5. Fuzzy Hyperbolic Partial Differential Equations with Generalized Differentiability

5.1 Introduction

A new technique using an adaptive fuzzy algorithm to obtain the solutions to a class of partial differential equations is presented in [22]. Bertone et al. [15] investigated the heat, wave and Poisson equations as classical models of partial differential equations with uncertain parameters, considering the parameters as fuzzy numbers.

In this chapter, we consider a higher order finite difference method for the fuzzy hyperbolic differential equations under generalized differentiability. An implicit difference approximation scheme is developed. Numerical results are provided to verify the accuracy of the proposed algorithm.

5.2 Fuzzy Hyperbolic Partial Differential Equations

Consider the fuzzy hyperbolic partial differential equation
\[ D_{tt}u(x,t) = \beta^2 D_{xx}u(x,t), \quad 0 < x < L, \quad 0 < t < T, \quad (5.1) \]

with the initial and boundary conditions

\[ u(x,0) = f(x), \quad D_{t}u(x,0) = g(x), \quad 0 < x < L, \]
\[ u(0,t) = K_1, \quad u(L,t) = K_2, \quad t > 0, \quad (5.2) \]

where \( \beta^2, K_1 \) and \( K_2 \) are fuzzy numbers.

**Definition 5.1.** Let \( u : [0,L] \times [0,T] \rightarrow \mathbb{R}_F \) and \( (k,l,m,n) \in \{1,2\} \). If \( D^2_{t,n,k}u \) and \( D^2_{x,m,l}u \) exist on \([0,L] \times [0,T]\) and \( D^2_{t,n,k}u(x,t) = \beta^2 D^2_{x,m,l}u(x,t) \) for all \((x,t) \in [0,L] \times [0,T]\) and satisfy (5.2), then \( u \) is said to be an \((n,k,m,l)\) solution for the fuzzy hyperbolic equation (5.1)-(5.2) on \([0,L] \times [0,T]\).

**Definition 5.2.** Let \( u : [0,L] \times [0,T] \rightarrow \mathbb{R}_F \) and \( (k,l,m,n) \in \{1,2\} \) and \( J_1 \) be such that \( J_1 \subset [0,L] \times [0,T] \). If \( D^2_{t,n,k}u \) and \( D^2_{x,m,l}u \) exist on \( J_1 \) and \( D^2_{t,n,k}u(x,t) = \beta^2 D^2_{x,m,l}u(x,t) \), for all \((x,t) \in J_1\), then \( u \) is said to be an \((n,k,m,l)\) solution for the fuzzy hyperbolic equation (5.1) on \( J_1 \).

(a) In (5.1), if both \( D_{tt}u \) and \( D_{xx}u \) are differentiable in the first form (1) or both \( D_{tt}u \) and \( D_{xx}u \) are differentiable in the second form (2), then we have eight systems, namely, \((1,1,1,1)\)-system, \((1,1,2,2)\)-system, \((2,2,1,1)\)-system, \((2,2,2,2)\)-system, \((1,2,1,2)\)-system, \((1,2,2,1)\)-system, \((2,1,2,1)\)-system and \((2,1,1,2)\)-system and these systems have the following form:

\[ D_{tt}u(x,t,r) = \beta^2 D_{xx}u(x,t,r), \]
\[ D_{tt}\tilde{u}(x,t,r) = \beta^2 D_{xx}\tilde{u}(x,t,r), \quad 0 < x < L, \quad t > 0, \quad r \in [0,1]. \quad (5.3) \]

The initial and boundary conditions for \((1,1,1,1)\), \((1,1,2,2)\), \((1,2,1,2)\) and \((1,2,2,1)\)-systems are
\[ u(x, 0, r) = f(x, r), \quad \overline{u}(x, 0, r) = \overline{f}(x, r), \quad 0 < x < L, \quad r \in [0, 1], \]
\[ D_t u(x, 0, r) = g(x, r), \quad D_t \overline{u}(x, 0, r) = \overline{g}(x, r), \quad 0 < x < L, \quad r \in [0, 1], \]
\[ u(0, t, r) = K_3(r), \quad u(l, t, r) = K_4(r), \quad 0 < t < T, \quad r \in [0, 1], \]
\[ \overline{u}(0, t, r) = \overline{K}_3(r), \quad \overline{u}(l, t, r) = \overline{K}_4(r), \quad 0 < t < T, \quad r \in [0, 1]. \] \tag{5.4}

The initial and boundary conditions for \((2,2,1,1), (2,2,2,2), (2,1,1,2)\) and \((2,1,2,1)\)-systems are

\[ u(x, 0, r) = f(x, r), \quad \overline{u}(x, 0, r) = \overline{f}(x, r), \quad 0 < x < L, \quad r \in [0, 1], \]
\[ D_t u(x, 0, r) = g(x, r), \quad D_t \overline{u}(x, 0, r) = \overline{g}(x, r), \quad 0 < x < L, \quad r \in [0, 1], \]
\[ u(0, t, r) = K_3(r), \quad u(l, t, r) = K_4(r), \quad 0 < t < T, \quad r \in [0, 1], \]
\[ \overline{u}(0, t, r) = \overline{K}_3(r), \quad \overline{u}(l, t, r) = \overline{K}_4(r), \quad 0 < t < T, \quad r \in [0, 1]. \] \tag{5.5}

(b) In (5.1), if \(D_{tt}u\) is differentiable in the second form (2) and \(D_{xx}u\) is differentiable in the first form (1) or \(D_{tt}u\) is differentiable in the first form (1) and \(D_{xx}u\) is differentiable in the second form (2), then we have eight systems, namely, \((1,2,1,1)\)-system, \((1,2,2,2)\)-system, \((2,1,1,1)\)-system, \((2,1,2,2)\)-system, \((1,1,2,1)\)-system, \((1,1,1,2)\)-system, \((2,2,2,1)\)-system and \((2,2,1,2)\)-system and these systems have the following form:

\[ D_{tt} \overline{u}(x, t, r) = \beta^2 D_{xx} \overline{u}(x, t, r), \]
\[ D_{tt} u(x, t, r) = \beta^2 D_{xx} u(x, t, r), \quad 0 < x < L, \quad 0 < t < T, \quad r \in [0, 1]. \] \tag{5.6}

The initial and boundary conditions for both \((1,2,1,1), (1,2,2,2), (1,1,2,1)\) and \((1,1,1,2)\)-systems are as in (5.4) and the initial and boundary conditions for both \((2,1,1,1), (2,1,2,2), (2,2,2,1)\) and \((2,2,1,2)\)-systems are as in (5.5).
5.3 Finite Difference Method

Assume that $u$ is a fuzzy function of the independent crisp variables $x$ and $t$. Subdivide the $x-t$ plane into sets of equal rectangles of sides $h, k$, by equally spaced grid lines parallel to $O_t$, defined by $x_i = ih, i = 0, 1, 2, ..., m$, and equally spaced grid lines parallel to $O_x$, defined by $t_j = jk, j = 0, 1, 2, ..., n$, where $m$ and $n$ are positive integers with $h = \frac{L}{m}$ and $k = \frac{T}{n}$.

Denote the exact value of $u$ and the approximate value of $v$ at the mesh point $(ih, jk)$ by $u_{i,j}$ and $v_{i,j}$ respectively. Also denote the parametric form of the fuzzy number $u_{i,j}$ and $v_{i,j}$ by $(u_{i,j}(r), \overline{u}_{i,j}(r))$ and $(v_{i,j}(r), \overline{v}_{i,j}(r))$ respectively.

By Taylor’s theorem and definition of compact finite difference, we get the approximate solution as follows

$$D_{xx}v_{i,j}(r) \simeq \frac{\delta^2_x}{(1 + \frac{\delta^2_x}{12})} u_{i,j}(r),$$

$$D_{xx}u_{i,j}(r) \simeq \frac{\delta^2_x}{(1 + \frac{\delta^2_x}{12})} \overline{u}_{i,j}(r),$$

(5.7)

where $\delta^2_x u_{i,j}(r) = u_{i+1,j}(r) - 2u_{i,j}(r) + u_{i-1,j}(r)$ and $\delta^2_x \overline{u}_{i,j}(r) = u_{i+1,j}(r) - 2u_{i,j}(r) + u_{i-1,j}(r)$ with a leading error of order $h^4$ and

$$D_{t}u_{i,j}(r) \simeq \frac{u_{i,j+1}(r) - u_{i,j}(r)}{k}, \quad D_{t}\overline{u}_{i,j}(r) \simeq \frac{\overline{u}_{i,j+1}(r) - \overline{u}_{i,j}(r)}{k}. \quad (5.8)$$

By the technique of Crank-Nicolson (C-N), the finite difference representation of (5.3) is

$$v_{i,j-1}(r) - \frac{2\nu_{i,j}(r) + \nu_{i,j+1}}{k^2}$$

$$= \frac{\beta^2}{2h^2} \left[ (v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}) + (v_{i-1,j-1} - 2v_{i,j-1} + v_{i+1,j-1}) \right], \quad (5.9)$$
\[
\frac{\overline{v}_{i,j-1}(r) - 2\overline{v}_{i,j}(r) + \overline{v}_{i,j+1}}{k^2} = \frac{\beta^2}{2h^2} \left[ (\overline{v}_{i,j+1} - 2\overline{v}_{i,j} + \overline{v}_{i+1,j+1}) + (\overline{v}_{i-1,j-1} - 2\overline{v}_{i,j-1} + \overline{v}_{i+1,j-1}) \right]. \quad (5.10)
\]

(5.9) and (5.10) can be written as
\[
-\eta\overline{v}_{i-1,j+1} + (1 + 2\eta)\overline{v}_{i,j+1} - \eta\overline{v}_{i+1,j+1}
= \eta\overline{v}_{i-1,j-1} - (1 - 2\eta)\overline{v}_{i,j-1} + 2\overline{v}_{i,j} + \eta\overline{v}_{i+1,j-1}, \quad (5.11)
\]
\[
-\eta\overline{v}_{i-1,j+1} + (1 + 2\eta)\overline{v}_{i,j+1} - \eta\overline{v}_{i+1,j+1}
= \eta\overline{v}_{i-1,j-1} - (1 - 2\eta)\overline{v}_{i,j-1} + 2\overline{v}_{i,j} + \eta\overline{v}_{i+1,j-1}. \quad (5.12)
\]

The finite difference representation of (5.7) is
\[
\frac{\nu_{i,j-1}(r) - 2\nu_{i,j}(r) + \nu_{i,j+1}}{k^2}
= \frac{\beta^2}{2h^2} \left[ (\nu_{i,j+1} - 2\nu_{i,j} + \nu_{i+1,j+1}) + (\nu_{i-1,j-1} - 2\nu_{i,j-1} + \nu_{i+1,j-1}) \right], \quad (5.13)
\]
\[
\frac{\overline{v}_{i,j-1}(r) - 2\overline{v}_{i,j}(r) + \overline{v}_{i,j+1}}{k^2}
= \frac{\beta^2}{2h^2} \left[ (\overline{v}_{i,j+1} - 2\overline{v}_{i,j} + \overline{v}_{i+1,j+1}) + (\overline{v}_{i-1,j-1} - 2\overline{v}_{i,j-1} + \overline{v}_{i+1,j-1}) \right]. \quad (5.14)
\]

(5.13) and (5.14) can be written as
\[
-\eta\overline{v}_{i-1,j+1} + \nu_{i,j+1} + 2\eta\overline{v}_{i,j+1} - \eta\overline{v}_{i+1,j+1}
= \eta\overline{v}_{i-1,j-1} - \nu_{i,j-1} - 2\eta\overline{v}_{i,j-1} + 2\nu_{i,j} + \eta\overline{v}_{i+1,j-1}. \quad (5.15)
\]
\[-\eta v_{i-1,j+1} + \overline{v}_{i,j+1} + 2\eta v_{i,j+1} - \eta v_{i+1,j+1} = \eta v_{i-1,j-1} - \overline{v}_{i,j-1} - 2\eta w_{i,j-1} + 2\overline{v}_{i,j} + \eta w_{i+1,j-1}, \quad (5.16)\]

where \( \eta = \frac{k^2 \beta^2}{2h^2} \), \( r \in [0,1] \) and \( v_{i,j} = (w_{i,j}(r), \overline{v}_{i,j}(r)) \) is the exact solution of the approximating difference equations.

**Matrix form of the difference schemes**

Matrix form of the difference schemes (5.11) and (5.12):

\[ \tilde{P}_1 w_{j+1} = \tilde{Q}_1 w_{j-1} + 2\tilde{R}_1 w_j + \tilde{G}_{j+1} \quad (5.17) \]

where the tridiagonal matrices \( \tilde{P}_1 \) and \( \tilde{Q}_1 \) are given by

\[ \tilde{P}_1 = \begin{bmatrix} 1 + 2\eta & -\eta & 0 \\
-\eta & 1 + 2\eta & -\eta \\
\vdots & \ddots & \ddots \\
-\eta & 1 + 2\eta & -\eta \\
0 & -\eta & 1 + 2\eta \end{bmatrix}_{m-1 \times m-1}, \]

\[ \tilde{Q}_1 = \begin{bmatrix} -(1 + 2\eta) & \eta & 0 \\
\eta & -(1 + 2\eta) & \eta \\
\vdots & \ddots & \ddots \\
\eta & -(1 + 2\eta) & \eta \\
0 & \eta & -(1 + 2\eta) \end{bmatrix}_{m-1 \times m-1}, \]

\( \tilde{R}_1 \) is identity matrix of order \( m-1 \) and the column matrix \( \tilde{G}_{j+1} \) of order \( (m-1) \times 1 \) is given by

\[ \tilde{G}_{j+1} = [\eta(w_{0,j-1} + w_{0,j+1}), 0, \ldots, 0, \eta(w_{m,j-1} + w_{m,j+1})]^T. \]
where \( w_j \in \{ v_j, \overline{v}_j \} \).

Matrix form of the difference schemes (5.15) and (5.16):

\[
\tilde{P}_2w_{j+1} = \tilde{Q}_2w_{j-1} + 2\tilde{R}_2w_j + \tilde{C}_{j+1}
\]

(5.18)

where

\[
\tilde{P}_2 = \begin{bmatrix}
1 & 0 & 0 & 2\eta & -\eta & 0 \\
0 & 1 & 0 & -\eta & 2\eta & -\eta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & -\eta & 2\eta & -\eta \\
0 & 0 & 1 & 0 & -\eta & 2\eta \\
2\eta & -\eta & 0 & 1 & 0 & 0 \\
-\eta & 2\eta & -\eta & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\eta & 2\eta & -\eta & 0 & 1 & 0 \\
0 & -\eta & 2\eta & 0 & 0 & 1
\end{bmatrix}_{2(m-1) \times 2(m-1)}
\]
\[ \hat{Q}_2 = \begin{bmatrix}
1 & 0 & 0 & -2\eta & \eta & 0 \\
0 & 1 & 0 & \eta & -2\eta & \eta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & \eta & -2\eta & \eta \\
0 & 0 & 1 & \eta & -2\eta & \eta \\
-2\eta & \eta & 0 & 1 & 0 & 0 \\
\eta & -2\eta & \eta & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\eta & -2\eta & \eta & 0 & 1 & 0 \\
0 & \eta & -2\eta & 0 & 0 & 1 \\
\end{bmatrix}_{2(m-1) \times 2(m-1)} \]

\( \hat{R}_2 \) is identity matrix of order \( 2(m-1) \) and the column matrix \( \hat{C}_{j+1} \) of order \( 2(m-1) \times 1 \) is given by

\[ \hat{C}_{j+1} = [\eta(v_{0,j-1} + v_{0,j+1}), 0, \ldots, 0, \eta(v_{m,j-1} + v_{m,j+1}), \eta(v_{0,j-1} + v_{0,j+1}), 0, \ldots, 0, \eta(v_{m,j-1} + v_{m,j+1})]^T. \]

### 5.4 Examples

**Example 5.1.** Consider the fuzzy hyperbolic problem

\[ D_t u(x, t) = 4 D_{xx} u(x, t), 0 < x < 1, t > 0, \quad (5.19) \]

with the initial and boundary conditions

\[ \begin{align*}
\bar{u}(x, 0) &= k \sin(\pi x), \quad D_t \bar{u}(x, 0) = 0, 0 \leq x \leq 1, \\
\bar{u}(0, t) &= \bar{u}(1, t) = 0, t > 0,
\end{align*} \quad (5.20) \]

78
and where \( \bar{k}[r] = (\overline{k}(r), \overline{\bar{k}}(r)) = (0.75 + 0.25r, 1.25 - 0.25r) \).

If \( u \) is a \((1,1,1,1)\)-solution of (5.19)-(5.21), then \( u \) satisfies the \((1,1,1,1)\)-system associated with (5.19). Similarly for other system. On the other hand, by direct calculation, the corresponding solution of the \((1,1,1,1)\), \((1,1,2,2)\), \((1,2,1,2)\), \((1,2,2,1)\), \((2,2,1,1)\), \((2,2,2,2)\), \((2,1,1,2)\) and \((2,1,2,1)\) systems has necessarily the following form

\[
u(x, t) = ((0.75 + 0.25r) \sin(\pi x) \cos(2\pi t), (1.25 - 0.25r) \sin(\pi x) \cos(2\pi t)), \quad (5.22)
\]

Figure 5.1: Solution (5.22) for different values of \( x \) and \( t \)

\((\bar{u}(t, r), \underline{u}(t, r))\) represents a fuzzy number when \( \sin(\pi x) \cos(2\pi t) \geq 0 \).

Hence (5.22) represents fuzzy number \((x, t) \in [0, 1] \times ([0, 1/4] \cup [3/4, 1])\). Solution of (5.22) is plotted in Figure 5.1 for different values \( x \) and \( t \). Now we find the range of \((1,1,1,1)\), \((1,1,2,2)\), \((1,2,1,2)\), \((1,2,2,1)\), \((2,2,1,1)\), \((2,2,2,2)\), \((2,1,1,2)\) and \((2,1,2,1)\)-solutions of the hyperbolic equation (5.19) separately.
Case (i):

**(1,1,1,1) and (1,1,2,2)-solutions**

The (1) derivative of (5.22) with respect to $t$ in this case is given by

$$D_{t,1}^1 u(x,t)$$

$$= (-2\pi(0.75 + 0.25r) \sin(\pi x) \sin(2\pi t), -2\pi(1.25 - 0.25r) \sin(\pi x) \sin(2\pi t)). \quad (5.23)$$

Then it is a fuzzy number when $\sin(\pi x) \sin(2\pi t) \leq 0$. Hence it is a fuzzy number for $(x,t) \in [0,1] \times [3/4,1]$. The (1) derivative of (5.23) with respect to $t$ is given by

$$D_{t,1,1}^2 u(x,t)$$

$$= (-4\pi^2(0.75 + 0.25r) \sin(\pi x) \cos(2\pi t), -4\pi^2(1.25 - 0.25r) \sin(\pi x) \cos(2\pi t)).$$

It is a fuzzy number when $\sin(\pi x) \cos(2\pi t) \leq 0$. Hence it is a fuzzy number in $(x,t) \in [0,1] \times \{3/4\}$. By the Definition 2.2, $D_{t,1,2}^2 u(x,t)$ does not exist. Hence $u$ in (5.22) is not (1,1,1,1) and (1,1,2,2)-solutions of the fuzzy differential equation (5.19).

**{(1,2,1,2)}-solution**

The (2) derivative of (5.23) with respect to $t$ is given by

$$D_{t,1,2}^2 u(x,t)$$

$$= (-4\pi^2(1.25 - 0.25r) \sin(\pi x) \cos(2\pi t), -4\pi^2(0.75 + 0.25r) \sin(\pi x) \cos(2\pi t)).$$

It is a fuzzy number when $\sin(\pi x) \cos(2\pi t) \geq 0$. Hence it is a fuzzy number for $(x,t) \in [0,1] \times [3/4,1]$. The (1) derivative of (5.22) with respect to $x$ in this case is given by

$$D_{x,1}^1 u(x,t)$$

$$= (\pi(0.75 + 0.25r) \cos(\pi x) \cos(2\pi t), \pi(1.25 - 0.25r) \cos(\pi x) \cos(2\pi t)). \quad (5.24)$$
It is a fuzzy number when \( \cos(\pi x) \cos(2\pi t) \geq 0 \). Hence it is a fuzzy number for \((x, t) \in [0, 1/2] \times ([0, 1/4] \cup [3/4, 1]) \) and \([1/2, 1] \times [1/4, 3/4] \). By the Definition 2.2, \( D_{x,1}^1 u(x, t) \) exists in \((x, t) \in [0, 1/4] \times [3/4, 1] \). Hence \( u \) in (5.22) is a \((1,2,1,2)\)-solution of the fuzzy differential equation (5.19). The (2) derivative of (5.24) with respect to \( x \) in this case is given by

\[
D_{x,1,2}^2 u(x, t) = [-\pi^2 (1.25 - 0.25r) \sin(\pi x) \cos(2\pi t), -\pi^2 (0.75 + 0.25r) \sin(\pi x) \cos(2\pi t)]. \quad (5.25)
\]

It is a fuzzy number when \( \sin(\pi x) \cos(2\pi t) \geq 0 \). Hence it is a fuzzy number for \((x, t) \in [0, 1/2] \times ([0, 1/4] \cup [3/4, 1]) \). Hence \((1,2,1,2)\)-solution exists in \((x, t) \in [0, 1/2] \times [3/4, 1] \).

\((1,2,2,1)\)-solution

The (2) derivative of (5.22) with respect to \( x \) in this case is given by

\[
D_{x,2}^1 u(x, t) = (\pi (1.25 - 0.25r) \cos(\pi x) \cos(2\pi t), \pi (0.75 + 0.25r) \cos(\pi x) \cos(2\pi t)). \quad (5.26)
\]

It is a fuzzy number when \( \cos(\pi x) \cos(2\pi t) \leq 0 \). Hence it is a fuzzy number for \((x, t) \in [0, 1/2] \times \{1/4, 3/4\} \) and \([1/2, 1] \times ([0, 1/4] \cup [3/4, 1]) \). By the Definition 2.2, \( D_{x,2}^2 u(x, t) \) exists in \((x, t) \in [1/2, 1] \times ([0, 1/4] \cup [3/4, 1]) \).

The (1) derivative of (5.26) with respect to \( x \) in this case is given by

\[
D_{x,2,1}^1 u(x, t) = (-\pi^2 (1.25 - 0.25r) \sin(\pi x) \cos(2\pi t), -\pi^2 (0.75 + 0.25r) \sin(\pi x) \cos(2\pi t)). \quad (5.27)
\]

It is a fuzzy number when \( \sin(\pi x) \cos(2\pi t) \geq 0 \). Hence it is a fuzzy number for \((x, t) \in [1/2, 1] \times ([0, 1/4] \cup [3/4, 1]) \). Hence \((1,2,2,1)\)-solution exists in \((x, t) \in [1/2, 1] \times [3/4, 1] \).
(2,2,1,1) and (2,2,2,2)-solutions

The (2) derivative of (5.22) with respect to \( t \) in this case is given by

\[
D^1_{t,2} u(x, t) = (-2\pi (1.25 - 0.25r) \sin(\pi x) \sin(2\pi t), -2\pi (0.75 + 0.25r) \sin(\pi x) \sin(2\pi t)). \tag{5.28}
\]

It is a fuzzy number when \( \sin(\pi x) \sin(2\pi t) \geq 0 \). Hence it is a fuzzy number for \((x, t) \in [0, 1] \times [0, 1/4]\). The (2) derivative of (5.28) with respect to \( t \) in this case is given by

\[
D^2_{t,2} u(x, t) = (-4\pi^2 (0.75 + 0.25r) \sin(\pi x) \cos(2\pi t), -4\pi^2 (1.25 - 0.25r) \sin(\pi x) \cos(2\pi t)).
\]

It is a fuzzy number when \( \sin(\pi x) \cos(2\pi t) \leq 0 \). Hence it is a fuzzy number for \((x, t) \in [0, 1] \times \{1/4, 3/4\}\). By the Definition 2.2, \( D^2_{t,2} u(x, t) \) does not exist. Hence \( u \) in (5.22) is not (2,2,1,1) and (2,2,2,2)-solutions of the fuzzy differential equation (5.19). The (1) derivative of (5.28) with respect to \( t \) in this case is given by

\[
D^1_{t,1} u(x, t) = (-4\pi^2 (1.25 - 0.25r) \sin(\pi x) \cos(2\pi t), -4\pi^2 (0.75 + 0.25r) \sin(\pi x) \cos(2\pi t)).
\]

It is a fuzzy number when \( \sin(\pi x) \cos(2\pi t) \geq 0 \). Hence it is a fuzzy number for \((x, t) \in [0, 1] \times [0, 1/4]\). From (5.25) and (5.27), \( D^2_{x,1,2} u(x, t) \) and \( D^2_{x,2,1} u(x, t) \) are fuzzy numbers for \((x, t) \in [0, 1/2] \times ([0, 1/4] \cup [3/4, 1]) \) and \([1/2, 1] \times ([0, 1/4] \cup [3/4, 1]) \). Hence (2,1,1,2) and (2,1,2,1)-solutions exist in \((x, t) \in [0, 1/2] \times [0, 1/4] \) and \((x, t) \in [1/2, 1] \times [0, 1/4] \) respectively. The error and order are defined as follows:

\[
\overline{E}_\infty(h, k) = \max_{0 \leq j \leq m} |\pi(x_j, t_n) - v_{j,n}|, \quad \text{order} = \frac{\overline{E}_\infty(h, k)}{\overline{E}_\infty(h/2, k/2)}.
\]
\[ E_\infty(h, k) = \max_{0 \leq j \leq m} |u(x_j, t_n) - v_{j,n}|, \quad \text{order} = \frac{E_\infty(h, k)}{E_\infty(h/2, k/2)}. \]

In Table 5.1-5.2, we have obtained the errors as well as their convergence order for the difference scheme in the case when \( h \) and \( k \) decrease simultaneously for different \( r \). Comparison of upper branch of numerical and exact solutions for \( r = 0.1 \) when \( m = n = 20 \) and \( m = n = 80 \) is given in Figure 5.2(a) and Figure 5.2(b) respectively. Comparison of numerical and exact solutions for \( r = 0.7 \) when \( m = n = 20 \) and \( m = n = 80 \) is given in Figure 5.3(a) and Figure 5.3(b) respectively. Comparison of lower branch of numerical and exact solutions for \( r = 0.2 \) when \( m = n = 20 \) and \( m = n = 40 \) is given in Figure 5.4(a) and Figure 5.4(b) respectively. Comparison of numerical and exact solutions for \( r = 0.7 \) when \( m = n = 20 \) and \( m = n = 80 \) is given in Figure 5.5(a) and Figure 5.5(b) respectively.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( k )</th>
<th>( E_\infty(h, k) )</th>
<th>order</th>
<th>( E_\infty(h, k) )</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
<td>( 6.79672 \times 10^{-6} )</td>
<td>3.91244</td>
<td>( 6.79672 \times 10^{-6} )</td>
<td>3.91244</td>
</tr>
<tr>
<td>( \frac{1}{40} )</td>
<td>( \frac{1}{40} )</td>
<td>( 4.51376 \times 10^{-7} )</td>
<td>3.99462</td>
<td>( 4.51376 \times 10^{-7} )</td>
<td>3.99462</td>
</tr>
<tr>
<td>( \frac{1}{80} )</td>
<td>( \frac{1}{80} )</td>
<td>( 2.83165 \times 10^{-8} )</td>
<td>*</td>
<td>( 2.83165 \times 10^{-8} )</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 5.1: The maximum errors and convergence orders of difference scheme when \( r = 0.1 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( k )</th>
<th>( E_\infty(h, k) )</th>
<th>order</th>
<th>( E_\infty(h, k) )</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
<td>( 4.8548 \times 10^{-6} )</td>
<td>3.91244</td>
<td>( 4.8548 \times 10^{-6} )</td>
<td>3.91244</td>
</tr>
<tr>
<td>( \frac{1}{40} )</td>
<td>( \frac{1}{40} )</td>
<td>( 3.22412 \times 10^{-7} )</td>
<td>3.99462</td>
<td>( 3.22412 \times 10^{-7} )</td>
<td>3.99462</td>
</tr>
<tr>
<td>( \frac{1}{80} )</td>
<td>( \frac{1}{80} )</td>
<td>( 2.02261 \times 10^{-8} )</td>
<td>*</td>
<td>( 2.02261 \times 10^{-8} )</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 5.2: The maximum errors and convergence orders of difference scheme when \( r = 0.7 \).

Case (ii):

Now we find the solution of the four systems \((1,1,1,2), (1,1,2,1), (1,2,1,1), (1,2,2,2)\). These systems have the following form
Figure 5.2: The comparison of exact solution (•) and numerical solution of the difference scheme defined by (4.23)(•) for the example at $r = 0.1$. 

Figure 5.3: The comparison of exact solution (•) and numerical solution of the difference scheme defined by (4.23)(•) for the example at $r = 0.7$. 

Figure 5.4: The comparison of exact solution (•) and numerical solution of the difference scheme defined by (4.23)(•) for the example at $r = 0.2$. 

\begin{align*}
D_{tt} \overline{u}(x, t, r) &= \beta^2 D_{xx} \overline{u}(x, t, r), \\
D_{tt} \overline{u}(x, t, r) &= \beta^2 D_{xx} \overline{u}(x, t, r), \quad 0 < x < L, \quad 0 < t < T, \quad r \in [0, 1]. \quad (5.29)
\end{align*}
Using Laplace transform of (5.25) with respect to \(t\) and the initial conditions (5.20), we get

\[
\begin{align*}
4D_{xx} \bar{u}_s(x, s) - s^2 \bar{u}_s(x, s) &= -(0.75 + 0.25r) \sin(\pi x), \\
4D_{xx} \bar{u}_u(x, s) - s^2 \bar{u}_u(x, s) &= -(1.25 - 0.25r) \sin(\pi x).
\end{align*}
\]

Adding and subtracting (5.30) and (5.31), we get the following equations respectively

\[
\begin{align*}
4D_{xx} (\bar{u}_s(x, s) + \bar{u}_u(x, s)) - s^2 (\bar{u}_s(x, s) + \bar{u}_u(x, s)) &= - \sin(\pi x), \\
4D_{xx} (\bar{u}_s(x, s) - \bar{u}_u(x, s)) + s^2 (\bar{u}_s(x, s) - \bar{u}_u(x, s)) &= 0.5(1 - r) \sin(\pi x).
\end{align*}
\]

By taking Laplace transform of the boundary condition in (5.21), we get the solutions of (5.32) and (5.33) as

\[
\begin{align*}
\bar{u}_s(x, s) + \bar{u}_u(x, s) &= \frac{2 \sin \pi x}{s^2 + (2\pi)^2}, \\
\bar{u}_s(x, s) - \bar{u}_u(x, s) &= \frac{0.5(1 - r) \sin(\pi x)}{s^2 - (2\pi)^2}.
\end{align*}
\]

From (5.34) and (5.35), we get

\[
\begin{align*}
\bar{u}_s(x, s) + \bar{u}_u(x, s) &= \frac{2 \sin \pi x}{s^2 + (2\pi)^2}, \\
\bar{u}_s(x, s) - \bar{u}_u(x, s) &= \frac{0.5(1 - r) \sin(\pi x)}{s^2 - (2\pi)^2}.
\end{align*}
\]

Figure 5.5: The comparison of exact solution (−) and numerical solution of the difference scheme defined by (4.23)(−) for the example at \(r = 0.7\).
\[ u_s(x, s) = \frac{\sin \pi x}{s^2 + (2\pi)^2} + \frac{0.25(1 - r)\sin(\pi x)}{s^2 - (2\pi)^2}, \quad (5.36) \]
\[ u_s(x, s) = \frac{\sin \pi x}{s^2 + (2\pi)^2} + \frac{0.25(r - 1)\sin(\pi x)}{s^2 - (2\pi)^2}. \quad (5.37) \]

Taking inverse Laplace transform of (5.36) and (5.37), we get

\[ u(x, t) = \left( \frac{\sin \pi x}{2\pi} (\sin 2\pi t + \frac{(r - 1)}{4} \sin 2\pi t), \frac{\sin \pi x}{2\pi} (\sin 2\pi t + \frac{(1 - r)}{4} \sin 2\pi t) \right). \quad (5.38) \]

It is a fuzzy number for \((x, t) \in [0, 1] \times [0, 1]\).

\((1,1,1,2)\)-solution

The (1) derivative of (5.38), with respect to \(t\) in this case is given by

\[ D^1_{t,1} u(x, t) = \pi \sin \pi x \left( \frac{\sin 2\pi t + \frac{(r - 1)}{4} \sin 2\pi t}{2\pi}, \frac{\sin 2\pi t + \frac{(1 - r)}{4} \sin 2\pi t}{2\pi} \right). \quad (5.39) \]

It is a fuzzy number when \(\sin(\pi x) \geq 0\). Hence it is a fuzzy number for \((x, t) \in [0, 1] \times [0, 1]\). The (1) derivative of (5.39) with respect to \(t\) in this case is given by

\[ D^2_{t,1,1} u(x, t) = 2\pi \sin \pi x \left( -\sin 2\pi t + \frac{(r - 1)}{4} \sin 2\pi t, -\sin 2\pi t + \frac{(1 - r)}{4} \sin 2\pi t \right). \quad (5.40) \]

It is a fuzzy number when \(\sin(\pi x) \geq 0\). Hence it is a fuzzy number for \((x, t) \in [0, 1] \times [0, 1]\). The (1) derivative of (5.38) with respect to \(x\) in this case is given by

\[ D^1_{x,1} u(x, t) = \frac{\cos \pi x}{2} \left( \sin 2\pi t + \frac{(r - 1)}{4} \sin 2\pi t, \sin 2\pi t + \frac{(1 - r)}{4} \sin 2\pi t \right). \quad (5.41) \]

It is a fuzzy number when \(\cos(\pi x) \geq 0\). Hence it is a fuzzy number for \((x, t) \in [0, 1/2] \times [0, 1]\). The (2) derivative of (5.41) with respect to \(x\) in this case is given by
\[ D^2_{x,1,2}u(x,t) = \left( \frac{-\pi \sin \pi x}{2} \left( \sin 2\pi t + \frac{(1-r)}{4} \sin h2\pi t \right), \right. \\
\left. \frac{-\pi \sin \pi x}{2} \left( \sin 2\pi t + \frac{(r-1)}{4} \sin h2\pi t \right) \right). \] (5.42)

It is a fuzzy number when \( \sin(\pi x) \geq 0 \). Hence it is a fuzzy number for \((x,t) \in [0,1/2] \times [0,1] \). Hence \((1,1,1,2)\)-solution exists in \((x,t) \in [0,1/2] \times [0,1] \).

**\(1,1,2,1\)-solution**

From (5.40), \( D^2_{t,1,1}u(x,t) \) is a fuzzy number for \((x,t) \in [0,1] \times [0,1] \). The (2) derivative of (5.38) with respect to \(x\) in this case is given by

\[ D^1_{x,2}u(x,t) = \frac{\cos \pi x}{2} \left( \sin 2\pi t + \frac{(1-r)}{4} \sin h2\pi t, \sin 2\pi t + \frac{(r-1)}{4} \sin h2\pi t \right) \] (5.43)

It is a fuzzy number when \( \cos(\pi x) \leq 0 \). Hence it is a fuzzy number for \((x,t) \in [1/2,1] \times [0,1] \). The (1) derivative of (5.43) with respect to \(x\) in this case is given by

\[ D^2_{x,2,1}u(x,t) = \left( \frac{-\pi \sin \pi x}{2} \left( \sin 2\pi t + \frac{(1-r)}{4} \sin h2\pi t \right), \right. \\
\left. \frac{-\pi \sin \pi x}{2} \left( \sin 2\pi t + \frac{(r-1)}{4} \sin h2\pi t \right) \right). \] (5.44)

It is a fuzzy number when \( \sin(\pi x) \geq 0 \). Hence it is a fuzzy number for \((x,t) \in [1/2,1] \times [0,1] \). Hence \((1,1,2,1)\)-solution exists in \((x,t) \in [1/2,1] \times [0,1] \).

**\(1,2,1,1\) and \(1,2,2,2\)-solutions**

The (2) derivative of (5.39) with respect to \(t\) in this case is given by

\[ D^2_{t,1,2}u(x,t) = \left( 2\pi \sin \pi x \left( -\sin 2\pi t + \frac{(1-r)}{4} \sin h2\pi t \right), \right. \\
\left. 2\pi \sin \pi x \left( -\sin 2\pi t + \frac{(r-1)}{4} \cos h2\pi t \right) \right). \] (5.45)

87
It is not a fuzzy number because \( \sin(\pi x) \geq 0 \). (1,2,1,1) and (1,2,2,2)-solutions do not exist. Solution in (5.38) is plotted in Figure 5.6 for different values \( x \) and \( t \). In a similar way, we find the solutions of the remaining four systems (2,1,1,1), (2,1,2,2), (2,2,1,2) and (2,2,2,1) which have the following form

\[
\begin{align*}
\frac{\sin \pi x}{2\pi} \left( \sin 2\pi t + \frac{(1-r)}{4} \sin h2\pi t \right), & \quad \frac{\sin \pi x}{2\pi} \left( \sin 2\pi t + \frac{(r-1)}{4} \sin h2\pi t \right)
\end{align*}
\]

It is not a fuzzy number for all \((x, t) \in [0, 1] \times [0, 1]\).

![Figure 5.6: Solution (5.38) for different values of \( x \) and \( t \)](image)