) \) and such that \( E \otimes - \) leaves the sequence exact whenever \( E \) is an \( n \)-absolutely pure right \( R \)-module. The class of all Gorenstein \( n \)-flat left \( R \)-modules is denoted by \( GF_n(R) \).

**Example 3.2.2.** (i) Any \( n \)-flat module is Gorenstein \( n \)-flat because there is an exact complex

\[
X = \cdots \to 0 \to F \xrightarrow{id} F \to 0 \to 0 \to \cdots
\]
with $F$ being $n$-flat and such that for every $n$-absolutely pure module $E$, the complex 

$$E \otimes X = 0 \longrightarrow E \otimes F = E \otimes F \longrightarrow 0$$

is exact.

(ii) In general, Gorenstein $n$-flat module need not be Gorenstein flat. For example, let $R$ be commutative domain and let $M$ be a Gorenstein 1-flat module. Then, by the definition of Gorenstein 1-flat module there exists an exact sequence of 1-flat left $R$-modules,

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

such that $M \cong \text{Im}(F_0 \rightarrow F^0)$ and such that $E \otimes -$ leaves the sequence exact whenever $E$ is a 1-absolutely pure right $R$-module. We know that 1-flat modules are exactly torsion free by Lemma 1.0.4 and not every torsion free module is flat. Further, 1-absolutely pure right $R$-modules are divisible by Lemma 1.0.5 and divisible modules need not be injective. Therefore, $M$ need not be Gorenstein flat. But however it is true when $R$ is a PID.

**Proposition 3.2.3.** If $(M_i)_{i \in I}$ is a family of Gorenstein $n$-flat modules, then $\bigoplus M_i$ is a Gorenstein $n$-flat module.

**Proof.** Since direct sum of $n$-flat modules is $n$-flat and tensor products commute with direct sum, the result follows. \Box

**Lemma 3.2.4.** Let $M$ be a Gorenstein $n$-flat left $R$-module. Then, $\text{Tor}_i(E, M) = 0$ for all $n$-absolutely pure right $R$-modules $E$ and all $i \geq 1$. The converse is true when $R$ is right $n$-coherent.
Proof. By hypothesis, there is a $E \otimes -$ exact exact sequence $\cdots \to F_1 \to F_0 \to M \to 0$ with each $F^i$ an $n$-flat module. Thus, $\text{Tor}_i(E, M) = 0$ for all $n$-absolutely pure right $R$-modules $E$ and all $i \geq 1$ by definition. The converse follows from Theorem 1.0.37.

By Lemma 3.2.4, we immediately have the following result:

Corollary 3.2.5. Suppose $R$ is a right $n$-coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of left $R$-modules. Then, if $A$ and $C$ are Gorenstein $n$-flat, so is $B$. If $B$ and $C$ are Gorenstein $n$-flat, so is $A$. If $A$ and $B$ are Gorenstein $n$-flat, then $C$ is Gorenstein $n$-flat if and only if $0 \to E \otimes A \to E \otimes B$ is exact for any $n$-absolutely pure right module $E$.

Lemma 3.2.6. Let $R$ be a right $n$-coherent ring. Then, the following are equivalent:

(i) $M$ is a Gorenstein $n$-flat $R$-module.

(ii) There exists an exact sequence $\mathcal{F} = 0 \to M \to F^0 \to F^1 \to \cdots$ with each $F^i$ $n$-flat such that $E \otimes \mathcal{F}$ leaves the sequence exact for any $n$-absolutely pure right module $E$.

Proof. $(i) \Rightarrow (ii)$. Clear from the definition of Gorenstein $n$-flat $R$-module.

$(ii) \Rightarrow (i)$. Follows from Lemma 1.0.22.

Next, we give an important result about direct limits.

Theorem 3.2.7. If $R$ is right $n$-coherent and if $M_0 \to M_1 \to M_2 \to \cdots$ is a sequence of Gorenstein $n$-flat modules, then the direct limit $\varinjlim M_m$ is again Gorenstein $n$-flat.
Proof. For each integer $m$, consider the right $n$-flat resolution $F_m$ of $M_m$ as follows:

\[
\begin{array}{cccccc}
F_0 & \rightarrow & 0 & \rightarrow & M_0 & \rightarrow & F_0^0 & \rightarrow & F_1^1 & \rightarrow & F_0^2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
F_1 & \rightarrow & 0 & \rightarrow & M_1 & \rightarrow & F_1^0 & \rightarrow & F_1^1 & \rightarrow & F_1^2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\end{array}
\]

For each map $M_m \rightarrow M_{m+1}$ can be lifted to a chain map $F_m \rightarrow F_{m+1}$ of complexes. Since we are dealing with sequences (and not arbitrary direct systems), each column in the diagram is again a direct system. Thus, it makes sense to apply the exact functor $\varprojlim$ to the above diagram, and doing so, we obtain an exact complex,

\[
F = \varprojlim F_m = 0 \rightarrow \varprojlim M_m \rightarrow \varprojlim F_m^0 \rightarrow \varprojlim F_m^1 \rightarrow \cdots ,
\]

where each module $F^k = \varprojlim F_m^k$, $k = 0, 1, 2, \ldots$ is an $n$-flat module. Note that when $E$ is an $n$-absolutely pure right $R$-module, then by Lemma 3.2.6, $E \otimes F_m$ is exact since $E^+ = \text{Hom}_\mathbb{Z}(E, \mathbb{Q}/\mathbb{Z})$ is an $n$-flat left $R$-module. Since $\varprojlim$ commutes with the homology functor, we also get exactness of

\[
E \otimes F \cong \varprojlim (E \otimes F_m).
\]

Thus, we have constructed the “right half", $F$, of a complete $n$-flat resolution for $\varprojlim M_m$. Since $M_m$ is Gorenstein $n$-flat, we also have

\[
\text{Tor}_i(E, \varprojlim M_m) \cong \varprojlim \text{Tor}_i(E, M_m) = 0
\]

for $i > 0$, and all $n$-absolutely pure right modules $E$. Thus $\varprojlim M_m$ is Gorenstein $n$-flat. \qed
Next, we introduce the notion of Gorenstein $n$-absolutely pure module as follows.

**Definition 3.2.8.** A right $R$-module $M$ is said to be Gorenstein $n$-absolutely pure, if there exists an exact sequence of $n$-absolutely pure right $R$-modules
\[ \cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots \]
such that $M \cong \text{Im}(A_0 \to A^0)$ and such that $\text{Hom}(E, -)$ leaves the sequence exact whenever $E$ is an $n$-absolutely pure right $R$-module.

**Remark 3.2.9.** (i) The class of Gorenstein $n$-absolutely pure right $R$-modules is closed under direct products by definition.

(ii) Clearly, $M$ is a Gorenstein $n$-flat left $R$-module if and only if the character module $M^+$ is a Gorenstein $n$-absolutely pure right $R$-module.

(iii) Let $R$ be commutative and $n$-coherent ring. If $M$ is a Gorenstein $n$-absolutely pure right $R$-module, then the character module $M^+$ is a Gorenstein $n$-flat left $R$-module.

**Lemma 3.2.10.** Let $M$ be a Gorenstein $n$-absolutely pure right $R$-module. Then, $\text{Ext}^i(E, M) = 0$ for all $n$-absolutely pure right $R$-modules $E$ and all $i \geq 1$. The converse is true when $R$ is right $n$-coherent.

**Proof.** By hypothesis, there is a $\text{Hom}(E, -)$ exact exact sequence $0 \to M \to A^0 \to A^1 \to \cdots$ with each $A^i$ is $n$-absolutely pure. Thus, $\text{Ext}^i(E, M) = 0$ for all $n$-absolutely pure right $R$-modules $E$ and all $i \geq 1$ by definition. The converse follows from Theorem 1.0.48. \qed

**Corollary 3.2.11.** Let $R$ be a right $n$-coherent ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ an exact sequence of right $R$-modules. Then:
(i) If $M_1$ and $M_3$ are Gorenstein $n$-absolutely pure, so is $M_2$.

(ii) If $M_1$ and $M_2$ are Gorenstein $n$-absolutely pure, so is $M_3$.

(iii) If $M_2$ and $M_3$ are Gorenstein $n$-absolutely pure, then $M_1$ is Gorenstein $n$-absolutely pure if and only if $\text{Ext}^1(E, M_1) = 0$ for all $n$-absolutely pure right $R$-modules $E$.

Thus, the class of Gorenstein $n$-absolutely pure right $R$-modules is closed under direct summands.

Proof: (i), (ii) and (iii) follows from Lemma 3.2.10. The last statement holds by (i), (ii), Remark 3.2.9 (i) and Proposition 1.0.24.

We are now going to give a connection between Gorenstein $n$-flat modules and Gorenstein $n$-absolutely pure modules.

Theorem 3.2.12. Let $R$ be right $n$-coherent then the following statements are equivalent:

(i) $M$ is a Gorenstein $n$-flat left $R$-module.

(ii) There is a $E \otimes -$ exact exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of $n$-flat left $R$-modules such that $M = \text{Ker}(F^0 \to F^1)$ where $E$ is any $n$-absolutely pure right $R$-module.

(iii) $M^+$ is a Gorenstein $n$-absolutely pure right $R$-module.

Proof: (i) $\Rightarrow$ (ii). Clear from the definition of a Gorenstein $n$-flat module.

(ii) $\Rightarrow$ (iii). Follows from Remark 3.2.9 (ii).

(iii) $\Rightarrow$ (i). Follows from Theorem 1.0.37 since $M^+$ is a Gorenstein $n$-absolutely pure $R$-module.
Corollary 3.2.13. Assume that $R$ is right $n$-coherent, and consider a short exact sequence of left $R$-modules $0 \to G' \to G \to M \to 0$, where $G$ and $G'$ are Gorenstein $n$-flat modules. If $\text{Tor}_1(E, M) = 0$ for all $n$-absolutely pure right $R$-modules $E$, then $M$ is Gorenstein $n$-flat.

Proof. Define $G^+ = \text{Hom}_\mathbb{Z}(G, \mathbb{Q}/\mathbb{Z})$ and $G'^+ = \text{Hom}_\mathbb{Z}(G', \mathbb{Q}/\mathbb{Z})$ which are Gorenstein $n$-absolutely pure module by Theorem 3.2.12. Apply $\text{Hom}_\mathbb{Z}(\cdot, \mathbb{Q}/\mathbb{Z})$ to the exact sequence $0 \to G' \to G \to M \to 0$, we have an exact sequence

$$0 \to M^+ \to G^+ \to G'^+ \to 0$$

and noting that we have an isomorphism,

$$\text{Ext}^1(E, \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_\mathbb{Z}(\text{Tor}_1(E, M), \mathbb{Q}/\mathbb{Z}) = 0$$

for all $n$-absolutely pure right $R$-modules $E$. Therefore, $M^+$ is Gorenstein $n$-absolutely pure. Since $R$ is right $n$-coherent, we conclude that $M$ is Gorenstein $n$-flat by Theorem 3.2.12. \qed

Corollary 3.2.14. Let $R$ be commutative and $n$-coherent. Then, the following are equivalent:

(i) Every Gorenstein $n$-flat left $R$-module is $n$-flat.

(ii) Every Gorenstein $n$-absolutely pure right $R$-module is $n$-absolutely pure.

Proof. (i) $\Rightarrow$ (ii). Let $M$ be a Gorenstein $n$-absolutely pure right $R$-module. Then, $M^+$ is a Gorenstein $n$-flat left $R$-module by Remark 3.2.9 (iii). So $M^+$ is $n$-flat by (i). Thus, $M$ is $n$-absolutely pure by Theorem 1.0.10.
Let $M$ be a Gorenstein $n$-flat left $R$-module. Then, $M^+$ is a Gorenstein $n$-absolutely pure right $R$-module by Theorem 3.2.12 and so $M^+$ is $n$-absolutely pure by (ii). Thus, $M$ is $n$-flat.

Next, we show that any pure submodule of a Gorenstein $n$-flat left $R$-module is a Gorenstein $n$-flat over a right $n$-coherent ring.

**Lemma 3.2.15.** Let $R$ be a right $n$-coherent ring. Then, any pure submodule of a Gorenstein $n$-flat left $R$-module is a Gorenstein $n$-flat.

**Proof.** Let $N$ be a pure submodule of a Gorenstein $n$-flat left $R$-module $M$. Then, the pure exact sequence $0 \to N \to M \to M/N \to 0$ induces a split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. So $N^+$ is Gorenstein $n$-absolutely pure by Corollary 3.2.11 since $M^+$ is Gorenstein $n$-absolutely pure by Theorem 3.2.12. Thus, $N$ is Gorenstein $n$-flat by Theorem 3.2.12 again.

**Proposition 3.2.16.** Let $R$ be a commutative and $n$-coherent ring. Then,

(i) $M$ is a Gorenstein $n$-flat left $R$-module if and only if $M^{++}$ is Gorenstein $n$-flat.

(ii) $M$ is a Gorenstein $n$-absolutely pure right $R$-module if and only if $M^{++}$ is Gorenstein $n$-absolutely pure.

**Proof.** It follows from Remark 3.2.9 (ii) and (iii).

### 3.3 Gorenstein $n$-flat covers

The aim of this section is to prove the existence of Gorenstein $n$-flat covers for every module over right $n$-coherent ring.

The following result is due to Proposition 1.0.14.
Proposition 3.3.1. Let $\mathcal{X}$ be a class of modules. In addition, suppose $\mathcal{X}$ is closed under direct limits. Then, for a left $R$-module $M$, the existence of an $\mathcal{X}$-precover of $M$ implies the existence of an $\mathcal{X}$-cover.

By Theorem 3.2.7 and Proposition 3.3.1, in order to find Gorenstein $n$-flat cover for a module $M$, we only need to find Gorenstein $n$-flat precover of $M$. Next, we see the definition of Gorenstein $n$-cotorsion module as follows:

Definition 3.3.2. A left $R$-module $L$ is called Gorenstein $n$-cotorsion if $\Ext^1(G, L) = 0$ for all Gorenstein $n$-flat left $R$-modules $G$.

Enochs and López-Ramos proved that the class of Gorenstein injective left modules over a left noetherian ring is a Kaplansky class by Proposition 1.0.36. Now, we prove the following proposition by using the same technique as in Proposition 1.0.36.

Proposition 3.3.3. Given a ring $R$, the class of all Gorenstein $n$-flat left $R$-modules is a Kaplansky class.

Proof. Let $M \in \mathcal{GF}_n$ and $x \in M$. We want to show that there is a Gorenstein $n$-flat submodule $S$ of $M$ containing $x$ such that $S$ and $M/S$ are Gorenstein $n$-flat.

We first recall that $M \in \mathcal{GF}_n$ means that there exist an exact sequence

$$
\cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{f} M \to 0,
$$

where every $F_i$'s are $n$-flat left $R$-module and such that $M = \text{Im}(F_0 \to F^0)$ and remains exact whenever $E \otimes -$ is applied for any $n$-absolutely pure right $R$-module $E$. Now since $F_0 \to M$ is surjective, there is $y \in F_0$ such that $f(y) = x$. Then, consider $\langle y \rangle \to F_0$ the inclusion and we have by Lemma
1.0.29, there is a cardinal \( N_0 \) and a pure submodule \( S_0 \subseteq F_0 \) such that \( \langle y \rangle \subseteq S_0 \) and \( \text{Card}(S_0) \leq N_0 \). Now, consider \( S_0 \cap \text{Ker}(f) \subseteq M \). Then, there exists \( D_1 \subseteq F_1 \) such that \( d_1(D_1) = S_0 \cap \text{Ker}(f) \). Again by Lemma 1.0.29, there is a pure submodule \( S_1 \) in \( F_1 \) and a cardinal \( N_1 \) such that \( D_1 \subseteq S_1 \) and \( \text{Card}(S_1) \leq N_1 \). Now, consider \( d_1(S_1) \subseteq F_0 \). Then, there exist a pure submodule \( S_2 \) in \( F_0 \) and a cardinal \( N_2 \) such that \( d_1(S_1) \subseteq S_2 \) and \( \text{Card}(S_2) \leq N_2 \). We now consider \( f(S_2) \) and \( S_2 \cap \text{Ker}(f) \subseteq F_0 \), then there exists \( D_2 \subseteq F_1 \) such that \( d_1(D_2) = S_2 \cap \text{Ker}(f) \). By Lemma 1.0.29, there is a pure submodule \( S_3 \) in \( F_1 \) and a cardinal \( N_3 \) such that \( D_2 \subseteq S_3 \) and \( \text{Card}(S_3) \leq N_0 \).

Now, start the process again going back by considering \( S_2 \cap \text{Ker}(f) \) and proceeding as before, going \( n \) steps forward, going back \( n + 1 \) steps and \( n + 1 \) forward again. Then, we take the union of all complexes constructed in the "zag-zig" process

\[
S^* = \cdots \rightarrow S_1 \rightarrow S_0 \rightarrow S \rightarrow 0,
\]

where \( S = f(S_0) \subseteq M \), which contains the element \( x \) and that by the construction, there exists a cardinal \( N \) such that \( \text{Card}(S) \leq N \). The previous complex is exact by its construction and it is formed by \( n \)-flat modules since all of them are pure submodules of \( n \)-flat modules.

Our aim is to construct \( S \subseteq M \) with an exact sequence as before and such that it remains exact when \( E \otimes - \) is applied to it for every \( n \)-absolutely module \( E \). By Corollary 1.0.21 every \( n \)-absolutely pure module \( E \) may be expressed as the direct limit of a family of \( n \)-absolutely pure submodules. Then, we consider the module \( I = \bigoplus E_i \). Now, if a sequence is such that \( I \otimes - \) leaves it exact, by the commutativity of the tensor products with direct sums, we get that \( E_i \otimes - \) will also leave the sequence exact and from the commutativity of direct limits.
and the tensor products our sequence will remain exact under $E \otimes -$ for every $n$-absolutely pure module $E$. Let us consider the complex

$$
\cdots \xrightarrow{d_1} I \otimes S_1 \xrightarrow{d_2} I \otimes S_0 \xrightarrow{f'} I \otimes S \to 0.
$$

This complex is a subcomplex of

$$
\cdots \xrightarrow{d_1} I \otimes F_1 \xrightarrow{d_2} I \otimes F_0 \xrightarrow{f} I \otimes M \to 0,
$$

which is exact since $M \in \mathcal{GF}_n$ and $I$ is a direct sum of $n$-absolutely pure modules. Suppose without lost of generality that $\text{Ker}(f') \neq \text{Im}(d_1')$, then there is a pure submodule $S_1'$ of $F_1$ and a cardinal $N_1$ such that $S_1 \subseteq S_1'$, $\text{Ker}(f') \subseteq \text{Im}(d_1|_{I \otimes S_1'})$ and $\text{Card}(S_1') \leq N_1$. Now, let $S'$ be the image of $S_1'$ under the morphism $F_1 \to F_0$, and let $S_0'$ be a pure submodule of $F_0$ and $N_2$ be a cardinal such that $S' \subseteq S_0'$, $\text{Im}(d_1|_{I \otimes S_1'}) \subseteq I \otimes S_0'$, and $\text{Card}(S_0') \leq N_2$. Then, let $S'$ be the image of $S_0'$ under the morphism $F_0 \to M$. Then, we go back again and start another “zig-zag” process with $\text{Ker}(f|_{I \otimes S_0'})$ and $\text{Im}(d_1|_{I \otimes S_1'})$.

Now, take the union of these complexes formed in the “zig-zag” process above to get a complex

$$
T^* = \cdots \to T_1 \to T_0 \to T \to 0.
$$

By the construction above, there is a cardinal $N$ such that $\text{Card}(T) \leq N$ and when $E \otimes -$ is applied to the complex $T^*$ we get an exact complex. But $T^*$ may not be exact. So, we apply again the “zig-zag” process, we used to get $S^*$ to get another exact sequence $S^{**}$ which may not remain exact when $E \otimes -$ is applied. So again we apply the “zig-zag” process, we used to get $T^*$ to the exact sequence $S^{**}$ to get a new $T^{**}$ that may not be exact but remains exact when $E \otimes -$ is applied to it. The “limit” over these two procedures gives us a module $S$, a cardinal $N$, and a complex $S^*$ as we desired.
Finally, note that $M/S$ is also in $\mathcal{GF}_n$ since the quotient complex $F^*/S^*$ is exact and it remains exact when $E \otimes -$ is applied to it since $F^*$ and $S^*$ satisfy the two conditions.

We are now in a position to give the main result of this section, which extends Theorem 1.0.39.

**Theorem 3.3.4.** Let $R$ be a right $n$-coherent ring. Then, $(\mathcal{GF}_n, \mathcal{GF}_n^\perp)$ is a hereditary perfect cotorsion theory.

**Proof.** By Theorems 3.2.7, 1.0.31, and Proposition 3.3.3, we get that $(\mathcal{GF}_n, \mathcal{GF}_n^\perp)$ is a perfect cotorsion theory. Since the class of Gorenstein $n$-flat modules is projectively resolving by Corollary 3.2.5, then we get that $(\mathcal{GF}_n, \mathcal{GF}_n^\perp)$ is hereditary.

By Theorem 3.3.4, we immediately have the following results.

**Corollary 3.3.5.** All left modules over a right $n$-coherent ring have Gorenstein $n$-flat covers.

**Corollary 3.3.6.** If $R$ is right $n$-coherent, then every module has Gorenstein $n$-cotorsion envelope.

**Lemma 3.3.7.** If

$$0 \to U \to F \xrightarrow{\phi} M \to 0$$

is an exact sequence of left $R$-modules with $U$ Gorenstein $n$-cotorsion and $F$ Gorenstein $n$-flat then $F \to M$ is a Gorenstein $n$-flat precover of $M$. Conversely, if $\phi$ is a Gorenstein $n$-flat cover of $M$ then, $\text{Ker}\phi$ is Gorenstein $n$-cotorsion.
3.3. GORENSTEIN $n$-FLAT COVERS

Proof. Let $F' \in \mathcal{GF}_n$, we shall prove that $\text{Hom}(F', F) \rightarrow \text{Hom}(F', M) \rightarrow 0$ is exact. Applying $\text{Hom}(F', -)$ in the exact sequence, we have $\text{Hom}(F', F) \rightarrow \text{Hom}(F', M) \rightarrow \text{Ext}^1(F', U) = 0$ is exact, since $U$ is Gorenstein $n$-cotorsion. Hence, $M$ has a Gorenstein $n$-flat precover. Conversely, if $\phi : F \rightarrow M$ is a Gorenstein $n$-flat precover of $M$ then, $\text{Ker} \phi$ is Gorenstein $n$-cotorsion by Lemma 1.0.15. 

Lemma 3.3.8. Let $R$ be right $n$-coherent ring. Then, every $n$-cotorsion module is Gorenstein $n$-cotorsion.

Proof. It follows from Lemma 1.0.40.

Proposition 3.3.9. Let $R$ be a right $n$-coherent ring and $M$ an $R$-module then, any Gorenstein $n$-flat cover of $M$ is an $n$-flat cover of $M$.

Proof. By Corollary 3.3.5, $M$ has Gorenstein $n$-flat covers. Let $g : G \rightarrow M$ be a Gorenstein $n$-flat cover and $f : F \rightarrow M$ be an $n$-flat cover of $M$. Then, we have the short exact sequences $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ and $0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0$, where $K = \text{Ker} g$ and $C = \text{Ker} f$. By Lemma 3.3.7, $K$ is Gorenstein $n$-cotorsion and $C$ is $n$-cotorsion by Proposition 2.2.5. So $C$ is also Gorenstein $n$-cotorsion by Lemma 3.3.8. Thus, the sequence $0 \rightarrow \text{Hom}(G, C) \rightarrow \text{Hom}(G, F) \rightarrow \text{Hom}(G, M) \rightarrow 0$ is exact, this implies that there exists a homomorphism $\alpha : G \rightarrow F$ such that $g = f\alpha$.

On the other hand, there exists a homomorphism $\beta : F \rightarrow G$ such that $f = f\beta$ since $g : G \rightarrow M$ is a Gorenstein $n$-flat cover of $M$. Therefore, by the definition of covers, $\beta\alpha$ and $\alpha\beta$ are isomorphisms, this yields that $\alpha$ and $\beta$ are isomorphisms. Hence $G \cong F$. 

54