CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

Homological algebra is at the root of modern techniques in many areas, including commutative algebra and algebraic geometry. While classical homological algebra can be viewed as based on projective, injective and flat modules, Gorenstein homological algebra is its relative version that uses Gorenstein projective, Gorenstein injective and Gorenstein flat modules. The methods of Gorenstein homological algebra play a part in commutative and non-commutative algebra, in algebraic geometry and in triangulated category theory. It also has applications to mathematical physics and to knot theory.

The subject of relative homological algebra was introduced by Eilenberg and Moore in their AMS Memoir Foundations of Relative Homological Algebra. We now have in hand more theorems guaranteeing the existence of (pre)covers and (pre)envelopes. Also, several new useful ideas have come into play since the appearance of Eilenberg and Moore’s work. Motivated by injective
envelopes and projective covers, many other varied notions of envelopes and covers have been defined and investigated in various settings. For instance, Fuchs in [26] and Warfield in [52] defined and studied pure injective envelopes and used them to describe algebraically compact abelian groups and modules. Enochs in [15, 16] defined torsion free coverings and proved the existence of torsion free coverings over any integral domain. Concerning envelopes and covers, there are two primary problems: (1) How can we define envelopes or covers in a general setting? (2) How can we prove the existence of the defined envelopes and covers? Enochs first in [17] noticed the categorical version of injective envelopes and then made a general definition of envelopes and covers by diagrams for a given class of modules. In this settings, all the existing envelopes and covers can be recovered by specializing the class of modules. Enochs conjectured that every module over any associative ring admits a flat cover. One of the reasons to believe this is true is because many properties of flat modules are highly dualized counterparts of those for injective modules. The notion of a cotorsion theory was introduced by Salce in [40] and its importance in homological algebra has been shown by its use in the proof of the “flat cover conjecture”.

The notion of Gorenstein flat modules was introduced and studied over Gorenstein rings by Enochs et al. [21], as a generalization of the notion of flat modules in the sense that an \( R \)-module is flat if and only if it is Gorenstein flat with finite flat dimension. Chen and Ding [10] generalized known characterizations of Gorenstein flat modules over Gorenstein rings to \( n \)-FC rings. Holm [32] relies on the use of character modules over coherent rings to translate results for Gorenstein injective modules to the setting of
1.1. BACKGROUND AND MOTIVATION

Gorenstein flat modules and so he has generalized the study of the Gorenstein flat dimension to coherent rings. These works motivated us to introduce Gorenstein $FI$-flat modules and to study their covers over $GFI_F$-closed rings.

Sather-Wagstaff et al. proved in [41] that iterating the process used to define Gorenstein projective modules exactly leads to the Gorenstein projective modules. Also, they established in [42] a stability of the subcategory of Gorenstein flat modules under a procedure to build $R$-modules from complete resolutions. Further, Samir Bouchiba et al. in [8] proved over a left $GF$-closed ring $R$, the stability of the Gorenstein flat modules under the very process used to define these entities. Wang and Liu in [51] proved that the two-degree strongly Gorenstein flat modules are nothing more than the strongly Gorenstein flat modules. Recently, Selvaraj et al. [47] proved that two-degree Gorenstein $n$-flat modules are again Gorenstein $n$-flat modules over a left $GF_n$-closed ring. Attracted by this work we have studied the stability of Gorenstein $FI$-flat modules over $GFI_F$-closed rings.

Tate cohomology was developed for representations of finite groups and the definition was extended by Avramov and Martsinkovsky [4] based on Tate projective resolutions (or complete resolutions), so that it can be used for finite modules of finite $G$-dimension over a Noetherian ring. Many authors have contributed to the development of this theory in different abelian categories. For example, Veliche [50] and Christensen and Jørgensen [11] have derived a Tate cohomology theory for complexes and Sather-Wagstaff et al. [41] constructed a theory of Tate cohomology in any abelian category $\mathcal{A}$ based on a so-called Tate $\mathcal{W}$-resolution, where $\mathcal{W}$ is a class of objects of $\mathcal{A}$. Iacob [35], Christensen and Jørgensen [11, 12] have nurtured a parallel theory of Tate homology. In
Recent times, a theory of Tate homology based on so-called Tate flat resolutions, was developed by Liang [36]. About relative homology, Sather-Wagstaff et al. [45] studied some base-change behavior of Gorenstein flat dimension and associated relative homology functors. Inspired by this, we have developed Tate homology of Gorenstein $FI$-flat modules and discussed the Tate homology of $FI$-cotorsion module of finite Gorenstein $FI$-flat dimension.

Foxby [25], Vasconcelos [49] and Golod [30] independently initiated the study of semidualizing modules, which are common generalizations of dualizing modules and finitely generated projective modules of rank one. Christensen [14] defined semidualizing complexes and studied them in the context of derived categories. Recently, Holm and White [34] extended the definition of the semidualizing module to a pair of arbitrary associative rings. Especially, they defined the so-called $C$-projective, $C$-injective and $C$-flat modules, to characterize the Auslander class $A_C(R)$ and the Bass class $B_C(R)$, with respect to a semidualizing module $C$. The notion of $C$-projective ($C$-injective, $C$-flat) modules is important for the study of the relative homological algebra with respect to semidualizing modules. For example, Holm and Jørgensen [31] used these modules to define $C$-Gorenstein injective (projective, flat) modules, introduced the notions of $C$-Gorenstein projective, $C$-Gorenstein injective and $C$-Gorenstein flat dimensions and investigated the properties of these dimensions. Further, White introduced in [53] the $G_C$-projective modules and gave a functorial description of the $G_C$-projective dimension of modules with respect to a semidualizing module $C$ over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [53]. This motivated us to carry out
a study on Gorenstein $FI$-flat dimension relative to a semidualizing module.

The above observations motivated our interest in the study of the Homological algebra and we have obtained some significant results in the following topics:

(1) Gorenstein $FI$-flat modules and their covers.

(2) Stability of Gorenstein $FI$-flat modules.

(3) Gorenstein $FI$-flat dimension and Tate Homology.

(4) Gorenstein $FI$-flat dimension and Relative Homology.

(5) Gorenstein $FI$-flat dimension relative to a semidualizing module.

1.2 Summary

This thesis presents all the results obtained by us on the above topics in six different chapters.

Chapter I of the thesis is intended to provide the basic definitions and results known already which will be needed in subsequent chapters.

In Chapter II, we introduce the concepts of Gorenstein $FI$-injective and Gorenstein $FI$-flat modules. Also, we prove that every $R$-module admits Gorenstein $FI$-flat (pre)covers over a $GFI_F$-closed ring.

In Chapter III, we are concerned with the stability of the class of Gorenstein $FI$-flat modules. We give an answer for the following natural question in the setting of a left $GFI_F$-closed ring $R$ : Given an exact sequence of Gorenstein $FI$-flat $R$-modules $G = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots$ such that the
complex $H \otimes_R G$ is exact for each Gorenstein $FI$-injective right $R$-module $H$, is the module $M := \text{Im}(G_0 \to G^0)$ a Gorenstein $FI$-flat module?

In Chapter IV, we investigate the Tate homology $\widehat{\text{tor}}$ of modules of finite Gorenstein $FI$-flat dimension. More precisely, a tight connection between Tate homology, relative homology and absolute homology is surfaced in this chapter. Further, the Tate homology $\widehat{\text{tor}}$ of a $FI$-cotorsion module of finite Gorenstein $FI$-flat dimension is discussed in this chapter.

Chapter VI presents some properties and behavior of finite Gorenstein $FI$-flat dimension through the methods of relative homological algebra.

In the final chapter, we study some properties of $G_C$-$FI$-flat modules, where $C$ is a semidualizing module and we investigate the relation between the $G_C$-yoke with the $C$-yoke of a module as well as the relation between the $G_C$-$FI$-flat resolution and the $FI$-flat resolution of a module over $GFI_F$-closed rings. We also obtain a criterion for computing the $G_C$-$FI$-flat dimension of modules.

### 1.3 Preliminaries

This section deals with a number of definitions and results which will be used throughout the thesis.

**Notations:**

Throughout this thesis, $R$ is an associative ring with identity and all $R$-modules are, if not specified otherwise, left $R$-modules.

- $R$-$\text{Mod}$ the class of all left $R$-modules.
- $\text{Mod}$-$R$ the class of all right $R$-modules.
• For an $R$-module $M$, we use $M^+$ to denote the character module $\text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})$ of $M$.

• Given a class $\mathcal{C}$ of left $R$-modules, we write

\[
\begin{align*}
\mathcal{C}^\perp &= \{ N \in R\text{-Mod} \mid \text{Ext}_R^1(M, N) = 0, \forall M \in \mathcal{C} \} \\
\perp \mathcal{C} &= \{ N \in R\text{-Mod} \mid \text{Ext}_R^1(N, M) = 0, \forall M \in \mathcal{C} \} \\
\mathcal{C}^\top &= \{ M \in \text{Mod-R} \mid \text{Tor}_R^1(M, C) = 0, \forall C \in \mathcal{C} \} \\
\mathcal{C}^\perp &= \{ M \in R\text{-Mod} \mid \text{Tor}_R^1(C, M) = 0, \forall M \in \mathcal{C} \}.
\end{align*}
\]

• $pd(M)$ is projective dimension of $M$.

• $id(M)$ is injective dimension of $M$.

• $fd(M)$ is flat dimension of $M$.

• $FP-id(M)$ is $FP$-injective dimension of $M$.

• $\mathcal{M}(R)$ is the category of left $R$-modules.

• $\mathcal{GP}(R)$ is the class of all Gorenstein projective left $R$-modules.

• $\mathcal{GI}(R)$ is the class of all Gorenstein injective right $R$-modules.

• $\mathcal{GF}(R)$ is the class of all Gorenstein flat left $R$-modules.

• $Gfd_R(M)$ is Gorenstein flat dimension of a left $R$-module $M$.

Basic Definitions and Results:

**Definition 1.3.1.** [18] An $R$-module $P$ is said to be projective if given an exact sequence $A \xrightarrow{\phi} B \rightarrow 0$ of $R$-modules and an $R$-homomorphism $f : P \rightarrow B$, there exists an $R$-homomorphism $g : P \rightarrow A$ such that $f = \phi \circ g$. Dually,
we have the definition of injective modules and an $R$-module $F$ is said to be flat if given any exact sequence $0 \to A \to B$ of right $R$-modules, the tensored sequence $0 \to A \otimes F \to B \otimes F$ is exact. $\mathcal{P}roj$ (resp. $\mathcal{I}nj$ and $\mathcal{F}$) denote the classes of projective (resp. injective and flat) $R$-modules.

Let $\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} M \to 0$ be a projective resolution of a left $R$-module $M$ and consider the deleted projective resolution $\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \to 0$. Then, for each left $R$-module $N$, the $i$th cohomology module of the complex $0 \to \text{Hom}_R(P_0, N) \xrightarrow{\text{Hom}_R(\delta_1, N)} \text{Hom}_R(P_1, N) \xrightarrow{\text{Hom}_R(\delta_2, N)} \cdots$ is denoted by $\text{Ext}_R^i(M, N)$ and is defined by $\text{Ext}_R^i(M, N) = \ker(\text{Hom}_R(\delta_{i+1}, N))/\text{im}(\text{Hom}_R(\delta_i, N))$. Similarly, let $\cdots \to P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} M \to 0$ be a projective resolution of a right $R$-module $M$ and $N$ be a left $R$-module. Then, the $i$th homology module of the complex $\cdots \to P_2 \otimes_R N \xrightarrow{\delta_2 \otimes_R N} P_1 \otimes_R N \xrightarrow{\delta_1 \otimes_R N} P_0 \otimes_R N \to 0$ is denoted by $\text{Tor}_R^i(M, N)$ and is defined by $\text{Tor}_R^i(M, N) = \ker(\delta_i \otimes_R N)/\text{im}(\delta_{i+1} \otimes_R N)$.

Let $M$ and $N$ be $R$-modules. $\text{Hom}(M, N)$ (resp. $\text{Ext}^i(M, N)$) means $\text{Hom}_R(M, N)$ (resp. $\text{Ext}^i_R(M, N)$), and similarly, $M \otimes N$ (resp. $\text{Tor}_i(M, N)$) denotes $M \otimes_R N$ (resp. $\text{Tor}_i^R(M, N)$) for an integer $i \geq 1$.

**Proposition 1.3.2.** [9] If $C$ is injective $R$-module, then $\text{Ext}^1(A, \text{Hom}(B, C)) \cong \text{Hom}(\text{Tor}_1(A, B), C)$. where $A$ and $B$ are $R$-modules.

**Theorem 1.3.3.** [39] Let $\{A_k\}_{k \in I}$ be a class of right $R$-modules and $B$ be any left $R$-module. Then

$$\text{Tor}_n(\bigoplus A_k, B) \cong \bigoplus \text{Tor}_n(A_k, B) \forall n \geq 0.$$
Lemma 1.3.4. [39] (Schanuel’s Lemma) Let
\[ 0 \to K \to P \to M \to 0 \]
\[ 0 \to K' \to P' \to M \to 0 \]
be exact where \( P \) and \( P' \) are projective. Then there is an isomorphism
\[ K \oplus P' \cong K' \oplus P. \]

Definition 1.3.5. [48] A left \( R \)-module \( M \) is called absolutely pure (or FP-injective) if \( \text{Ext}^1(N, M) = 0 \) for all finitely presented left \( R \)-modules \( N \). \( \text{FP} \) denotes the class of all absolutely pure left \( R \)-modules.

Definition 1.3.6. [37] A left \( R \)-module \( M \) is called FI-injective if \( \text{Ext}^1(G, M) = 0 \) for any FP-injective left \( R \)-module \( G \).

Definition 1.3.7. [37] A right \( R \)-module \( N \) is said to be FI-flat if \( \text{Tor}^1(N, G) = 0 \) for any FP-injective left \( R \)-module \( G \).

Definition 1.3.8. [37] Let \( M \) be a left \( R \)-module. The FP-injective dimension of \( M \), denoted by \( \text{FP-id}(M) \), is defined to be the smallest nonnegative integer \( n \) such that \( \text{Ext}^{n+1}(F, M) = 0 \) for every finitely presented left \( R \)-module \( F \). If no such \( n \) exists, set \( \text{FP-id}(M) = \infty \).

Definition 1.3.9. Let \( R \) be a ring. For a left \( R \)-module \( N \), let \( \text{fd}(N) \) denote the smallest integer \( n \geq 0 \) such that \( \text{Tor}_{n+1}(N, N) = 0 \) for every finitely presented left \( R \)-module \( K \) and call \( \text{fd}(N) \) the flat dimension of \( N \). If no such \( n \) exists, set \( \text{fd}(N) = \infty \).

Definition 1.3.10. For a left \( R \)-Module \( M \), the FI-injective dimension \( \text{FI-id}(M) \) of \( M \) to the smallest integer \( n \geq 0 \) such that \( \text{Ext}^{n+1}(N, M) = 0 \) for any FP-injective left \( R \)-module \( N \). If no such \( n \) exists set \( \text{FI-id}(M) = \infty \).
1.3. PRELIMINARIES

Definition 1.3.11. [18] A ring $R$ is said to be right (resp. left) coherent if every finitely generated right (resp. left) ideal of $R$ is finitely presented. A ring $R$ is said to be coherent if it is both left and right coherent.

Proposition 1.3.12. [37] The following hold for a left coherent ring $R$.

1 A left $R$-module $M$ is injective if and only if $M$ is FI-injective and $FP-id(M) < 1$.

2 A right $R$-module $N$ is flat if and only if $N$ is FI-flat and $fd(N) < 1$.

Theorem 1.3.13. [39] A right $R$-module $B$ is flat if and only if its character module $B^+$ is an injective left $R$-module.

Lemma 1.3.14. [37] A right $R$-module $M$ is FI-flat if and only if its character module $M^+$ is an FI-injective left $R$-module.

Definition 1.3.15. [17] Let $I$ be a directed set, that is, $I$ is a partially ordered set such that for any $i, j \in I$ there is a $k \in I$ with $i, j \leq k$. Let $\{M_i\}_{i \in I}$ be a family of $R$-modules and suppose for each pair $i, j \in I$ with $i \leq j$ there is an $R$-homomorphism $f_{ji} : M_i \to M_j$ such that

1. $f_{ii} = 1_{M_i}$ for each $i \in I$

2. if $i \leq j \leq k$, then $f_{kj} \circ f_{ji} = f_{ki}$.

Then we say that the $R$-modules $M_i$ together with the homomorphism $f_{ji}$ form a direct system which is denoted $((M_i), (f_{ji}))$.

Definition 1.3.16. [17] The direct limit of a direct system $((M_i), (f_{ji}))$ of $R$-modules is an $R$-module $M$ with $R$-homomorphisms $g_i : M_i \to M$ for $i \in I$ with $g_i = g_j \circ f_{ji}$ whenever $i \leq j$ and such that if $(N, \{h_i\})$ is another such
family, then there is a unique $R$-homomorphism $f : M \to N$ such that $f \circ g_i = h_i$ for all $i \in I$. The direct limit of $(M, \{g_i\})$ is denoted by $\lim\limits_{\longrightarrow} M_i$.

**Definition 1.3.17.** [18] A submodule $A$ of an $R$-module $B$ is said to be a pure submodule if $0 \to M \otimes A \to M \otimes B$ is exact for all right $R$-modules $M$, equivalently, if $\text{Hom}(N, B) \to \text{Hom}(N, B/A) \to 0$ is exact for all finitely presented $R$-modules $N$. An exact sequence $0 \to A \to B \to B/A \to 0$ is said to be pure exact if $A$ is a pure submodule of $B$.

An $R$-module $H$ is said to be pure injective if for every pure exact sequence $0 \to A \to B$ of $R$-modules, $\text{Hom}(B, H) \to \text{Hom}(A, H) \to 0$ is exact. Clearly, every injective module is pure injective.

**Definition 1.3.18.** [17] Let $C$ be a class of $R$-modules and $X$ an $R$-module. We say that a homomorphism $f : C \to X$ is a $C$-precover if $C \in C$ and the abelian group homomorphism $\text{Hom}(C', f) : \text{Hom}(C', C) \to \text{Hom}(C', X)$ is surjective for each $C' \in C$. A $C$-precover $f : C \to X$ is called a $C$-cover if every endomorphism $g : C \to C$ such that $fg = f$ is an isomorphism. Dually, we have the definition of a $C$-(pre) envelope. $C$-covers ($C$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

**Definition 1.3.19.** [32] Let $M$ be an $R$-module and $C$ be class of left $R$-modules. Then

1. A left $C$-resolution of $M$ is an exact sequence $\cdots \to C_1 \to C_0 \to M \to 0$ with $C_i \in C$ for all $i \geq 0$.

2. A right $C$-resolution of $M$ is an exact sequence $0 \to M \to C_0 \to C_1 \to \cdots$ with $C_i \in C$ for all $i \geq 0$. 
1.3. PRELIMINARIES

A left (right) resolution of $M$ is proper if $\text{Hom}_R(C, -)$ leaves left (right) resolution of $M$ exact whenever $C \in \mathcal{C}$.

**Theorem 1.3.20.** [18] Let $\mathcal{X}$ be a precovering class closed under finite direct sums of an abelian category $\mathcal{AB}$ and $0 \to M' \to M \to M'' \to 0$ be a $\text{Hom}(\mathcal{X}, -)$ exact complex of objects of $\mathcal{AB}$. Then,

(a) If $T$ is a covariant functor, there is a long exact sequence $\cdots \to L_nT(M'') \to L_{n-1}T(M') \to L_{n-1}T(M) \to \cdots \to L_0T(M') \to L_0T(M) \to L_0T(M'') \to 0$.

(b) If $T$ is a contravariant functor, there is a long exact sequence $0 \to R^0T(M'') \to R^0T(M) \to R^0T(M') \to \cdots \to R^{n-1}T(M'') \to R^{n-1}T(M) \to R^{n-1}T(M') \to R^nT(M'') \to \cdots$.

**Theorem 1.3.21.** [54] Let $\mathcal{X}$ be a class of left $R$-modules and $\mathcal{X}$ is closed under direct limits. If $M$ has an $\mathcal{X}$-precover, then $M$ has an $\mathcal{X}$-cover.

**Lemma 1.3.22.** [54] Let $\phi : X \to M$ be an $\mathcal{X}$-cover of $M$, and assume that $\mathcal{X}$ is closed under extensions. Set $K = \text{Ker}(\phi)$. Then, $\text{Ext}^1(X', K) = 0$ for any $X' \in \mathcal{X}$.

**Theorem 1.3.23.** [29]

(a) Let $\mathcal{X}$ be any class of left $R$-modules. Then, every module has a $\uparrow \mathcal{X}$-cover.

(b) Let $\mathcal{Y}$ be any class consisting of character modules (of left $R$-modules). Then, every module has a $\downarrow \mathcal{Y}$-cover.

**Definition 1.3.24.** [18] A pair of classes of $R$-modules $(\mathcal{A}, \mathcal{B})$ is said to be a cotorsion pair if $\mathcal{A} = \downarrow \mathcal{B}$ and $\downarrow \mathcal{B} = \mathcal{A}$. Two simple examples of cotorsion pair in the category of $R$-modules are $(\text{Proj}, R - \text{Mod})$ and $(R - \text{Mod}, \text{Inj})$. 

12
Definition 1.3.25. [18] A cotorsion pair \((A, B)\) is said to be complete if for any \(R\)-module \(X\) there are exact sequences \(0 \to X \to B \to A \to 0\) and \(0 \to B' \to A' \to X \to 0\) with \(B, B' \in B\) and \(A, A' \in A\). A cotorsion pair \((A, B)\) is said to be perfect if every \(R\)-module has an \(A\)-cover and a \(B\)-envelope. A cotorsion pair \((A, B)\) is said to be hereditary if whenever \(0 \to A' \to A \to A'' \to 0\) is exact with \(A, A'' \in A\) then \(A'\) is also in \(A\), or equivalently, if \(0 \to B' \to B \to B'' \to 0\) is exact with \(B', B \in B\) then \(B''\) is also in \(B\).

Proposition 1.3.26. [7] The flat cotorsion theory in the category of the left \(R\)-modules (for some ring \(R\)) is cogenerated by a set.

Corollary 1.3.27. [22] Let \(E\) be an injective \(R\)-module. Then, \(E\) may be expressed as the direct limit of a directed system of submodules of \(E\) which are in the smallest class of \(R\)-modules.

Lemma 1.3.28. [21] If \(R\) is right coherent and \(0 \to M \to F^0 \to F^1 \to \cdots\) is a flat resolvent of the left \(R\)-module \(M\), then for any injective right \(R\)-module \(E\), \(0 \to E \otimes M \to E \otimes F^0 \to E \otimes F^1 \to \cdots\) is an exact sequence.

Definition 1.3.29. [32] Let \(R\) be a ring and let \(\mathfrak{X}\) be a class of left \(R\)-modules.

1. \(\mathfrak{X}\) is closed under extensions: If for every short exact sequence of left \(R\)-modules \(0 \to A \to B \to C \to 0\), the conditions \(A\) and \(C\) are in \(\mathfrak{X}\) implies \(B\) is in \(\mathfrak{X}\).

2. \(\mathfrak{X}\) is closed under kernels of epimorphisms: If for every short exact sequence of left \(R\)-modules \(0 \to A \to B \to C \to 0\), the conditions \(B\) and \(C\) are in \(\mathfrak{X}\) implies \(A\) is in \(\mathfrak{X}\).

3. \(\mathfrak{X}\) is projectively resolving: If it contains all projective modules and it is closed under both extensions and kernels of epimorphisms. i.e., for every
short exact sequence of $R$-modules $0 \to A \to B \to C \to 0$ with $C \in \mathfrak{X}$, the conditions $A \in \mathfrak{X}$ and $B \in \mathfrak{X}$ are equivalent.

**Proposition 1.3.30.** [32] Let $\mathfrak{X}$ be a class of $R$-modules which is either projectively resolving, or injective resolving. If $\mathfrak{X}$ is closed under countable direct sums, or closed under countable direct products, then $\mathfrak{X}$ is also closed under direct summands.

**Definition 1.3.31.** [18] Let $\mathcal{C}, \mathcal{D}, \text{and } \mathcal{E}$ be abelian categories and $T : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be an additive functor contravariant in the first variable and covariant in the second. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes of objects of $\mathcal{C}$ and $\mathcal{D}$ respectively. Then, $T$ is said to be right balanced by $\mathfrak{X} \times \mathfrak{Y}$ if for each object $M$ of $\mathcal{C}$, there is a $T(-, \mathfrak{Y})$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ with each $F_i \in \mathfrak{X}$, and if for every object $N$ of $\mathcal{D}$, there is a $T(\mathfrak{X}, -)$ exact complex $0 \to N \to G^0 \to G^1 \to \cdots$ with $G^i \in \mathfrak{Y}$. If, on the other hand, the complex $\cdots \to F_1 \to F_0 \to M \to 0$ is $T(\mathfrak{Y}, -)$ exact and the complex $0 \to N \to G^0 \to G^1 \to \cdots$ is $T(-, \mathfrak{X})$ exact, then $T$ is said to be left balance by $\mathfrak{Y} \times \mathfrak{X}$.

**Definition 1.3.32.** [36] A complex

$$
\cdots \to X_1 \overset{\delta^X_1}{\to} X_0 \overset{\delta^X_0}{\to} X_{-1} \to \cdots
$$

of $R$-modules will be denoted by $(X, \delta^X)$ or simply $X$. We frequently (and without warning) identify $R$-modules with complexes concentrated in degree 0. The $n$th cycle (resp. homology) of $X$ is defined as $\text{Ker} \delta^X_n$ (resp. $\text{Ker} \delta^X_n / \text{Im} \delta^X_{n+1}$) and it is denoted by $Z_n(X)$ (resp. $H_n(X)$). For any $m \in \mathbb{Z}$, $\Sigma^m X$ denotes the complex with the degree-$n$ term $(\Sigma^m X)_n = X_{n-m}$ and whose boundary operators are $(-1)^m \delta^X_{n-m}$. We set $\Sigma M = \Sigma^1 M$. 

14
If \( X \) and \( Y \) are both complexes, then by a morphism \( \alpha : X \to Y \) we mean a sequence \( \alpha_n : X_n \to Y_n \) such that \( \alpha_{n-1}\delta_n^X = \delta_n^Y \alpha_n \) for each \( n \in \mathbb{Z} \). A quasi isomorphism, indicated by the symbol “\( \simeq \)”, is a morphism of complexes that induces an isomorphism in homology. The mapping cone \( \text{Cone}(\alpha) \) of \( \alpha \) is defined as \( \text{Cone}(\alpha)_n = Y_n \oplus X_{n-1} \) with \( n \)th boundary operator \( \delta_{n}^{\text{Cone}(\alpha)} = \begin{pmatrix} \delta_n^Y & \alpha_{n-1} \\ 0 & -\delta_{n-1}^X \end{pmatrix} \). A complex \( X \) is contractible if there is an isomorphism \( \alpha \) of complexes such that \( X \cong \text{Cone}(\alpha) \).

**Fact 1.3.33.** If \( X \) is a contractible complex, then \( X \) is exact and \( M \otimes - \) exacts it for any right \( R \)-module \( M \).

We recall from [20] it is convenient to assume that for a complex \( C \), we have \( C_i \cap C_j = \emptyset \) when \( i \neq j \). So then, we take \( x \in C \) to mean \( x \in \bigcup_{i \in \mathbb{Z}} C_i \). We note that in actual practice we may have \( C_i \cap C_j \neq \emptyset \) with \( i \neq j \). The cardinality of a complex \( C \) is denoted by \( |C| \) or \( \text{Card}(C) \) and is defined to be \( \sum_{i \in \mathbb{Z}} |C_i| \), i.e. \( |\bigcup_{i \in \mathbb{Z}} C_i| \).

**Definition 1.3.34.** [23] A class \( \mathcal{K} \) of \( R \)-modules is a Kaplansky class if there exists a cardinal \( \mathcal{N} \) such that for every \( M \in \mathcal{K} \) and for each \( x \in M \), there exists a submodule \( F \) of \( M \) such that \( x \in F \subset M, F, M/F \in \mathcal{K} \) and \( \text{Card}(F) \leq \mathcal{N} \).

**Lemma 1.3.35.** [18] Let \( M \) and \( N \) be \( R \)-modules. Then, there is a cardinal number \( \aleph_\alpha \) dependent on \( \text{Card}(N) \) and \( \text{Card}(R) \) such that for any morphism \( f : N \to M \), there is a pure submodule \( S \) of \( M \) such that \( f(N) \subset S \) and \( \text{Card}(S) \leq \aleph_\alpha \).

**Theorem 1.3.36.** [18] Let \( M \) and \( N \) be \( R \)-modules and suppose \( M \) is the union of a continuous chain of submodules \( (M_\alpha)_{\alpha<\lambda} \). Then, if \( \text{Ext}^1(M_0, N) = 0 \) and \( \text{Ext}^1(M_{\alpha+1}/M_\alpha, N) = 0 \) whenever \( \alpha + 1 < \lambda \), then \( \text{Ext}^1(M, N) = 0 \).
Theorem 1.3.37. [23] Let \( \mathcal{K} \) be a Kaplansky class. If \( \mathcal{K} \) contains the projective modules and it is closed under extensions and direct limits then \( (\mathcal{K}, \mathcal{K}^\perp) \) is a perfect cotorsion theory.

Definition 1.3.38. [18] A module \( M \) is said to be Gorenstein projective if there exists a \( \text{Hom}(\cdot, \text{Proj}) \) exact exact sequence

\[
\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots
\]

of projective modules such that \( M = \text{Ker}(P^0 \to P^1) \).

Definition 1.3.39. [18] A module \( M \) is said to be Gorenstein injective if there exists a \( \text{Hom}(\text{Inj}, \cdot) \) exact exact sequence

\[
\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots
\]

of injective modules such that \( M = \text{Ker}(E^0 \to E^1) \).

Definition 1.3.40. [38] A left \( R \)-module \( M \) is said to be Gorenstein flat, if there exists an exact sequence of flat left \( R \)-modules,

\[
\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots
\]

such that \( M \cong \text{Im}(F_0 \to F^0) \) and such that \( B \otimes - \) leaves the sequence exact whenever \( B \) is an injective right \( R \)-module.

Definition 1.3.41. [20] The Gorenstein flat dimension of a left \( R \)-module \( M \) is defined by \( Gfd_R(M) \leq m \) if and only if \( M \) has a resolution of Gorenstein flat modules of length \( m \). If there is no such \( m \) exists, set \( Gfd_R(M) = \infty \).

Definition 1.3.42. [6] An \( R \)-module \( M \) is called a strongly Gorenstein flat module if there exists an exact sequence of \( R \)-modules,

\[
0 \to M \to F \to M \to 0
\]
such that $F$ is a flat $R$-module and $I \otimes -$ leaves this sequence exact whenever $I$ is an injective right module over $R$.

**Definition 1.3.43.** [38] A left $R$-module $M$ is said to be Gorenstein $FP$-injective if there exists a $\text{Hom}(FP, -)$ exact sequence

$$\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

of injective left $R$-modules such that $M = \text{Ker}(E^0 \rightarrow E^1)$. We denote $\mathcal{GF}(R)$ is the class of all Gorenstein $FP$-injective left $R$-modules.

**Theorem 1.3.44.** [32] The class $\mathcal{GP}(R)$ of all Gorenstein projective $R$-modules is projectively resolving. Furthermore, $\mathcal{GP}(R)$ is closed under arbitrary direct sums and under direct summands.

**Proposition 1.3.45.** [23] If $R$ is a left noetherian ring and $GI$ is the class of all Gorenstein injective left $R$-modules then $GI$ is a Kaplansky class.

**Proposition 1.3.46.** [23] Given a ring $R$, the class of Gorenstein flat modules is a Kaplansky class.

**Theorem 1.3.47.** [32] For any (left) $R$-module $M$, we consider the following conditions.

(1) $M$ is a Gorenstein flat (left) $R$-module;

(2) The Pontryagin dual $\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective right $R$-module;

(3) $M$ admits a co-proper right flat resolution and $\text{Tor}_i(I, M) = 0$ for all injective right $R$-modules $I$, and all integers $i > 0$. 
Then, $(1) \Rightarrow (2)$. If $R$ is right coherent, then also $(2) \Rightarrow (3) \Rightarrow (1)$, and hence all three conditions are equivalent.

**Proposition 1.3.48.** [32] The class $\mathcal{GF}(R)$ is closed under arbitrary direct sums.

**Theorem 1.3.49.** [22] Let $R$ be a right coherent ring. Then, $(\mathcal{GF}, \mathcal{GF}^\perp)$ is a hereditary perfect cotorsion theory.

**Corollary 1.3.50.** [27] Let $R$ be a right coherent ring. Then, every cotorsion $R$-module with finite flat dimension is Gorenstein cotorsion.

**Proposition 1.3.51.** [32] Assume that $R$ is right coherent. If $T$ is a Gorenstein flat $R$-module, then $\text{Ext}^i(T, K) = 0$ for all integers $i > 0$, and all cotorsion $R$-modules $K$ with finite flat dimension.

**Theorem 1.3.52.** [32] Assume that $R$ is right coherent. If any two of the modules $M, M'$ or $M''$ in a short exact sequence $0 \to M' \to M \to M'' \to 0$ have finite Gorenstein flat dimension, then so has the third.

**Proposition 1.3.53.** [32] Assume that $R$ is right coherent. If $(M_\lambda)_{\lambda \in \Lambda}$ is any family of (left) $R$-modules, then we have an equality,

$$Gfd_R(\coprod M_\lambda) = \sup\{Gfd_R M_\lambda | \lambda \in \Lambda\}.$$ 

**Theorem 1.3.54.** [32] Assume that $R$ is right coherent ring, and that $M$ is an $R$-module with finite Gorenstein flat dimension $m$. Then, $M$ admits a surjective Gorenstein flat precover $\phi : T \to M$, where $K = \text{Ker} \phi$ satisfies $fd_R K = m - 1$ (if $m = 0$, this should be interpreted as $K = 0$). In particular, $M$ admits a proper left Gorenstein flat resolution of length $m$. 

18
**Theorem 1.3.55.** [32] If $R$ is right coherent, then the class $\mathcal{G}\mathcal{F}(R)$ of Gorenstein flat $R$-modules is projectively resolving and closed under direct summands. Furthermore, if $M_0 \to M_1 \to M_2 \to \cdots$ is a sequence of Gorenstein flat modules, then the direct limit $\varinjlim M_n$ is again Gorenstein flat.

**Theorem 1.3.56.** [32] Assume that $R$ is right coherent. Let $M$ be a (left) $R$-module with finite Gorenstein flat dimension, and let $m \geq 0$ be an integer. Then, the following four conditions are equivalent:

(a) $Gfd_R(M) \leq m$;

(b) $\text{Tor}_i(L, M) = 0$ for all right $R$-modules $L$ with finite $id_R(L)$, and all $i > m$;

(c) $\text{Tor}_i(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > n$;

(d) For every exact sequence $0 \to K_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0$, where $G_0, \cdots, G_{m-1}$ are Gorenstein flats, then also $K_n$ is Gorenstein flat.

**Proposition 1.3.57.** [6] Let $M$ be an $R$-module. Then, the following are equivalent:

(1) $M$ is a strongly Gorenstein flat module;

(2) There exists a short exact sequence $0 \to M \to F \to M \to 0$, where $F$ is a flat module and $\text{Tor}_1(M, I) = 0$ for any injective module $I$;

(3) There exists a short exact sequence $0 \to M \to F \to M \to 0$, where $F$ is a flat module and $\text{Tor}_1(M, I') = 0$ for any module $I'$ with finite injective dimension;
(4) There exists a short exact sequence \( 0 \to M \to F \to M \to 0 \), where \( F \) is a flat module such that the sequence \( 0 \to M \otimes I \to F \otimes I \to M \otimes I \to 0 \) is exact for any injective module \( I \);

(5) There exists a short exact sequence \( 0 \to M \to F \to M \to 0 \), where \( F \) is a flat module such that the sequence \( 0 \to M \otimes I' \to F \otimes I' \to M \otimes I' \to 0 \) is exact for any module \( I' \) with finite injective dimension.

**Theorem 1.3.58.** [6] If a module is Gorenstein flat, then it is a direct summand of a strongly Gorenstein flat modules.

**Theorem 1.3.59.** [38] Let \( R \) be a left coherent ring. Then, the following are equivalent for a left \( R \)-module \( M \):

1. \( M \) is Gorenstein \( FP \)-injective;
2. \( M \) has an exact left \( FP \)-resolution and \( \text{Ext}^i(F,M) = 0 \) for all \( FP \)-injective left \( R \)-modules \( F \) and all \( i \geq 1 \);
3. \( M \) has an exact left \( FP \)-resolution and \( \text{Ext}^i(F,M) = 0 \) for all left \( R \)-modules \( F \) with \( \mathcal{FP}\text{-id}(F) < \infty \) and all \( i \geq 1 \); Moreover, if \( \mathcal{FP}\text{-id}(R) < \infty \), then the above conditions are equivalent to
4. \( \text{Ext}^i(F,M) = 0 \) for all \( FP \)-injective left \( R \)-modules \( F \) and all \( i \geq 1 \).

**Definition 1.3.60.** [5] A ring \( R \) is said to be left \( GF \)-closed if \( GF(R) \) is closed under extensions.

**Theorem 1.3.61.** [5] The following conditions are equivalent for a ring \( R \):

1. \( R \) is left \( GF \)-closed;
2. The class \( GF(R) \) is projectively resolving;
(3) For every short exact sequence of left $R$-modules $0 \to G_1 \to G_0 \to M \to 0$, where $G_0$ and $G_1$ are Gorenstein flat. If $\text{Tor}_1(I, M) = 0$ for all injective right $R$-modules $I$, then $M$ is Gorenstein flat.

**Lemma 1.3.62.** [4] Let $T \xrightarrow{\gamma} P \xrightarrow{\pi} M$ and $T' \xrightarrow{\gamma'} P' \xrightarrow{\pi'} M'$ be complete resolutions. For each homomorphism of modules $\mu : M \to M'$ there exists a unique up to homotopy morphism $\overline{\mu}$, making the right hand square of the diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\gamma} & P \\
\downarrow{\overline{\mu}} & & \downarrow{\overline{\mu}} \\
T' & \xrightarrow{\gamma'} & P'
\end{array}
\begin{array}{ccc}
P & \xrightarrow{\mu} & M \\
\downarrow{\mu} & & \downarrow{\mu} \\
M' & & M'
\end{array}
$$

commute, and for each choice of $\overline{\mu}$ there exists a unique up to homotopy morphism $\widehat{\mu}$, making the left hand square commute up to homotopy. Finally, if $\mu = 1_M$, then $\overline{\mu}$ and $\widehat{\mu}$ are homotopy equivalences.

**Lemma 1.3.63.** [4] Let $0 \to M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \to 0$ be a GF-exact sequence of $R$-modules. Fix proper Gorenstein flat resolution $\gamma : G \to M$ and $\gamma' : G' \to M'$, and fix flat resolutions $\pi : F \to M$ and $\pi' : F' \to M'$. There exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
0 & \to & F \\
\downarrow{\phi} & & \downarrow{\phi'} \\
0 & \to & G
\end{array}
\begin{array}{ccc}
\xrightarrow{\overline{\mu}} & \xrightarrow{\overline{\mu'}} & \xrightarrow{\overline{\mu''}} \\
\downarrow{\phi''} & & \downarrow{\phi''} \\
0 & \to & G''
\end{array}
\begin{array}{ccc}
0 & \to & M \\
\downarrow{\gamma} & & \downarrow{\gamma'} \\
0 & \to & M'
\end{array}
\begin{array}{ccc}
\xrightarrow{\mu} & \xrightarrow{\mu'} & \xrightarrow{\mu''} \\
\downarrow{\gamma''} & & \downarrow{\gamma''} \\
0 & \to & M''
\end{array}
\to 0$$
where the top and middle rows are degree-wise split exact, $\gamma'$ is a proper Gorenstein flat resolution, $\pi' = \gamma'\phi'$ is a FI-flat resolution, and $\pi = \gamma\phi$ and $\pi'' = \gamma''\phi''$.

**Proposition 1.3.64.** [36] Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of right $R$-modules. If $N$ is an $R$-module admitting a Tate flat resolution, then there is a long exact sequence

$$\cdots \rightarrow \hat{\text{tor}}_i(M', N) \rightarrow \hat{\text{tor}}_i(M, N) \rightarrow \hat{\text{tor}}_i(M'', N) \rightarrow \hat{\text{tor}}_{i-1}(M', N) \rightarrow \cdots.$$ 

**Definition 1.3.65.** [53] A degree wise finite projective (resp. free) resolution of an $R$-module $M$ is a projective (resp. free) resolution $P$ of $M$ such that each $P_i$ is a finitely generated projective (resp. free). Note that $M$ admits a degree wise finite projective resolution if and only if it admits a degree wise finite free resolution. However, it is possible for a module to admit a bounded degree wise finite projective resolution but not to admit a bounded degree wise finite free resolution. For example, if $R = k_1 \oplus k_2$, where $k_1$ and $k_2$ are fields, then $M = k_1 \oplus 0$ is a projective $R$-module, but it does not admit a bounded free resolution.

**Definition 1.3.66.** [34] Let $R$ and $S$ be rings. An $(S, R)$-bimodule $C$ is called semidualizing if the following conditions are satisfied:

1. $S_C$ admits a degree wise finite $S$-projective resolution;
2. $C_R$ admits a degree wise finite $R^{op}$-projective resolution;
3. The homothety map $S_S \rightarrow \text{Hom}_{R^{op}}(C, C)$ is an isomorphism;
4. The homothety map $R_R \rightarrow \text{Hom}_S(C, C)$ is an isomorphism;
1.3. PRELIMINARIES

(5) $\text{Ext}^i_S(C, C) = 0$ for any $i \geq 1$;

(6) $\text{Ext}^i_{R^{\text{op}}}(C, C) = 0$ for any $i \geq 1$.

**Definition 1.3.67.** [34] Let $C$ be a semidualizing module for a ring $R$. An $R$-module is $C$-projective if it has the form $C \otimes P$ for some projective module $P$. An $R$-module is called $C$-injective if it has the form $\text{Hom}(C, I)$ for some injective module $I$. Set

$$\mathcal{P}_C(R) = \{C \otimes P \mid P \text{ is } R\text{-projective}\},$$

and

$$\mathcal{I}_C(R) = \{\text{Hom}(C, I) \mid I \text{ is } R\text{-injective}\}.$$

**Lemma 1.3.68.** [31] If $I$ is an injective $R$-module, then $\text{Hom}_R(C, I)$ and $I$ are $C$-Gorenstein injective.

**Definition 1.3.69.** [34] An $R$-module is called $C$-flat if it has the form $C \otimes F$ for some flat module $F$. Set $\mathcal{F}_C(R) = \{C \otimes F \mid F \text{ is } R\text{-flat}\}$.

**Definition 1.3.70.** [55] An $R$-module is $C$-FP-injective if it has the form $\text{Hom}(C, E)$ for some FP-injective module $E$. Set $\mathcal{FP}_C(R) = \{\text{Hom}(C, E) \mid E \text{ is } R\text{-FP-injective}\}$.

**Definition 1.3.71.** [53] The Auslander class $\mathcal{A}_C(R)$ with respect to $C$ consists of all modules $M$ satisfying:

(A1) $\text{Tor}_i(C, M) = 0$ for any $i \geq 1$;

(A2) $\text{Ext}^i(C, C \otimes M) = 0$ for any $i \geq 1$; and

(A3) The natural evaluation homomorphism $\mu_M : M \to \text{Hom}(C, C \otimes M)$ is an isomorphism.
The Bass class $B_C(R)$ with respect to $C$ consists of all modules $N$ satisfying:

(B1) $\text{Ext}^i(C, N) = 0$ for any $i \geq 1$;

(B2) $\text{Tor}_i(C, \text{Hom}(C, N)) = 0$ for any $i \geq 1$; and

(B3) The natural evaluation homomorphism $\nu_N : C \otimes \text{Hom}(C, N) \to N$ is an isomorphism.

**Proposition 1.3.72.** [34] Let $S_C R$ be a semidualizing bimodule. Then the following hold:

(a) The classes $\mathcal{A}_C(R)$ and $B_C(S)$ are closed under direct summands, products, coproducts, and filtered colimits.

(b) The classes $A^f_C(R)$ and $B^f_C(S)$ are closed under finite direct sums and direct summands.

**Lemma 1.3.73.** [34] Let $S_C R$ be a semidualizing bimodule. For modules $R U$ and $S V$, the following hold:

(a) $V \in F_C(S)$ if and only if $V \in B_C(S)$ and $\text{Hom}_S(C, V)$ is flat over $R$.

(b) $V \in P_C(S)$ if and only if $V \in B_C(S)$ and $\text{Hom}_S(C, V)$ is projective over $R$.

(c) $U \in I_C(R)$ if and only if $U \in A_C(R)$ and $C \otimes U$ is injective over $S$.

**Proposition 1.3.74.** [34] Let $S_C R$ be a semidualizing bimodule. Then the following hold:

(a) The class $F_C(S)$ is covering on the category of $S$-modules.

(b) The class $P_C(S)$ is precovering on the category of $S$-modules.
(c) The class $\mathcal{I}_C(R)$ is enveloping on the category of $R$-modules.

(d) If $R$ is right coherent and $C$ is faithfully semidualizing, then the class $\mathcal{F}_C(S)$ is preenveloping on the category of $S$-modules.

(e) If $S$ is left noetherian and $C$ is faithfully semidualizing, then the class $\mathcal{I}_C(R)$ is covering on the category of $R$-modules.

Definition 1.3.75. [53] An $R$-module $M$ is $G_C$-projective if there exists a complete $\mathcal{P}C$-resolution is an exact sequence of $R$-modules

$$
\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow C \otimes P^0 \rightarrow C \otimes P^1 \rightarrow \cdots,
$$

where $P_i$’s and $P^i$’s are projective such that $M \cong \text{coker}(P_0 \rightarrow C \otimes P^0)$ and $\text{Hom}_R(-, C \otimes P)$ leaves the above sequence exact whenever $P$ is a projective $R$-module.

Theorem 1.3.76. [53] The class of $G_C$-projectives is projectively resolving and closed under direct summands. The class of finite $G_C$-projective $R$-modules is closed under summands. The class of $G_C$-projective $R$-modules admitting a degree-wise finite projective resolution is finite projectively resolving.

Definition 1.3.77. [42] Let $C$ be a semidualizing right $R$-module.

1. A complete $\mathcal{F}_C$-resolution of $R$-modules is of the form

$$
\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow C \otimes F^0 \rightarrow C \otimes F^1 \rightarrow \cdots,
$$

where $F_i$’s and $F^i$’s are flat.

2. An $R$-module $M$ is $G_C$-flat if there exists a complete $\mathcal{F}_C$-resolution is an exact sequence of $R$-modules
\[ \cdots \to F_1 \to F_0 \to C \otimes F^0 \to C \otimes F^1 \to \cdots , \]

where \( F_i \)'s and \( F^i \)'s are flat such that \( M \cong \operatorname{coker}(F_0 \to C \otimes F^0) \) and \( \operatorname{Hom}_R(\operatorname{Hom}_R(C, I) \otimes -) \) leaves the above sequence exact whenever \( I \) is an injective right \( R \)-module. \( \mathcal{GF}_C(R) \) denotes the class of all \( G_C \)-flat \( R \)-modules.

**Lemma 1.3.78.** [42] Let \( C \) be a semidualizing \( R \)-module. If \( X \) is a complete \( \mathcal{F}_C \)-resolution, then \( \operatorname{Coker}(\delta^X_m) \in \mathcal{GF}_C(R) \) for each \( m \in \mathbb{Z} \).

**Definition 1.3.79.** [42] Let \( C \) be a semidualizing left \( R \)-module.

1. A complete \( \mathcal{I}_C \mathcal{I} \)-resolution of \( R \)-modules is of the form

\[ \cdots \to I_1 \to I_0 \to I^0 \otimes C \to I^1 \otimes C \to \cdots , \]

where \( I_i \)'s and \( I^i \)'s are injective right \( R \)-modules.

2. A right \( R \)-module \( M \) is \( G_C \)-injective if there exists a complete \( \mathcal{I}_C \mathcal{I} \)-resolution is an exact sequence of \( R \)-modules

\[ \cdots \to I_1 \to I_0 \to I^0 \otimes C \to I^1 \otimes C \to \cdots , \]

where \( I_i \)'s and \( I^i \)'s are injective right \( R \)-modules such that \( M \cong \operatorname{coker}(I_0 \to I^0 \otimes C) \) and \( \operatorname{Hom}_R(\operatorname{Hom}_R(C, I), -) \) leaves the above sequence exact whenever \( I \) is an injective right \( R \)-module. \( \mathcal{GI}_C(R) \) denotes the class of all \( G_C \)-injective right \( R \)-modules.

**Lemma 1.3.80.** [42] Let \( C \) be a semidualizing \( R \)-module. If \( M \) is an \( R \)-module, then \( M \) is in \( \mathcal{GF}_C(R) \) if and only if its Pontryagin dual \( M^+ \) is in \( \mathcal{GI}_C(R) \).
Proposition 1.3.81. [42] Let \( C \) be a semidualizing \( R \)-module. The category \( \mathcal{GF}_C(R) \) is closed under kernels of epimorphisms, extensions and summands.

Definition 1.3.82. [31] If \( C \) is any \( R \)-module, then the direct sum \( R \oplus C \) can be equipped with the product:

\[
(a, c)(a', c') = (aa', ac' + a'c).
\]

This turns \( A \oplus C \) into a ring which is called the trivial extension of \( R \) by \( C \) and denoted by \( R \ltimes C \).

Proposition 1.3.83. [31] If \( E \) is a faithfully injective \( R \)-module, and \( M \) is any homologically right-bounded \( R \)-complex, then

\[
\text{Gfd}_{R \ltimes C}(M, E) = \text{Gfd}_{R \ltimes C}M.
\]

Proposition 1.3.84. [31] Let \( M \) be a \( R \)-module and \( C \) be a semidualizing \( R \)-module. Then the following hold:

1. \( M \) is \( C \)-Gorenstein injective if and only if \( M \) is Gorenstein injective over \( R \ltimes C \).

2. \( M \) is \( C \)-Gorenstein flat if and only if \( M \) is Gorenstein flat over \( R \ltimes C \).