
4.1 Introduction

Ma et al. [48] introduced the numerical method for solving fuzzy differential equations. Subsequently Friedman et al. [31], Bede [12] and Abbasbandy et al. [67] discussed the numerical solutions of fuzzy differential equations. In [18], Buckley and Feuring proposed a procedure to examine the solution for fuzzy partial differential equations (FPDEs). Their proposed method works only for elementary partial differential equations. In [5, 6], Allahviranloo used a numerical method (explicit finite difference method) to solve FPDEs, that is based on Seikkala derivative.

In this chapter, we consider a higher order finite difference method for the fuzzy parabolic differential equations under generalized differentiability. An implicit difference approximation scheme is developed. Numerical results are provided to verify the accuracy of the proposed algorithm.
4.2 Fuzzy Partial Differential Equations

Consider the fuzzy partial differential equation (FPDE)

\[ \varphi(D_x, D_y)u(x, y) = F(x, y, K), \quad (4.1) \]

subject to certain boundary conditions where the operator \( \varphi(D_x, D_y) \) is a polynomial in \( D_x \) and \( D_y \) with constant coefficient where \( D_x \) and \( D_y \) stand for the partial differential coefficients with respect to \( x \) and \( y \) respectively and \( F(x, y, K) \) is the fuzzy function with \( K = (k_1, \ldots, k_n) \), where \( k_i \) is a triangular fuzzy number in some interval \( J_i, 1 \leq i \leq n \). Let \( I_1 = [0, M_1], I_2 = [0, M_2] \). The fuzzy function \( u \) maps \( I_1 \times I_2 \) into fuzzy numbers.

**Case 1:** If we consider \( \varphi(D_x, D_y)u(x, y) \) in (4.1) by using the derivative in the first form (1), then we have

\[ \varphi(D_x, D_y)u(x, y) = (\varphi(D_x, D_y)u(x, y, r), \varphi(D_x, D_y)\bar{u}(x, y, r)) \]

and we have to solve the system of partial differential equations

\[ \begin{align*}
\varphi(D_x, D_y)u(x, y, r) &= F(x, y, r) = \min \{F(x, y, k)|k \in K(r)\}, \quad (4.2) \\
\varphi(D_x, D_y)\bar{u}(x, y, r) &= \bar{F}(x, y, r) = \max \{F(x, y, k)|k \in K(r)\}, \quad (4.3)
\end{align*} \]

for all \((x, y) \in I_1 \times I_2\) and all \( r \in [0, 1] \). We append equations (4.2) and (4.3) with any boundary conditions, for example, if they were \( u(0, y) = C_1, u(M_1, y) = C_2, u(x, 0) = C_3 \) and \( u(x, M_2) = C_4 \), then we add

\[ \begin{align*}
u(0, y, r) &= C_1(r), & u(M_1, y, r) &= C_2(r) \\
\bar{u}(x, 0, r) &= C_3(r), & \bar{u}(x, M_2, r) &= C_4(r)
\end{align*} \quad (4.4) \]
to equation (4.2) and
\[ \bar{u}(0, y, r) = C_1(r), \quad \bar{u}(M_1, y, r) = \bar{C}_2(r), \]
\[ \bar{u}(x, 0, r) = \bar{C}_3(r), \quad \bar{u}(x, M_2, r) = \bar{C}_4(r), \]
(4.5)
to equation (4.3) where \( C_i = (C_i(r), \bar{C}_i(r)), i = 1, 2, 3, 4. \)

**Case 2**: If we consider \( \varphi(D_x, D_y)u(x, y) \) in (4.1) by using the derivative in the second form (2), then we have
\[ \varphi(D_x, D_y)u(x, y) = (\varphi(D_x, D_y)\bar{u}(x, y, r), \varphi(D_x, D_y)\bar{u}(x, y, r)) \]
and we should solve the system of partial differential equations
\[ \varphi(D_x, D_y)\bar{u}(x, y, r) = \bar{F}(x, y, r) = \max\{F(x, y, k)|k \in K[r]\}, \]
\[ \varphi(D_x, D_y)\bar{u}(x, y, r) = \underline{F}(x, y, r) = \min\{F(x, y, k)|k \in K[r]\}, \]
(4.6)
(4.7)
for all \((x, y) \in I_1 \times I_2\) and all \(r \in [0, 1]\). We append equations (4.6) and (4.7) with boundary conditions (4.4) and (4.5) respectively.

We use the following notations
\( D^1_{i,n}u \) and \( D^1_{x,n}u (n \in \{1, 2\}) \) denote \((n)\)- differentiable of \(u\) with respect to \(t\) and \(x\) respectively.
\( D^2_{i,n,k}u \) and \( D^2_{x,m,l}u (k, l, m, n \in \{1, 2\}) \) denote \((k)\)- differentiable of \(D^1_{i,n}u\) with respect to \(t\) and \((l)\)- differentiable of \(D^1_{x,m}u\) with respect to \(x\) respectively.
4.3 Fuzzy Parabolic Partial Differential Equations

Consider the fuzzy parabolic partial differential equation

\[ D_t u(x, t) = \alpha^2 D_{xx} u(x, t), \quad 0 < x < L, \quad 0 < t < T, \] (4.8)

with the initial and boundary conditions

\[ u(x, 0) = f(x), \quad 0 < x < L, \quad u(0, t) = K_1, \quad u(L, t) = K_2, \quad t > 0, \] (4.9)

**Definition 4.1.** Let \( u : [0, L] \times [0, T] \rightarrow \mathbb{R}_F \) and \((n, m, l) \in \{1, 2\}\). If \( D_{t,n}^1 u \) and \( D_{x,m,l}^2 u \) exist on \([0, L] \times [0, T]\) and \( D_{t,n}^1 u(x, t) = \alpha^2 D_{x,m,l}^2 u(x, t) \) for all \((x, t) \in [0, L] \times [0, T]\) and satisfy (4.9), then \( u \) is said to be an \((n, m, l)\) solution for the fuzzy parabolic equation (4.8)-(4.9) on \([0, L] \times [0, T]\).

**Definition 4.2.** Let \( u : [0, L] \times [0, T] \rightarrow \mathbb{R}_F \) and \((n, m, l) \in \{1, 2\}\) and \( J_1 \) be such that \( J_1 \subset [0, L] \times [0, T]\). If \( D_{t,n}^1 u \) and \( D_{x,m,l}^2 u \) exist on \( J_1 \) and \( D_{t,n}^1 u(x, t) = \alpha^2 D_{x,m,l}^2 u(x, t) \), for all \((x, t) \in J_1\), then \( u \) is said to be an \((n, m, l)\) solution for the fuzzy parabolic equation (4.8) on \( J_1 \).

(a) In (4.8), if both \( D_t u \) and \( D_{xx} u \) are differentiable in the first form (1) or if both \( D_t u \) and \( D_{xx} u \) are differentiable in the second form (2), then we have four systems, namely, \((1,1,1)\)-system, \((1,2,2)\)-system, \((2,1,2)\)-system and \((2,2,1)\)-system and these systems have the following form

\[ D_t \underline{u}(x, t, r) = \alpha^2 D_{xx} \underline{u}(x, t, r), \]

\[ D_t \bar{u}(x, t, r) = \alpha^2 D_{xx} \bar{u}(x, t, r), \quad 0 < x < L, \quad 0 < t < T, \quad r \in [0, 1]. \] (4.10)
(b) In (4.8), if \( D_t u \) is differentiable in the second form (2) and \( D_{xx} u \) is differentiable in the first form (1) or if \( D_t u \) is differentiable in the first form (1) and \( D_{xx} u \) is differentiable in the second form (2), then we have four systems, namely, (1,1,2)-system, (1,2,1)-system, (2,1,1)-system and (2,2,2)-system and these systems have the following form

\[
D_t \overline{u}(x, t, r) = \alpha^2 D_{xx} \overline{u}(x, t, r),
\]

\[
D_t \overline{u}(x, t, r) = \alpha^2 D_{xx} \overline{u}(x, t, r), \quad 0 < x < L, \quad 0 < t < T, \quad r \in [0, 1]. \tag{4.11}
\]

In both cases, the initial and boundary conditions are

\[
\underline{u}(x, 0, r) = f(x, r), \quad \overline{u}(x, 0, r) = \overline{f}(x, r), \quad 0 < x < L, \quad r \in [0, 1],
\]

\[
\underline{u}(0, t, r) = K_1(r), \quad \underline{u}(l, t, r) = K_2(r), \quad 0 < t < T, \quad r \in [0, 1],
\]

\[
\overline{u}(0, t, r) = \overline{K}_1(r), \quad \overline{u}(l, t, r) = \overline{K}_2(r), \quad 0 < t < T, \quad r \in [0, 1]. \tag{4.12}
\]

### 4.4 Finite Difference Method

Assume that \( u \) is a fuzzy function of the independent crisp variables \( x \) and \( t \). Subdivide the \( x-t \) plane into sets of equal rectangles of sides \( h, k \), by equally spaced grid lines parallel to \( O_t \), defined by \( x_i = ih, \quad i = 0, 1, 2, ..., m \), and equally spaced grid lines parallel to \( O_x \), defined by \( t_j = jk, \quad j = 0, 1, 2, ..., n \), where \( m \) and \( n \) are positive integers with \( h = \frac{L}{m} \) and \( k = \frac{T}{n} \).

Denote the exact value of \( u \) and the approximate value of \( v \) at the mesh point \((ih, jk)\) by \( u_{i,j} \) and \( v_{i,j} \) respectively. Also denote the parametric form of the fuzzy number \( u_{i,j} \) and \( v_{i,j} \) by \((\underline{u}_{i,j}(r), \overline{u}_{i,j}(r))\) and \((\underline{v}_{i,j}(r), \overline{v}_{i,j}(r))\) respectively.

By Taylor’s theorem and definition of compact finite difference, we get the approximate solution as follows
\[ D_{xx} u_{i,j}(r) \simeq \frac{\delta_z^2}{(1 + \frac{\delta_z^2}{12})} u_{i,j}(r), \]
\[ D_{xx} \bar{u}_{i,j}(r) \simeq \frac{\delta_z^2}{(1 + \frac{\delta_z^2}{12})} \bar{u}_{i,j}(r), \]  
\( \text{(4.13)} \)

where \( \delta_z^2 \bar{u}_{i,j}(r) = u_{i+1,j}(r) - 2u_{i,j}(r) + u_{i-1,j}(r) \) and \( \delta_z^2 u_{i,j}(r) = u_{i+1,j}(r) - 2u_{i,j}(r) + u_{i-1,j}(r) \) with a leading error of order \( h^4 \) and

\[ D_t u_{i,j}(r) \simeq \frac{u_{i,j+1}(r) - u_{i,j}(r)}{k}, \quad D_t \bar{u}_{i,j}(r) \simeq \frac{\bar{u}_{i,j+1}(r) - \bar{u}_{i,j}(r)}{k}. \]  
\( \text{(4.14)} \)

By the technique of Crank-Nicolson (C-N), the finite difference representation of (4.10) is

\[ \frac{u_{i,j+1}(r) - u_{i,j}(r)}{k} = \frac{\beta^2}{2h^2} \left( \frac{\delta_z^2}{(1 + \frac{\delta_z^2}{12})} (u_{i,j}(r) + u_{i,j+1}(r)) \right), \]  
\( \text{(4.15)} \)

\[ \frac{\bar{u}_{i,j+1}(r) - \bar{u}_{i,j}(r)}{k} = \frac{\beta^2}{2h^2} \left( \frac{\delta_z^2}{(1 + \frac{\delta_z^2}{12})} (\bar{u}_{i,j}(r) + \bar{u}_{i,j+1}(r)) \right). \]  
\( \text{(4.16)} \)

(4.15) and (4.16) can be written as

\[ (1 - \mu) u_{i-1,j+1}(r) + (10 + 2\mu) u_{i,j+1}(r) + (1 - \mu) u_{i+1,j+1}(r) = (1 + \mu) u_{i-1,j}(r) \]
\[ + (10 - 2\mu) u_{i,j}(r) + (1 + \mu) u_{i+1,j}(r), \]  
\( \text{(4.17)} \)

\[ (1 - \mu) \bar{u}_{i-1,j+1}(r) + (10 + 2\mu) \bar{u}_{i,j+1}(r) + (1 - \mu) \bar{u}_{i+1,j+1}(r) = (1 + \mu) \bar{u}_{i-1,j}(r) \]
\[ + (10 - 2\mu) \bar{u}_{i,j}(r) + (1 + \mu) \bar{u}_{i+1,j}(r). \]  
\( \text{(4.18)} \)
The finite difference representation of (4.11) is

\[ \frac{v_{i,j+1}(r) - v_{i,j}(r)}{k} = \frac{\beta^2}{2h^2} \left( \frac{\delta_x^2}{(1 + \frac{\delta_x^2}{12})} (v_{i,j}(r) + v_{i,j+1}(r)) \right), \quad (4.19) \]

\[ \frac{\overline{v}_{i,j+1}(r) - \overline{v}_{i,j}(r)}{k} = \frac{\beta^2}{2h^2} \left( \frac{\delta_x^2}{(1 + \frac{\delta_x^2}{12})} (\overline{v}_{i,j}(r) + \overline{v}_{i,j+1}(r)) \right). \quad (4.20) \]

(4.19) and (4.20) can be written as

\[ \psi_{i-1,j+1}(r) + 10\psi_{i,j+1}(r) + \psi_{i+1,j+1}(r) - \mu \overline{\psi}_{i-1,j+1}(r) + 2\mu \overline{\psi}_{i,j+1}(r) - \mu \overline{\psi}_{i+1,j+1}(r) \]

\[ = \psi_{i-1,j}(r) + 10\psi_{i,j}(r) + \psi_{i+1,j}(r) + \mu \overline{\psi}_{i-1,j}(r) - 2\mu \overline{\psi}_{i,j}(r) + \mu \overline{\psi}_{i+1,j}(r), \quad (4.21) \]

\[ -\mu \overline{\psi}_{i-1,j+1}(r) + 2\mu \overline{\psi}_{i,j+1}(r) - \mu \overline{\psi}_{i+1,j+1}(r) + \overline{\psi}_{i-1,j+1}(r) + 10\overline{\psi}_{i,j+1}(r) + \overline{\psi}_{i+1,j+1}(r) \]

\[ = \mu \overline{\psi}_{i-1,j}(r) - 2\mu \overline{\psi}_{i,j}(r) + \mu \overline{\psi}_{i+1,j}(r) + \overline{\psi}_{i-1,j}(r) + 10\overline{\psi}_{i,j}(r) + \overline{\psi}_{i+1,j}(r), \quad (4.22) \]

where \( \mu = \frac{6k\beta^2}{h^2} \), \( r \in [0, 1] \) and \( \psi_{i,j} = (\psi_{i,j}(r), \overline{\psi}_{i,j}(r)) \) is the exact solution of the approximating difference equations.

**Matrix form of the difference schemes**

Matrix form of the difference schemes (4.17) and (4.18):

\[ P_1 w_{j+1} = Q_1 w_j + G_{j+1} \quad (4.23) \]
where

\[
P_1 = \begin{bmatrix}
10 + 2\mu & 1 - \mu \\
1 - \mu & 10 + 2\mu & 1 - \mu \\
& \ddots & \ddots & \ddots \\
& 1 - \mu & 10 + 2\mu & 1 - \mu \\
& & 1 - \mu & 10 + 2\mu
\end{bmatrix}_{m-1 \times m-1},
\]

\[
Q_1 = \begin{bmatrix}
10 - 2\mu & 1 + \mu \\
1 + \mu & 10 - 2\mu & 1 + \mu \\
& \ddots & \ddots & \ddots \\
& 1 + \mu & 10 - 2\mu & 1 + \mu \\
& & 1 + \mu & 10 - 2\mu
\end{bmatrix}_{m-1 \times m-1},
\]

\[
\nu_{j+1} = \begin{bmatrix}
\nu_{1,j+1}, \nu_{2,j+1}, \ldots, \nu_{m-1,j+1}
\end{bmatrix}^T, \quad \overline{\nu}_{j+1} = \begin{bmatrix}
\overline{\nu}_{1,j+1}, \overline{\nu}_{2,j+1}, \ldots, \overline{\nu}_{m-1,j+1}
\end{bmatrix}^T,
\]

\[
G_{j+1} = \begin{bmatrix}
-(1 - \mu)w_{0,j+1} + (1 + \mu)w_{0,j}, 0, \ldots, 0, -(1 - \mu)w_{m,j+1} + (1 + \mu)w_{m,j}
\end{bmatrix}^T,
\]

where \( w_j \in \{v_j, \overline{v}_j\} \).

Matrix form of the difference schemes (4.21) and (4.22):

\[
P_2 \begin{bmatrix}
\nu_{j+1}(r) \\
\overline{\nu}_{j+1}(r)
\end{bmatrix} = Q_2 \begin{bmatrix}
\nu_j(r) \\
\overline{\nu}_j(r)
\end{bmatrix} + C_{j+1}
\]

(4.24)

where \( P_2 = \begin{pmatrix} A & -B \\ -B & A \end{pmatrix}, \quad Q_2 = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \) and
\[ A = \begin{bmatrix} 10 & 1 \\ 1 & 10 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 10 & 1 \\ 1 & 10 \end{bmatrix} \]_{m-1 \times m-1}, \]

\[ B = \begin{bmatrix} -2\mu & \mu \\ \mu & -2\mu & \mu \\ \vdots & \vdots & \vdots \\ \mu & -2\mu & \mu \\ \mu & -2\mu \end{bmatrix} \]_{m-1 \times m-1}, \]

\[ v_{j+1} = [v_{1,j+1}, v_{2,j+1}, \ldots, v_{m-1,j+1}]^T, \quad v_{j+1} = [v_{1,j+1}, v_{2,j+1}, \ldots, v_{m-1,j+1}]^T, \]

\[ C_{j+1} = [v_{0,j+1}, 0, \ldots, 0, v_{m-1,j+1}, v_{0,j+1}, 0, \ldots, 0, v_{m-1,j+1}]^T. \]

### 4.5 Examples

**Example 4.1.** Consider the fuzzy parabolic equation

\[ D_t u(x, t) = D_{xx} u(x, t), \quad 0 < x < 1, \quad 0 < t < 1, \]  

with the initial condition

\[ u(x, 0) = k \sin(\pi x), \quad 0 \leq x \leq 1, \]

where \( k = (\underline{k}(r), \bar{k}(r)) = (r - 1, 1 - r) \) and with the boundary conditions

\[ u(0, t) = u(1, t) = 0, \quad 0 < t < 1. \]
If $u$ is a $(1,1,1)$-solution of (4.25)-(4.27), then $u$ satisfies the $(1,1,1)$-system associated with (4.25). Similarly for the other system. On the other hand, by direct calculation, the corresponding solution of the $(1,1,1), (1,2,2), (2,1,2)$ and $(2,2,1)$-systems has necessarily the following expression

$$u(x, t) = ((r - 1)e^{-\pi^2 t} \sin(\pi x), (1 - r)e^{-\pi^2 t} \sin(\pi x)).$$ \hspace{1cm} (4.28)$$

$u(x, t)$ represents a fuzzy number when $\sin(\pi x) \geq 0$. Hence (4.28) represents a fuzzy number when $(x, t) \in [0, 1] \times [0, 1]$. Solution in (4.28) is plotted in Figure 4.1 for different values $x$ and $t$. Now we find the range of $(1,1,1), (1,2,2), (2,1,2)$ and $(2,2,1)$-solution of the parabolic equation (4.25) separately.

Figure 4.1: Solution in (4.28) for different values of $x$ and $t$
(1,1,1) and (1,2,2)-solutions

The (1) derivative of (4.28) with respect to \( t \) is given by

\[
D^1_{t,1} u(x, t) = (-\pi^2(r-1)e^{-\pi^2 t} \sin(\pi x), -\pi^2(1-r)e^{-\pi^2 t} \sin(\pi x)).
\]

It is not a fuzzy number because \( \sin(\pi x) \geq 0 \), for all \( x \in [0, 1] \). Therefore \( u(x, t) \) in (4.28) is not (1) differentiable with respect to \( t \). Hence (1, 1, 1) solution and (1, 2, 2) solution do not exist.

(2,1,1) and (2,2,1)-solutions

The (2) derivative of (4.28) with respect to \( t \) is given by

\[
D^1_{t,2} u(x, t) = (-\pi^2(1-r)e^{-\pi^2 t} \sin(\pi x), -\pi^2(r-1)e^{-\pi^2 t} \sin(\pi x)).
\]

It is a fuzzy number when \( \sin(\pi x) \geq 0 \). Hence it is a fuzzy number for \( (x, t) \in [0, 1] \times [0, 1] \).

The (1) derivative of (4.28) with respect to \( x \) is given by

\[
D^1_{x,1} u(x, t) = (\pi(r-1)e^{-\pi^2 t} \cos(\pi x), \pi(1-r)e^{-\pi^2 t} \cos(\pi x)).
\]

It is a fuzzy number when \( \cos(\pi x) \geq 0 \). Hence it is a fuzzy number for \( (x, t) \in [0, 1/2] \times [0, 1] \). Again the (2) derivative of \( D^1_{x,1} u(x, t) \) with respect to \( x \) is given by

\[
D^2_{x,1,2} u(x, t) = (\pi^2(r-1)e^{-\pi^2 t} \sin(\pi x), \pi^2(1-r)e^{-\pi^2 t} \sin(\pi x)).
\]

It is a fuzzy number when \( \sin(\pi x) \geq 0 \). Hence it is a fuzzy number for \( (x, t) \in [0, 1/2] \times [0, 1] \). Therefore (2, 1, 2)-solution of the fuzzy differential equation exists for \( (x, t) \in [0, 1/2] \times [0, 1] \). The (2) derivative of (4.28) with respect to \( x \) is given by \( D^1_{x,2} u(x, t) = (\pi(1-r)e^{-\pi^2 t} \cos(\pi x), \pi(r-1)e^{-\pi^2 t} \cos(\pi x)) \). It is a fuzzy number
when $\cos(\pi x) \leq 0$. Hence it is a fuzzy number for $(x, t) \in [1/2, 1] \times [0, 1]$. The (1) derivative of $D^1_{x,t}u(x,t)$ with respect to $x$ is given by

$$D^2_{x,t}u(x,t) = (\pi^2(r-1)e^{-\pi^2t} \sin(\pi x), \pi^2(1-r)e^{-\pi^2t} \sin(\pi x)).$$

(4.29)

It is a fuzzy number when $\sin(\pi x) \geq 0$. Hence it is a fuzzy number for $(x, t) \in [1/2, 1] \times [0, 1]$. Therefore $(2, 2, 1)$-solution of the fuzzy differential equation exists for $(x, t) \in [1/2, 1] \times [0, 1]$.

The error and order are defined as follows

$$E_{\infty}(h, k) = \max_{0 \leq j \leq m} |u(x_j, t_n) - v_{j,n}|, \quad \text{Order} = \frac{E_{\infty}(h, k)}{E_{\infty}(h/2, k/4)}.$$  

$$E_{\infty}(h, k) = \max_{0 \leq j \leq m} |u(x_j, t_n) - v_{j,n}|, \quad \text{Order} = \frac{E_{\infty}(h, k)}{E_{\infty}(h/2, k/4)}.$$  

In Table 4.1-4.3, we have obtained the errors as well as their convergence order for the difference scheme in the case when $h$ and $k$ decrease simultaneously for different $r$. Comparison of lower branch numerical and exact solutions for $r = 0.5$ when $m = n = 20$, $m = n = 40$ and $m = n = 80$ are given in Figure 4.2(a), Figure 4.2(b) and Figure 4.2(c) respectively. Comparison of upper branch numerical and exact solutions for $r = 0.7$ when $m = n = 20$, $m = n = 40$ and $m = n = 80$ are given in Figure 4.3(a), Figure 4.3(b) and Figure 4.3(c) respectively.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$k$</th>
<th>$E_{\infty}(h,k)$</th>
<th>Order</th>
<th>$\bar{E}_{\infty}(h,k)$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/20$</td>
<td>$1/20$</td>
<td>$6.79672 \times 10^{-6}$</td>
<td>3.91244</td>
<td>$6.79672 \times 10^{-6}$</td>
<td>3.91244</td>
</tr>
<tr>
<td>$1/40$</td>
<td>$1/40$</td>
<td>$4.51376 \times 10^{-7}$</td>
<td>3.99462</td>
<td>$4.51376 \times 10^{-7}$</td>
<td>3.99462</td>
</tr>
<tr>
<td>$1/80$</td>
<td>$1/80$</td>
<td>$2.83165 \times 10^{-8}$</td>
<td>*</td>
<td>$2.83165 \times 10^{-8}$</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 4.1: The maximum errors and convergence orders of difference scheme when $r = 0.3$.
Table 4.2: The maximum errors and convergence orders of difference scheme when \( r = 0.5 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( k )</th>
<th>( E_\infty(h, k) )</th>
<th>Order</th>
<th>( E_\infty(h, k) )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
<td>( 4.8548 \times 10^{-6} )</td>
<td>3.91244</td>
<td>( 4.8548 \times 10^{-6} )</td>
<td>3.91244</td>
</tr>
<tr>
<td>( \frac{1}{30} )</td>
<td>( \frac{1}{20} )</td>
<td>( 3.22412 \times 10^{-7} )</td>
<td>3.99462</td>
<td>( 3.22412 \times 10^{-7} )</td>
<td>3.99462</td>
</tr>
<tr>
<td>( \frac{1}{30} )</td>
<td>( \frac{1}{30} )</td>
<td>( 2.02261 \times 10^{-8} )</td>
<td>*</td>
<td>( 2.02261 \times 10^{-8} )</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 4.3: The maximum errors and convergence orders of difference scheme when \( r = 0.7 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( k )</th>
<th>( E_\infty(h, k) )</th>
<th>Order</th>
<th>( E_\infty(h, k) )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
<td>( 2.91288 \times 10^{-6} )</td>
<td>3.91244</td>
<td>( 2.91288 \times 10^{-6} )</td>
<td>3.91244</td>
</tr>
<tr>
<td>( \frac{1}{30} )</td>
<td>( \frac{1}{20} )</td>
<td>( 1.93447 \times 10^{-7} )</td>
<td>3.99462</td>
<td>( 1.93447 \times 10^{-7} )</td>
<td>3.99462</td>
</tr>
<tr>
<td>( \frac{1}{30} )</td>
<td>( \frac{1}{30} )</td>
<td>( 1.21356 \times 10^{-8} )</td>
<td>*</td>
<td>( 1.21356 \times 10^{-8} )</td>
<td>*</td>
</tr>
</tbody>
</table>

Figure 4.2: The comparison of exact solution (−) and numerical solution of the difference scheme defined by (4.23)(−) for the example at \( r = 0.5 \).

Figure 4.3: The comparison of exact solution (−) and numerical solution of the difference scheme defined by (4.23)(−) for the example at \( r = 0.7 \).

Now by using the Laplace transform, we find the solution of the remaining four systems \((1, 1, 2), (1, 2, 1), (2, 1, 1)\) and \((2, 2, 2)\).
These systems have the following form

\[ D_t u(x, t, r) = \beta^2 D_{xx} u(x, t, r), \]
\[ D_t \overline{u}(x, t, r) = \beta^2 D_{xx} \overline{u}(x, t, r), \quad 0 < x < 1, \; 0 < t < T, \; r \in [0, 1]. \] (4.30)

We denote \( L[u(x, t); s] = u_*(x, s). \) By taking the Laplace transform of (4.30) with respect to \( t \) and using the initial conditions (4.26), we get

\[ D_{xx} u_*(x, s) = s u_*(x, s) - (r - 1) \sin(\pi x), \] (4.31)
\[ D_{xx} \overline{u}_*(x, s) = s \overline{u}_*(x, s) - (1 - r) \sin(\pi x). \] (4.32)

Adding and subtracting (4.31) and (4.32), we get the following equations respectively

\[ D_{xx} (u_*(x, s) + \overline{u}_*(x, s)) - s(u_*(x, s) + \overline{u}_*(x, s)) = 0, \]
\[ D_{xx} (\overline{u}_*(x, s) - u_*(x, s)) + s(\overline{u}_*(x, s) - u_*(x, s)) = 2(1 - r) \sin(\pi x). \]

The solutions of above two equations are

\[ u_*(x, s) + \overline{u}_*(x, s) = a_1 e^{\sqrt{s}x} + b_1 e^{-\sqrt{s}x}, \]
\[ \overline{u}_*(x, s) - u_*(x, s) = a_2 \cos \sqrt{s}x + b_2 \sin \sqrt{s}x + \frac{2(r - 1) \sin \pi x}{\pi^2 - s}. \]

By taking Laplace transform of boundary condition in (4.27), we get \( a_1 = 0, b_1 = 0, a_2 = 0 \) and \( b_2 = 0. \) Therefore

\[ \overline{u}_*(x, s) = \frac{(1 - r) \sin \pi x}{s - \pi^2}, \quad u_*(x, s) = \frac{(r - 1) \sin \pi x}{s - \pi^2}. \] (4.33)
Taking inverse Laplace transform of (4.33), we get

\[ u(x, t) = (r - 1)e^{\pi^2 t} \sin \pi x, (1 - r)e^{\pi^2 t} \sin \pi x). \]  \hspace{1cm} (4.34)

\( u(x, t) \) represents a fuzzy number when \( \sin(\pi x) \geq 0 \). Hence (4.34) represents a fuzzy number when \( (x, t) \in [0, 1] \times [0, 1] \). Solution in (4.34) is plotted in Figure 4.4 for different values of \( x \) and \( t \). Now we find the range of \( (1, 1, 2), (1, 2, 1), (2, 1, 1) \) and \( (2, 2, 2) \)-solution of the parabolic equation (4.25) separately. The (1) derivative of (4.34) with respect to \( t \) is given by 

\[ (1,1,2)\text{-solution} \]

The (1)-derivative of (4.34) with respect to \( t \) is given by the \( D_{t,1} u(x, t) = (\pi^2(r - 1)e^{\pi^2 t} \sin(\pi x), \pi^2(1 - r)e^{\pi^2 t} \sin(\pi x)). \) It is a fuzzy number when \( \sin(\pi x) \geq 0 \). Hence it is a fuzzy number for \( (x, t) \in [0, 1] \times [0, 1] \).
The (1)-derivative of (4.34) with respect to $x$ is given by

$$D_{x,1}^1 u(x,t) = (\pi(r - 1)e^{\pi^2 t}\cos(\pi x), \pi(1 - r)e^{\pi^2 t}\cos(\pi x)).$$

It is a fuzzy number when $\cos(\pi x) \geq 0$. Hence it is a fuzzy number for $(x,t) \in [0, 1/2] \times [0, 1]$. Again the (2)-derivative of $D_{x,1}^1 u(x,t)$ with respect to $x$ is given by

$$D_{x,1,2}^2 u(x,t) = (-\pi^2(1 - r)e^{\pi^2 t}\sin(\pi x), -\pi^2(1 - r)e^{\pi^2 t}\sin(\pi x)).$$

It is a fuzzy number when $\sin(\pi x) \geq 0$. Hence it is a fuzzy number for $(x,t) \in [0, 1/2] \times [0, 1]$. Therefore (1,1,2)-solution of the fuzzy differential equation exists for $(x,t) \in [0, 1/2] \times [0, 1]$.

**(1,2,1)-solution**

The (2)-derivative of (4.34) with respect to $x$ is given by

$$D_{x,2}^1 u(x,t) = (\pi(1 - r)e^{\pi^2 t}\cos(\pi x), \pi(1 - r)e^{\pi^2 t}\cos(\pi x)). \quad (4.35)$$

Then it is a fuzzy number when $\cos(\pi x) \leq 0$. Hence it is a fuzzy number in $(x,t) \in [1/2, 1] \times [0, 1]$. Again the (1)-derivative of (4.35) with respect to $x$ is given by

$$D_{x,2,1}^2 u(x,t) = (-\pi^2(1 - r)e^{\pi^2 t}\sin(\pi x), -\pi^2(1 - r)e^{\pi^2 t}\sin(\pi x)). \quad (4.36)$$

Then it is a fuzzy number when $\sin(\pi x) \geq 0$. Hence it is a fuzzy number in $(x,t) \in [1/2, 1] \times [0, 1]$. Therefore (1,2,1)-solution of the fuzzy differential equation exists for $(x,t) \in [1/2, 1] \times [0, 1]$. 

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(2,1,1) and (2,2,2)-solutions

The (2)-derivative of (4.34) with respect to $t$ is given by

$$D_{i,2}^1u(x, t) = (\pi^2(1-r)e^{\pi^2t}\sin(\pi x), \pi^2(r-1)e^{\pi^2t}\sin(\pi x)).$$

(4.37)

It is a fuzzy number when $\sin(\pi x) \leq 0$. Hence it is not a fuzzy number for all $(x, t) \in [0, 1] \times [0, 1]$. Hence (2, 1, 1) solution and (2, 2, 2) solution do not exist.