2. Analytical and Numerical Solutions of One Dimensional Fractional Sub-Diffusion Equation with Neumann Boundary Conditions

2.1 Introduction

Diffusion equations model the diffusive and thermodynamic phenomena and describe the spread of particles (ions, molecules, etc.) from the area of higher concentration to the area of lower concentration. The random walk of these particles is called the Brownian motion. In many thermal problems, the temperature of the body is related to the heat flux. One example of such a problem is the temperature of electrical windings which strongly depends on the heat flux generated by the currents. In such a case, the dynamical relation between the temperature and the heat flux is described by heat diffusion equation. Diffusion not described by normal diffusion in the long time limit is known as anomalous. Fractional diffusion equation is a generalization of the classical diffusion equation which models anomalous diffusive phenomena. Some particles of the diffusing species get stuck for long time thus giving rise to slow diffusion which is referred to as anomalous sub-diffusion. Anomalous super-diffusion
appears when the particles move faster than in random diffusion. In normal
diffusion, the mean-square displacement \( <x^2(t)> \) of the diffusing particles is
proportional to the time \( t \), that is, \( <x^2(t)> \approx Dt \) where the diffusion coefficient
\( D \) is constant. Anomalous diffusion takes the nonlinear form \( <x^2(t)> \approx D_a t^a \)
where \( D \) is the generalized diffusion coefficient and \( a \) is anomalous diffusion
exponent. Anomalous super-diffusion corresponds to the case where the
mean-square displacement grows super-linearly in time whereas anomalous
sub-diffusion leads to sub-linear growth.

Fractional diffusion equations are useful for applications in which a cloud of
particles spreads faster than predicted by the classical equation. Kilbas et al. [19]
considered methods and results for fractional diffusion equations. A number of
numerical methods for solving different types of time fractional diffusion equations
has been proposed [4–6, 16, 25, 57, 60, 61, 63]. Chen et al. [4, 5] and Zhuang et al.
[63] used finite difference method based on Grunwald-Letnikov definition of
fractional derivatives and Gao et al. [16] and Zhao et al. [61] concentrated on \( L1 \)
discretization. Yuste and Acedo [60] combined the forward time centered space
(FTCS) method with the Grunwald-Letnikov discretization of the
Riemann-Liouville derivative to obtain an explicit FTCS scheme for solving the
fractional diffusion equation. Zhuang and Liu [63] proposed the implicit difference
approximation for the time fractional diffusion equation. Chen et al. [4] studied
the Fourier method for fractional diffusion describing sub-diffusion.

In this chapter, we solve the fractional sub-diffusion equation with Neumann
boundary conditions using the implicit difference approximation scheme and do the
stability and convergence analysis.

The model problem considered here is

\[
\frac{\partial u(x,t)}{\partial t} = 0D_t^{1-\gamma} \left[ K_\gamma \frac{\partial^2 u(x,t)}{\partial x^2} \right] + f(x,t), \quad 0 < x < L, \quad 0 < t \leq T, \quad (2.1)
\]
with initial condition
\[ u(x, 0) = w(x), \quad 0 \leq x \leq L \]  
(2.2)

and boundary conditions
\[ u_x(0, t) = \phi(t), \quad u_x(L, t) = \xi(t), \quad 0 < t \leq T, \]  
(2.3)

where \( K_\gamma \) is the generalized diffusion constant, \( 0 < \gamma < 1 \), \( 0D_t^{1-\gamma}u(x, t) \) denotes Riemann-Liouville time fractional derivative of order \( 1-\gamma \) of the function \( u(x, t) \) defined by (1.19), that is,
\[
0D_t^{1-\gamma}u(x, t) = \frac{1}{\Gamma(1-\gamma)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\gamma} \frac{\partial u(x, \tau)}{\partial \tau} d\tau
\]

and \( f(x, t), w(x), \phi(t), \xi(t) \) are known smooth functions. We assume that the equations (2.1)-(2.3) have a unique solution \( u(x, t) \in C^{2,1}_{x,t}([0, L] \times [0, T]) \).

Ervin et al. [13] obtained some results on existence and uniqueness of solutions of fractional diffusion equations. By using following lemma, we obtain the equivalent form (2.1).

**Lemma 2.1.** Let \( 0 < \gamma < 1 \). Then

(i) \( 0D_t^{\gamma-1} \left( \frac{\partial u}{\partial t} \right) = 0D_t^{\gamma} (u(x, t) - u(x, 0)) \).

(ii) \( 0D_t^{\gamma-1} \left( 0D_t^{1-\gamma}u(x, t) \right) = u(x, t) - 0D_t^{\gamma}u(x, \tau)|_{\tau=0} = u(x, t) \).

**Proof.**

(i) \[
0D_t^{\gamma-1} \left( \frac{\partial u}{\partial t} \right) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \frac{\partial u(x, \tau)}{\partial \tau} d\tau
\]
\[
= \frac{-t^{-\gamma}}{\Gamma(1-\gamma)} u(x, 0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{-\gamma-1} u(x, \tau) d\tau
\]
\[
= -0D_t^{\gamma}u(x, 0) + 0D_t^{\gamma}u(x, t)
\]
\((ii)_{0D}_t^{\gamma-1} \left(_{0D}_t^{1-\gamma} u(x,t) \right) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \frac{\partial}{\partial \tau}_{0D}_\tau^{\gamma-1} u(x,\tau) d\tau \)

\[= \frac{1}{\Gamma(1-\gamma)} \left( \left( (t-\tau)^{-\gamma} {_{0D}}_\tau^{\gamma-1} u(x,\tau) \right)' \right)_{\tau=0} - \gamma \int_0^t (t-\tau)^{-\gamma-1} \frac{\partial}{\partial \tau}_{0D}_\tau^{\gamma-1} u(x,\tau) d\tau \]

\[= \frac{-t^{-\gamma}}{\Gamma(1-\gamma)} \lim_{\tau \to 0} \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau-\xi)^{\gamma-1} u(x,\xi) d\xi + {_{0D}}_0^{\gamma-1}_{0D}_t^{\gamma} u(x,t) \]

\[= \frac{-t^{-\gamma}}{\Gamma(1-\gamma)} \lim_{\tau \to 0} \frac{1}{\Gamma(\gamma)} \left( \frac{-(\tau-\xi)^{\gamma}}{\gamma} u(x,\xi)'_\tau + \int_0^\tau (\tau-\xi)^{\gamma} \frac{\partial u(x,\xi)}{\partial \xi} d\xi \right) + u(x,t) \]

\[= u(x,t). \]

Applying \(_{0D}_t^{\gamma-1}\) on both sides of (2.1), we get

\[_{0D}_t^{\gamma-1} \left( \frac{\partial u}{\partial t} \right) = {_{0D}}_t^{\gamma-1} \left( _{0D}_t^{1-\gamma} \left( K_\gamma \frac{\partial^2 u}{\partial x^2} \right) \right) + g(x,t), \]

where \(g(x,t) = _{0D}_t^{\gamma-1} f(x,t).\) By applying Lemma 2.1 (i), we get

\[_{0D}_t^{\gamma}(u(x,t) - u(x,0)) = {_{0D}}_t^{\gamma-1} \left( _{0D}_t^{1-\gamma} \left( K_\gamma \frac{\partial^2 u}{\partial x^2} \right) \right) + g(x,t). \]

By applying Lemma 2.1 (ii), we get

\[_{0D}_t^{\gamma}[u(x,t) - u(x,0)] = K_\gamma \frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t).\] (2.4)

Using the relationship between Caputo fractional derivative and Riemann-Liouville fractional derivative (1.11), (2.4) can be rewritten as

\[\_0^C_{0D}_t^{\gamma} u(x,t) = K_\gamma \frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t), \]

where \(_0^C_{0D}_t^{\gamma} u(x,t)\) denotes the caputo time fractional derivative of order \(\gamma\) of the function \(u(x,t)\) defined by (1.21).
2.2 Analytical Solution of FSDE

Applying finite cosine transform \([15]\) with respect to the spatial variable \(x\), we have

\[
\frac{C}{C(t)} = \frac{\Gamma(1 - \gamma)}{-K^2(\gamma)} U(n, 0) = -K^2(\gamma) U(n, t) + H(t) + G(n, t),
\]

where \(a = \frac{\pi}{L}, \quad H(t) = \frac{-2K^2}{L} \left[(-1)^{n+1} u_x(L, t) + u_x(0, t)\right]\) and \(U(n, t), U(n, 0), G(n, t)\) are finite cosine transform of \(u(x, t), u(x, 0),\) and \(g(x, t)\) respectively. Applying Laplace transform with respect to the time variable \(t\), we have

\[
\bar{U}(n, s) = \frac{s^{-\gamma}}{s^\gamma + K^2(\gamma)} U(n, 0) + \frac{\bar{H}(s) + \bar{G}(n, s)}{s^\gamma + K^2(\gamma)}.
\]

Applying inverse Laplace transform, we have

\[
U(n, t) = E_{\gamma,1}[-K^2(\gamma)t^\gamma] U(n, 0) + (H(t) + G(n, t)) \ast t^{-\gamma} E_{\gamma,\gamma}[-K^2(\gamma)t^\gamma], \quad (2.5)
\]

where \(E_{\alpha,\beta}\) is the Mittag Leffler function and \(f(t) \ast g(t)\) is the convolution of these functions and is defined by

\[
f(t) \ast g(t) = \int_0^t f(\tau)g(t - \tau)d\tau.
\]

Applying inverse finite cosine transform, we have

\[
u(x, t) = \frac{U(0, t)}{2} + \sum_{n=1}^{\infty} U(n, t) \cos(nax),
\]

where \(U(n, t)\) is given in (2.5).
2.3 Derivation of Numerical Scheme

The exact solution $u$ of FSDE at the point $(x_j, t_k)$ is denoted by $u^k_j$ and the corresponding solution vector is denoted by $u^k = u(t_k) = (u^k_0, u^k_1, ..., u^k_m)^T$. The exact solution of an approximating difference scheme at the same point will be denoted by $U^k_j$ and the corresponding solution vector is denoted by $U^k = U(t_k) = (U^k_0, U^k_1, ..., U^k_m)^T$. Using the relationship between the Grunwald - Letnikov formula and the Riemann - Liouville fractional derivatives [40, 41], we can approximate the fractional derivative by

$$0D_t^\gamma[u^k_j - u^0_j] = \tau^{-\gamma} \sum_{i=0}^{k} \lambda^\gamma_i [u^{k-i}_j - u^0_j] + O(\tau), \quad (2.6)$$

where $\lambda^\gamma_i$ is given in (1.28).

We use the first and second order central difference schemes for the first and second order spatial derivatives respectively [48]:

$$\frac{\partial u^k_j}{\partial x} = \frac{u^k_{j+1} - u^k_{j-1}}{2h} + O(h^2), \quad (2.7)$$

$$\frac{\partial^2 u^k_j}{\partial x^2} = \frac{u^k_{j+1} - 2u^k_j + u^k_{j-1}}{h^2} + O(h^2). \quad (2.8)$$

2.4 An Implicit Difference Approximation Scheme

The initial boundary value problem (2.2), (2.3) and (2.4) can be approximated by the following implicit difference approximation scheme. For $1 \leq k \leq n$, we have

$$\tau^{-\gamma} \left[ U^k_j + \sum_{i=1}^{k-1} \lambda^\gamma_i U^{k-i}_j - \sum_{i=0}^{k-1} \lambda^\gamma_i U^0_j \right] = \frac{K^2}{h^2} [U^k_{j+1} - 2U^k_j + U^k_{j-1}] + g^k_j, \quad 0 \leq j \leq m, \quad (2.9)$$
\[ \frac{U^k_{j} - U^k_{j-1}}{2h} = \phi(t_k), \quad (2.10) \]
\[ \frac{U^k_{m+1} - U^k_{m}}{2h} = \xi(t_k), \quad (2.11) \]
\[ U^0_j = w(x_j), \quad 0 \leq j \leq m, \]

where \( g^k_j = g(x_j, t_k) \). Substitute (2.10) and (2.11) in (2.9) when \( j = 0 \) and \( j = m \) respectively, to have

\[ \tau^{-\gamma} \left[ U^k_0 + \sum_{i=1}^{k-1} \lambda_i^k U^k_{0} - \sum_{i=0}^{k-1} \lambda_i^0 U^0_{0} \right] = \frac{2K}{h^2} [U^k_1 - U^k_0] - \frac{2K}{h} \phi(t_k) + g^k_0, \]
\[ \tau^{-\gamma} \left[ U^k_j + \sum_{i=1}^{k-1} \lambda_i^k U^k_{j} - \sum_{i=0}^{k-1} \lambda_i^0 U^0_{j} \right] = \frac{K}{h^2} [U^k_{j+1} - 2U^k_j + U^k_{j-1}] + g^k_j, \quad 1 \leq j \leq m - 1, \]
\[ \tau^{-\gamma} \left[ U^k_m + \sum_{i=1}^{k-1} \lambda_i^k U^k_{m} - \sum_{i=0}^{k-1} \lambda_i^0 U^0_{m} \right] = \frac{2K}{h^2} [U^k_{m-1} - U^k_m] + \frac{2K}{h} \xi(t_k) + g^k_m, \]
\[ U^0_j = w(x_j), \quad 0 \leq j \leq m, \quad (2.12) \]

Introduce the scaling parameter \( \mu = K \gamma / h^2 \) and after rearranging the terms, we have

\[ (1 + 2\mu) U^k_0 = - \sum_{i=1}^{k-1} \lambda_i^k U^k_{0} + \sum_{i=0}^{k-1} \lambda_i^0 U^0_{0} + 2\mu U^1_0 - 2h \mu \phi(t_k) + \gamma g^k_0, \]
\[ (1 + 2\mu) U^k_j = - \sum_{i=1}^{k-1} \lambda_i^k U^k_{j} + \sum_{i=0}^{k-1} \lambda_i^0 U^0_{j} + \mu [U^k_{j+1} + U^k_{j-1}] + \gamma g^k_j, \quad 1 \leq j \leq m - 1, \]
\[ (1 + 2\mu) U^k_m = - \sum_{i=1}^{k-1} \lambda_i^k U^k_{m} + \sum_{i=0}^{k-1} \lambda_i^0 U^0_{m} + 2\mu U^k_{m-1} + 2h \mu \xi(t_k) + \gamma g^k_m, \]
\[ U^0_j = w(x_j), \quad 0 \leq j \leq m, \quad (2.12) \]
or

\[(1 + 2\mu)U_k^k - 2\mu U_k^1 = -\sum_{i=0}^{k-1} \lambda_i^k U_0^{k-i} + \sum_{i=1}^{k-1} \lambda_i^k U_0^i - 2h\mu\phi(t_k) + \tau^\gamma g_0^k,\]

\[-\mu U_{j-1}^k + (1 + 2\mu) U_j^k - \mu U_{j+1}^k = -\sum_{i=1}^{k-1} \lambda_i^j U_j^{k-i} + \sum_{i=0}^{k-1} \lambda_i^j U_j^i + \tau^\gamma g_j^k,\]

\[-2\mu U_{m-1}^k + (1 + 2\mu) U_m^k = -\sum_{i=1}^{k-1} \lambda_i^m U_m^{k-i} + \sum_{i=0}^{k-1} \lambda_i^m U_m^i + 2h\mu\xi(t_k) + \tau^\gamma g_m^k,\]

\[U_j^0 = w(x_j), \quad 0 \leq j \leq m.\]

(2.13)

### 2.5 Matrix Form of the IDAS

We give the matrix form of the IDAS (2.13) by

\[
\bar{A}U^k = -\sum_{i=1}^{k-1} \lambda_i^k U_0^{k-i} + \sum_{i=0}^{k-1} \lambda_i^k U_0^i + \bar{G}^k, \quad 1 \leq k \leq n,
\]

where

\[
\bar{A} = \begin{pmatrix}
1 + 2\mu & -2\mu \\
-\mu & 1 + 2\mu & -\mu \\
& \ddots & \ddots & \ddots \\
& -\mu & 1 + 2\mu & -\mu \\
& -2\mu & 1 + 2\mu
\end{pmatrix}_{(m+1) \times (m+1)}
\]

\[
\bar{G}^k = \begin{pmatrix}
2h\mu\phi(t_k) + \tau^\gamma g_0^k \\
\tau^\gamma g_1^k \\
\vdots \\
\tau^\gamma g_{m-1}^k \\
2h\mu\xi(t_k) + \tau^\gamma g_m^k
\end{pmatrix}_{(m+1) \times 1}, \quad 1 \leq k \leq n.
\]
For the solvability of the scheme, we have the following theorem.

**Theorem 2.1.** The difference equations (2.13) has a unique solution.

**Proof.** Because, for any \( \mu = K \frac{\gamma}{h^2} > 0 \), the coefficient matrix \( \bar{A} \) for the difference equation is strictly diagonally dominant. Consequently the matrix \( \bar{A} \) is nonsingular, thus is invertible. Hence completes the proof of the theorem. \( \square \)

### 2.6 Theoretical Analysis of the IDAS

#### 2.6.1 The Local Truncation Error

The local truncation error of IDAS (2.13) is

\[
R^k_j = \tau^{-\gamma} \left[ u^k_j + \sum_{i=1}^{k-1} \lambda_i^k u^{k-i}_j - \sum_{i=0}^{k-1} \lambda_i^k u^0_j \right] - \frac{K \gamma}{h^2} [u^k_{j+1} - 2u^k_j + u^k_{j-1}] - g^k_j
\]

\[
= \tau^{-\gamma} \left[ u^k_j + \sum_{i=1}^{k-1} \lambda_i^k u^{k-i}_j - \sum_{i=0}^{k-1} \lambda_i^k u^0_j \right] - oD^k_i [u^k_j - u^0_j]
\]

\[
+ K \gamma \left[ \frac{\partial^2 u^k_j}{\partial x^2} - \frac{1}{h^2} (u^k_{j+1} - 2u^k_j + u^k_{j-1}) \right]
\]

\[
= O(\tau + h^2), \quad 1 \leq j \leq m - 1, \quad 1 \leq k \leq n. \tag{2.14}
\]

Using the Taylor series expansion of \( u^k_1 \) about the point \( (x_0, t_k) \), we get

\[
R^k_0 = \tau^{-\gamma} \left[ u^k_0 + \sum_{i=1}^{k-1} \lambda_i^k u^{k-i}_0 - \sum_{i=0}^{k-1} \lambda_i^k u^0_0 \right] - \frac{2K \gamma}{h^2} [u^k_1 - u^k_0 - h \phi(t_k)] - g^k_0
\]

\[
= \tau^{-\gamma} \left[ u^k_0 + \sum_{i=1}^{k-1} \lambda_i^k u^{k-i}_0 - \sum_{i=0}^{k-1} \lambda_i^k u^0_0 \right] - oD^k_i [u^k_1 - u^0_0]
\]

\[
= O(\tau + h), \quad 1 \leq k \leq n. \tag{2.15}
\]

Similarly, using the Taylor series expansion of \( u^k_{m-1} \) about the point \( (x, t_k) \), we get

\[
R^k_m = O(\tau + h), \quad 1 \leq k \leq n. \tag{2.16}
\]
2.6.2 Stability

We introduce some relevant notations and properties. Suppose that
\[ u^k = \{ u^k_j | 0 \leq j \leq m, 0 \leq k \leq n \} \] and \[ v^k = \{ v^k_j | 0 \leq j \leq m, 0 \leq k \leq n \} \] are two grid functions on \( \Omega_h \times \Omega_r \). We introduce the following notations:
\[
(u^k_j)_x = \frac{u^k_{j+1} - u^k_j}{h}, \quad (u^k_j)_{xx} = \frac{u^k_{j+1} - 2u^k_j + u^k_{j-1}}{h},
\]
\[
\langle u, v \rangle = h \left[ \frac{u^k_0 v^k_0}{2} + \sum_{j=1}^{m} u^k_j v^k_j + \frac{u^k_m v^k_m}{2} \right], \quad \| u \| = \langle u, u \rangle^{1/2}.
\]

Let \( \tilde{U}^k_j \) be the approximate solution of (2.13) and \( \epsilon^k_j = U^k_j - \tilde{U}^k_j, \ 0 \leq j \leq m, 0 \leq k \leq n \), denote the corresponding error.

**Lemma 2.2.** The coefficients \( \lambda^\gamma_i \ (i = 0, 1, 2, \ldots) \) defined by (1.28) satisfy

(i) \( \lambda^\gamma_0 = 1, \ \lambda^\gamma_1 = -\gamma, \ \lambda^\gamma_i < 0, \ i = 1, 2, \ldots \)

(ii) \( \sum_{i=0}^{\infty} \lambda^\gamma_i = 0. \)

(iii) \( \sum_{i=0}^{k-1} \lambda^\gamma_i > 0 \) and consequently \( -\sum_{i=1}^{k} \lambda^\gamma_i < 1, \) for all \( k \geq 1. \)

**Lemma 2.3.** Suppose that \( \epsilon^k \) \( (0 \leq k \leq n) \) is error of IDAS (2.13). Then we have
\[
\| \epsilon^k \|^2 \leq -\sum_{i=1}^{k-1} \lambda^\gamma_i \| \epsilon^{k-i} \|^2 + \sum_{i=0}^{k-1} \lambda^\gamma_i \| \epsilon^0 \|^2, \ k = 1, 2, \ldots n.
\]

**Proof.** For the IDAS defined by (2.12), for \( 1 \leq k \leq n \), its error satisfies
\[
(1 + 2\mu) \epsilon^k_0 = -\sum_{i=1}^{k-1} \lambda^\gamma_i \epsilon^{k-i}_0 + \sum_{i=0}^{k-1} \lambda^\gamma_i \epsilon^0_i + 2\mu \epsilon^k_1 \quad (2.17)
\]
Multiplying (2.17), (2.18) and (2.19) by $h$, we have

\[(1 + 2\mu)\epsilon_j^k = - \sum_{i=1}^{k-1} \lambda_i^2 \epsilon_{j-i}^k + \frac{\sum_{i=1}^{k-1} \lambda_i^2 \epsilon_{i}^k + \mu e_{j+1}^k + \epsilon_{j-1}^k}{2}, \quad 1 \leq j \leq m - 1 \quad (2.18)\]

\[(1 + 2\mu)\epsilon_m^k = - \sum_{i=1}^{k-1} \lambda_i^2 \epsilon_{m}^k + \frac{\sum_{i=0}^{k-1} \lambda_i^2 \epsilon_{m}^0 + 2\mu \epsilon_{m-1}^k}{2}. \quad (2.19)\]

Multiplying (2.17), (2.18) and (2.19) by $\frac{\epsilon_0^k}{2}$, $h \epsilon_j^k$ and $\frac{\epsilon_m^k}{2}$ respectively and adding (2.17)-(2.19), we have

\[h \left[ \sum_{j=1}^{m-1} (\epsilon_j^k)^2 + \frac{(\epsilon_0^k)^2}{2} + \frac{(\epsilon_m^k)^2}{2} \right] = h \sum_{j=1}^{k-1} \lambda_j^2 \left[ \sum_{i=1}^{m-1} \epsilon_{j-i}^k \epsilon_j^k + \frac{\epsilon_0^k \epsilon_0^k + \epsilon_m^k \epsilon_m^k}{2} \right] - h \sum_{j=1}^{k-1} \lambda_j^2 \left[ \sum_{i=1}^{m-1} \left[ (\epsilon_{j-i}^k)^2 + (\epsilon_j^k)^2 \right] + \frac{(\epsilon_0^k)^2}{2} + \frac{(\epsilon_m^k)^2}{2} \right] + \mu h \left[ \sum_{j=1}^{m-1} (\epsilon_{j+1}^k + \epsilon_{j-1}^k + \epsilon_0^k + \epsilon_m^k) \right]. \quad (2.20)\]

We have

\[h \left[ \sum_{j=1}^{m-1} (\epsilon_j^k)^2 + \frac{(\epsilon_0^k)^2}{2} + \frac{(\epsilon_m^k)^2}{2} \right] = \|\epsilon^k\|^2. \quad (2.21)\]

From Lemma 2.2, it follows that

\[-h \sum_{i=1}^{k-1} \lambda_i^2 \left[ \sum_{j=1}^{m-1} \epsilon_{j-i}^k \epsilon_j^k + \frac{\epsilon_0^k \epsilon_0^k + \epsilon_m^k \epsilon_m^k}{2} \right] \leq -h \sum_{i=1}^{k-1} \lambda_i^2 \left[ \sum_{j=1}^{m-1} \left[ (\epsilon_{j-i}^k)^2 + (\epsilon_j^k)^2 \right] + \frac{(\epsilon_0^k)^2}{2} + \frac{(\epsilon_m^k)^2}{2} \right] \leq - h \sum_{i=1}^{k-1} \lambda_i^2 \left[ \|\epsilon_{j-i}^k\|^2 + \|\epsilon_j^k\|^2 \right], \quad (2.22)\]

\[h \sum_{i=0}^{k-1} \lambda_i^2 \left[ \sum_{j=1}^{m-1} \epsilon_0^k \epsilon_j^k + \frac{\epsilon_0^k \epsilon_0^k + \epsilon_m^k \epsilon_m^k}{2} \right] \leq h \sum_{i=0}^{k-1} \lambda_i^2 \left[ \sum_{j=1}^{m-1} \left[ (\epsilon_0^k)^2 + (\epsilon_j^k)^2 \right] + \frac{(\epsilon_0^k)^2}{2} + \frac{(\epsilon_m^k)^2}{2} \right] \leq \frac{1}{2} \sum_{i=0}^{k-1} \lambda_i^2 \left[ \|\epsilon_0^k\|^2 + \|\epsilon_j^k\|^2 \right]. \quad (2.23)\]
and
\[
\mu h \left[ \sum_{j=1}^{m-1} [\epsilon_{j+1}^k + \epsilon_{j-1}^k + \epsilon_0^k + \epsilon_{m-1}^k] \right] \\
\leq \frac{\mu h}{2} \sum_{j=1}^{m-1} [(\epsilon_{j+1}^k)^2 + 2(\epsilon_{j}^k)^2 + (\epsilon_{0}^k)^2 + (\epsilon_{m-1}^k)^2] \\
= 2\mu \|\epsilon^k\|^2.
\] (2.24)

Taking into account (2.20)-(2.24), we conclude that
\[
\|\epsilon^k\|^2 \leq -\sum_{i=1}^{k-1} \lambda_i^\gamma \|\epsilon^{k-i}\|^2 + \sum_{i=0}^{k-1} \lambda_i^\gamma \|\epsilon^0\|^2, \quad k = 1, 2, \ldots n.
\]

**Theorem 2.2.** The IDAS defined by (2.13) is unconditionally stable, that is, \(\|\epsilon^k\|^2 \leq \|\epsilon^0\|^2, \quad k = 1, 2, \ldots n\).

**Proof.** We will use mathematical induction to complete the proof. From Lemma 2.3, for \(k=1\), we have
\[
\|\epsilon^1\|^2 \leq \|\epsilon^0\|^2.
\]
Suppose that \(\|\epsilon^k\|^2 \leq \|\epsilon^0\|^2, \quad k = 1, 2, \ldots n - 1\). Then
\[
\|\epsilon^n\|^2 \leq -\sum_{i=1}^{n-1} \lambda_i^\gamma \|\epsilon^{n-i}\|^2 + \sum_{i=0}^{n-1} \lambda_i^\gamma \|\epsilon^0\|^2
\]
\[
\leq \|\epsilon^0\|^2 \left[ -\sum_{i=1}^{n-1} \lambda_i^\gamma + \sum_{i=0}^{n-1} \lambda_i^\gamma \right] = \|\epsilon^0\|^2
\]
which proves that the scheme (2.13) is unconditionally stable. \(\square\)

### 2.6.3 Convergence

Let \(e_j^k = u_j^k - U_j^k\) and \(e^k = \{e_j^k|0 \leq j \leq m, 0 \leq k \leq n\}\) a grid function on \(\Omega_h \times \Omega_r\).

**Lemma 2.4.** [5] If \(\lambda_i^\gamma\) are defined by (1.28), then we have \(-\sum_{i=1}^{\infty} \lambda_i^\gamma > \frac{1}{n^\Gamma(1-\gamma)}, \quad n \in \mathbb{N}\).
Proof. Let \( v_n = -n^\gamma \sum_{i=n}^{\infty} \lambda_i^\gamma \). By Lemma 2.2 and (1.28), we have \( \sum_{i=0}^{\infty} \lambda_i^\gamma = 0 \), \( \sum_{i=0}^{n} \lambda_i^\gamma = \sum_{i=0}^{n} (i-\gamma-1) = \binom{n-\gamma}{n} \) and \( -\sum_{i=0}^{\infty} \lambda_i^\gamma = \sum_{i=0}^{n-1} \lambda_i^\gamma = \binom{n-1-\gamma}{n-1} \) from which it follows that

\[
-\sum_{i=n}^{\infty} \lambda_i^\gamma = \binom{n-1-\gamma}{n} = \frac{n}{n-\gamma} = 1 + \frac{\gamma}{n-\gamma}.
\]  

(2.25)

For \( x \in (-1, 1) \),

\[
(1 + x)^\gamma = 1 + \gamma x + \frac{\gamma(\gamma - 1)}{2!} x^2 + \frac{\gamma(\gamma - 1)(\gamma - 2)}{3!} x^3 + \ldots,
\]

since \( 0 < \gamma < 1 \), we have

\[
\frac{(n + 1)^\gamma}{n^\gamma} = \left(1 + \frac{1}{n}\right)^\gamma = 1 + \frac{\gamma}{n} + \frac{\gamma(\gamma - 1)}{2!} \left(\frac{1}{n}\right)^2 + \frac{\gamma(\gamma - 1)(\gamma - 2)}{3!} \left(\frac{1}{n}\right)^3 + \ldots
\]<

\[
1 + \frac{\gamma}{n}
\]

(2.26)

Using (2.25) and (2.26) we get

\[
-\sum_{i=n}^{\infty} \lambda_i^\gamma > \frac{(n + 1)^\gamma}{n^\gamma}
\]  

(2.27)

for which \( v_n > v_{n+1} \), since \( \frac{t^{-\gamma}}{\Gamma(1-\gamma)} = O(1) \) we have

\[
\lim_{\tau \to 0, n \tau = t} \tau^{-\gamma} \sum_{i=0}^{n} \lambda_i^\gamma = \frac{1}{\Gamma(1-\gamma)}.
\]

\[
\lim_{n \to \infty} v_{n+1} = -\lim_{n \to \infty} (n + 1)^\gamma \sum_{i=n+1}^{\infty} \lambda_i^\gamma
\]

\[
= -\lim_{n \to \infty} \left(\frac{n + 1}{n}\right)^\gamma n^\gamma \sum_{i=n+1}^{\infty} \lambda_i^\gamma
\]

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\[
\lim_{n \to \infty} v_{n+1} = - \lim_{n \to \infty} n^\gamma \sum_{i=n+1}^{\infty} \lambda_i^\gamma \\
= \lim_{0 \to n} n^\gamma \sum_{i=n+1}^{\infty} \lambda_i^\gamma \\
= \frac{1}{\Gamma(1 - \gamma)} \quad (2.28)
\]

From (2.27)- (2.28) we conclude that \( v_n > \frac{1}{\Gamma(1 - \gamma)}, n \in \mathbb{N} \). Hence proved the lemma.

**Lemma 2.5.** If \( e_n^j = u_n^j - U_n^j \), then there exists a positive constant \( C_3 \) such that \( \| e_n^\| ^2 \leq C_3 (\sum_{i=n}^{\infty} \lambda_i^\gamma)^{-1} \tau^\gamma (\tau + h^{1.5})^2 \), for all \( n \in \mathbb{N} \).

**Proof.** According to the results from discussing the consistency of the IDAS defined by (2.13), for \( 1 \leq k \leq n \), we get the following error equations

\[
(1 + 2\mu) e_0^k = - \sum_{i=1}^{k-1} \lambda_i^\gamma e_0^{k-i} + \sum_{i=0}^{k-1} \lambda_i^\gamma e_0^0 + 2\mu e_{i-1}^{k} + \tau^\gamma R_0^k, \quad (2.29)
\]

\[
(1 + 2\mu) e_j^k = - \sum_{i=1}^{k-1} \lambda_i^\gamma e_j^{k-i} + \sum_{i=0}^{k-1} \lambda_i^\gamma e_j^0 + \mu[e_{j+1}^{k} + e_{j-1}^k] + \tau^\gamma R_j^k, \quad 1 \leq j \leq m - 1, (2.30)
\]

\[
(1 + 2\mu) e_m^k = - \sum_{i=1}^{k-1} \lambda_i^\gamma e_m^{k-i} + \sum_{i=0}^{k-1} \lambda_i^\gamma e_m^0 + 2\mu e_{m-1}^{k} + \tau^\gamma R_m^k, \quad (2.31)
\]

\[ e_j^0 = 0, \quad 1 \leq j \leq m. \]

Multiplying (2.29), (2.30) and (2.31) by \( \frac{h}{2} e_0^k \), \( he_j^k \) and \( \frac{h}{2} e_m^k \) respectively and adding (2.25)-(2.27), we have

\[
(1 + 2\mu) h \left[ \sum_{j=1}^{m-1} (e_j^{k})^2 + \frac{(e_0^k)^2}{2} + \frac{(e_m^k)^2}{2} \right] = - h \sum_{i=1}^{k-1} \lambda_i^\gamma \left[ \sum_{j=1}^{m-1} e_{j-i}^{k} e_j^0 + \frac{e_0^{k-i} e_0^0}{2} + \frac{e_{m-i}^{k} e_m^0}{2} \right] \\
+ \mu h \left[ \sum_{j=1}^{m-1} (e_{j+1}^k + e_{j-1}^k + e_{j}^0 + e_{m-1}^k + e_{m-1}^k)^2 \right] + h \tau^\gamma \left[ \sum_{j=1}^{m-1} R_j^{k} e_j^0 + \frac{R_0^k e_0^0}{2} + \frac{R_m^k e_m^0}{2} \right]. \quad (2.32)
\]

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From the proof of Lemma 2.3, it follows that

\[
  h \left[ \sum_{j=1}^{m-1} \left( e_j^k \right)^2 + \frac{(e_0^k)^2}{2} + \frac{(e_m^k)^2}{2} \right] = \| e^k \|^2, \tag{2.33}
\]

\[
  -h \sum_{i=1}^{k-1} \lambda_i^\gamma \left[ \sum_{j=1}^{m-1} e_j^k \left( e_j^k - e_{0_i}^k \right) \right] \sum_{i=1}^{m-1} \left( e_{m_i}^k - e_m^k \right) \leq -\frac{1}{2} \sum_{i=1}^{k-1} \lambda_i^\gamma \left[ \| e^{K-i} \|^2 + \| e^k \|^2 \right]. \tag{2.34}
\]

\[
  \mu h \left[ \sum_{j=1}^{m-1} \left( e_{j+1}^k + e_{j-1}^k \right) + e_1^k e_0^k + e_{m-1}^k e_m^k \right] \leq 2\mu \| e^k \|^2. \tag{2.35}
\]

Using Lemma 2.4, we have

\[
  h^{\gamma} \left[ \sum_{j=1}^{m-1} R_j^k e_j^k + \frac{R_0^k e_0^k}{2} + \frac{R_m^k e_m^k}{2} \right] \leq h^{\gamma} \left[ \sum_{j=1}^{m-1} \left( R_j^k \right)^2 + \frac{(-\sum_{i=k}^{\infty} \lambda_i^\gamma) }{2} \left( e_j^k \right)^2 \right]
  + \frac{\tau^{\gamma}}{(-2 \sum_{i=k}^{\infty} \lambda_i^\gamma)} \left( R_0^k \right)^2 + \frac{(-\sum_{i=k}^{\infty} \lambda_i^\gamma) }{2} \left( e_0^k \right)^2 + \frac{\tau^{\gamma}}{(-2 \sum_{i=k}^{\infty} \lambda_i^\gamma)} \left( R_m^k \right)^2 + \frac{(-\sum_{i=k}^{\infty} \lambda_i^\gamma) }{2} \left( e_m^k \right)^2
  = \frac{h^{\gamma}}{(2 \sum_{i=k}^{\infty} \lambda_i^\gamma)} \left[ \sum_{j=1}^{m-1} \left( R_j^k \right)^2 + \frac{(-\sum_{i=k}^{\infty} \lambda_i^\gamma) }{2} \left( e_j^k \right)^2 \right] - \frac{1}{2} \sum_{i=k}^{\infty} \lambda_i^\gamma \| e^k \|^2
  = \frac{h^{\gamma} (m-1) C_2 (\tau + h^2)^2 + C_1 (\tau + h)^2} {2 \sum_{i=k}^{\infty} \lambda_i^\gamma} - \frac{1}{2} \sum_{i=k}^{\infty} \lambda_i^\gamma \| e^k \|^2
  \leq \frac{h^{\gamma} \gamma \Gamma (1 - \gamma) C_2 (\tau + h^{1.5})^2 - \frac{1}{2} \sum_{i=k}^{\infty} \lambda_i^\gamma \| e^k \|^2}{\gamma \Gamma (1 - \gamma)}
  = C_3 \tau^\gamma (\tau + h^{1.5})^2 + \frac{1}{2} \sum_{i=0}^{k-1} \lambda_i^\gamma \| e^k \|^2. \tag{2.36}
\]

Taking into account (2.32)-(2.36), we conclude that

\[
  \| e^k \|^2 \leq -\sum_{i=1}^{k-1} \lambda_i^\gamma \| e^{K-i} \|^2 + C_3 \tau^\gamma (\tau + h^{1.5})^2. \tag{2.37}
\]
It follows from (2.37) by induction that
\[ \|e_k\|_2^2 \leq C_3 \left( - \sum_{i=k}^{\infty} \lambda_i^\gamma \right)^{-1} \tau^\gamma (\tau + h^{1.5})^2, \quad \text{for} \quad 1 \leq k \leq n. \] (2.38)

In fact, for \( k=1 \), (2.38) is fulfilled obviously. Suppose that
\[ \|e_k\|_2^2 \leq C_3 \left( - \sum_{i=k}^{\infty} \lambda_i^\gamma \right)^{-1} \tau^\gamma (\tau + h^{1.5})^2, \quad k = 1, 2, \ldots, n-1, \]
\[ \|e^n\|_2^2 \leq - \sum_{i=1}^{n-1} \lambda_i^\gamma \|e^n-i\|^2 + C_3 \tau^\gamma (\tau + h^{1.5})^2 \]
\[ \leq - \sum_{i=1}^{n-1} \lambda_i^\gamma C_3 \left( - \sum_{s=n-i}^{\infty} \lambda_s \right)^{-1} \tau^\gamma (\tau + h^{1.5})^2 + C_3 \tau^\gamma (\tau + h^{1.5})^2 \]
\[ \leq - \sum_{i=1}^{n-1} \lambda_i^\gamma C_3 \left( - \sum_{s=n}^{\infty} \lambda_s \right)^{-1} \tau^\gamma (\tau + h^{1.5})^2 + C_3 \tau^\gamma (\tau + h^{1.5})^2 \]
\[ = \left( 1 + \sum_{i=n}^{\infty} \lambda_i^\gamma \right) C_3 \left( - \sum_{s=n}^{\infty} \lambda_s \right)^{-1} \tau^\gamma (\tau + h^{1.5})^2 + C_3 \tau^\gamma (\tau + h^{1.5})^2 \]
and \[ \|e^n\|_2^2 = C_3 \left( - \sum_{s=n}^{\infty} \lambda_s \right)^{-1} \tau^\gamma (\tau + h^{1.5})^2. \]

\[ \square \]

**Theorem 2.3.** The IDAS defined by (2.13) is convergent and there exists a positive constant \( C \) such that \( \|e^k\| \leq C(\tau + h^{1.5}) \), for \( 1 \leq k \leq n \).

**Proof.** Using Lemma 2.4 and Lemma 2.5, we get \( \|e^k\| \leq C(\tau + h^{1.5}) \). \( \square \)

### 2.7 Numerical Experiments

In order to demonstrate the effectiveness of our difference scheme, we present two examples.
Example 2.1. Fractional diffusion equation describing sub-diffusion with non-homogeneous term

\[
\frac{\partial u(x,t)}{\partial t} = \frac{1}{0} D_t^{1-\gamma} \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right] + e^t \left[ (1 + \gamma) t^\gamma - \frac{\Gamma[2 + \gamma]}{\Gamma[1 + 2\gamma]} t^{2\gamma} \right], \quad x \in [0,1], t \in (0,1], \quad (2.39)
\]

\[
u(x,0) = 0, \quad (2.40)
\]

\[
u_x(0,t) = t^{\gamma+1}, \quad \nu_x(1,t) = e t^{\gamma+1}. \quad (2.41)
\]

The exact solution of (2.39)-(2.41) is, \( u(x,t) = e^t t^{\gamma+1} \).

Table 2.1: The errors and convergence orders of the scheme (2.13) for different \( \gamma \).

| \( \gamma \) | \( ||e(1/4,1/4)|| \) | \( ||e(1/8,1/8)|| \) | \( ||e(1/16,1/16)|| \) | \( r(1/4,1/4) \) | \( r(1/8,1/8) \) |
|---|---|---|---|---|---|
| 0.1 | 0.0148453 | 0.00948268 | 0.00531433 | 0.646639 | 0.835407 |
| 0.2 | 0.0416035 | 0.023245 | 0.0122521 | 0.839784 | 0.923891 |
| 0.3 | 0.0730891 | 0.0391855 | 0.0202663 | 0.899336 | 0.951237 |
| 0.4 | 0.109053 | 0.0573049 | 0.029609 | 0.928299 | 0.964762 |
| 0.5 | 0.149586 | 0.0776336 | 0.0394442 | 0.946222 | 0.973215 |
| 0.6 | 0.194844 | 0.100214 | 0.0508727 | 0.959235 | 0.979411 |
| 0.7 | 0.245029 | 0.125098 | 0.0632233 | 0.969894 | 0.98453 |
| 0.8 | 0.300377 | 0.152348 | 0.0767483 | 0.979404 | 0.989164 |
| 0.9 | 0.361165 | 0.182033 | 0.0914204 | 0.988458 | 0.993612 |

Table 2.2: The errors and convergence orders of the scheme (2.13) for different \( \tau \).

| \( \tau \) | \( ||e(h,\tau)|| \) | Order |
|---|---|---|
| \( 1/2 \) | 0.300762 | 0.951642 |
| \( 1/4 \) | 0.155507 | 0.974696 |
| \( 1/8 \) | 0.0791294 | 0.987091 |
| \( 1/16 \) | 0.0399203 | _ |
Figure 2.1: Exact solution and IDAS numerical solution at $t = 1$ with different $h$ and $\tau$ (solid line is the exact solution and dots denote numerical solutions of IDFS)
Suppose that the error of our scheme satisfies $\|e(\tau, h)\| = O(\tau^p + h^q)$, and $\|e(\tau, h)\|$ means the error $\|e^n\|$ computed with mesh sizes $\tau$ and $h$. If $\tau$ is small enough, then $\|e(\tau, h)\| = O(h^q)$. Consequently $\frac{\|e(\tau, h)\|}{\|e(\tau/2, h/2)\|} \approx 2^q$ and $q \approx \log_2 \left( \frac{\|e(\tau, h)\|}{\|e(\tau/2, h)\|} \right)$. On the other hand, if $h$ is small enough, then $\|e(\tau, h)\| = O(h^p)$ and we have $\frac{\|e(\tau, h)\|}{\|e(\tau, h/2)\|} \approx 2^p$ and $p \approx \log_2 \left( \frac{\|e(\tau, h)\|}{\|e(\tau/2, h)\|} \right)$. Some numerical results are put in Table 2.1, Table 2.2 and Table 2.3. In Table 2.1, the experimental order of convergence $r(\tau, h)$ is computed by the formula $r(\tau, h) = \log_2 \left( \frac{\|e(\tau, h)\|}{\|e(\tau/2, h)\|} \right)$. We list the errors and convergence order when $\alpha = 0.5$ and $h = 1/1000$ in Table 2.2. In Table 2.3, we list the errors and convergence order when $\alpha = 0.5$ and $\tau = 1/10000$. From Table 2.2 and Table 2.3, we have, for the initial boundary value problem (2.39)-(2.41), $\|e(\tau, h)\| = O(\tau + h^2)$. The computational results at $t = 1$ and $\alpha = 0.5$ with different $h$ and $\tau$ are shown in Figure 2.1 (a) and Figure 2.1 (b). In these two figures, solid line denotes the exact solution and dotted line denotes the numerical solutions of IDAS.

Example 2.2. Fractional diffusion equation describing sub-diffusion with homogeneous term

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2}, \quad 0 \leq x \leq 1, \quad 0 < t \leq 1,
\]
\[ u(x, 0) = \cos(2\pi x), \quad (2.43) \]
\[ u_x(0, t) = 0, \quad u_x(1, t) = 0. \quad (2.44) \]

The exact solution of (2.42)-(2.44) is, \( u(x, t) = E_\gamma(-4\pi^2 t^\gamma) \cos(2\pi x) \).

Table 2.4: The errors and convergence orders of the scheme (2.13) for different \( \tau \).

| \( \tau \) | \( ||e(h, \tau)|| \) | Order |
|---|---|---|
| \( \frac{1}{2} \) | 0.00254795 | 1.22934 |
| \( \frac{1}{4} \) | 0.00108673 | 1.10344 |
| \( \frac{1}{8} \) | 0.000505772 | 1.04908 |
| \( \frac{1}{16} \) | 0.000244427 | — |

Table 2.5: The errors and convergence orders of the scheme (2.13) for different \( h \).

| \( h \) | \( ||e(h, \tau)|| \) | Order |
|---|---|---|
| \( \frac{1}{4} \) | 0.00236345 | 2.13168 |
| \( \frac{1}{8} \) | 0.000539321 | 2.0027 |
| \( \frac{1}{16} \) | 0.000134578 | 1.89061 |
| \( \frac{1}{32} \) | 0.0000362949 | — |

We list the errors and convergence order when \( \alpha = 0.5 \) and \( h = 1/1000 \) in Table 2.4. In Table 2.5, we list the errors and convergence order when \( \alpha = 0.5 \) and \( \tau = 1/10000 \). From Table 2.4 and Table 2.5, we have, for the initial boundary value problem (2.42)-(2.44), \( ||e(\tau, h)|| = O(\tau + h^2) \).