5. Numerical Solution of One Dimensional Fractional Heat Equation with Variable Coefficients

5.1 Introduction

Fundamental solution of the space-time fractional diffusion equation is given in [29]. A number of numerical methods for solving different types of space and space-time fractional diffusion equations has been proposed [6, 25, 34–36, 51, 55, 57]. For fractional diffusion equations which involve a derivative of order $\alpha$, $(1 < \alpha \leq 2)$, Meerschaert et al. [34] studied fractional advection dispersion equation with the standard Grunwald-Letnikov definition of fractional derivative of order $\alpha$ by the explicit and implicit Euler methods and the Crank-Nicholson method which are all unconditionally unstable and the shifted Grunwald formula which allows the implicit Euler method (and also the Crank-Nicholson method) to be unconditionally stable. Therefore Meerschaert et al. [34–36, 55] used shifted Grunwald formula and Liu et al. [25] and Yang et al. [57] used shifted Grunwald formula, L2 approximation and Matrix transform methods for the fractional derivative of order $\alpha$ ($1 < \alpha \leq 2$). Sousa [50] concluded that the approximation based on the shifted Grunwald-
Letnikov definition is the best option when considering first order approximation and that the approximation obtained from the spline interpolation performs better for second order approximation. Sousa [51] presented a numerical method for space fractional diffusion equation with a second order approximation for the fractional derivative of order $\alpha$.

Chen et al. [6] used $L1$ discretization for the fractional time derivative in the one dimensional space-time fractional diffusion equation. For that equation, the numerical scheme presented in this chapter increases the order of convergence in the fractional time derivative of order $\gamma$ from $(2 - \gamma)$ to 2.

5.2 One Dimensional Space and Time Fractional Heat Equation

Consider the following one dimensional space-time fractional heat equation

$$\frac{\partial u(x, t)}{\partial t} = _0D_t^{1-\gamma} \left[ c(x, t) \frac{\partial^\alpha}{\partial |x|^\alpha} \right] u(x, t) + f(x, t), 0 < x < L, 0 < t \leq T, \quad (5.1)$$

with initial condition

$$u(x, 0) = w(x), \quad 0 \leq x \leq L \quad (5.2)$$

and boundary condition

$$u(0, t) = 0, \quad u(L, t) = 0, \quad 0 \leq t \leq T, \quad (5.3)$$

where $c(x, t) \geq 0$ is the variable coefficient, $0 < \gamma \leq 1$, $1 < \alpha \leq 2$, $\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha}$ denotes Riesz space fractional derivative of order $\alpha$ of the function $u(x, t)$ defined by [19]

$$\frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} = -\kappa_\alpha \left( _0D_x^\alpha + x D_L^\alpha \right) u(x, t),$$
where the coefficient \( \kappa_\alpha = \frac{1}{2 \cos\left(\frac{\alpha \pi}{2}\right)} \) \( \alpha \)

\( 0 D^\alpha_t u(x, t) \) and \( x D^\alpha_L u(x, t) \) denote left and right Riemann-Liouville space fractional derivative of order \( \alpha \) of the function \( u(x, t) \) defined by (1.19) and (1.20) respectively; \( f(x, t) \) and \( w(x) \) are known smooth functions and \( 0 D^{1-\gamma}_t u(x, t) \) denotes left Riemann-Liouville time fractional derivative of order \( 1 - \gamma \) of the function \( u(x, t) \) defined by (1.19). Using Lemma 2.1 and the relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative (1.11), (5.1) can be written as

\[
C_0 D^\gamma_t u(x, t) = \left( c(x, t) \frac{\partial^\alpha}{\partial |x|^\alpha} \right) u(x, t) + g(x, t),
\]

(5.4)

where \( g(x, t) = 0 D^{\gamma-1}_t f(x, t) \), \( 0 D^{\gamma-1}_t f(x, t) \) denotes Riemann-Liouville time fractional integral operator of order \( 1 - \gamma \) of the function \( f(x, t) \) defined by (1.17) and \( C_0 D^\gamma_t u(x, t) \) denotes Caputo time fractional derivative of order \( \gamma \) of the function \( u(x, t) \) defined by (1.21).

### 5.3 Approximation Operator of the Riesz Space Fractional Derivative

The approximation operator of the Riesz space fractional derivative is defined by [6]

\[
\left. \frac{\partial^\alpha u(x, t)}{\partial |x|^\alpha} \right|_{x=x_j} = \delta_{\alpha, x_j} u(x_j, t) + O(h^2),
\]

where

\[
\delta_{\alpha, x_j} u(x_j, t) = \frac{-\kappa_\alpha}{\Gamma(4 - \alpha) h^\alpha} \sum_{i=0}^m g^\alpha_{j,i} u(x_i, t) = \frac{-\kappa_\alpha}{\Gamma(4 - \alpha) h^\alpha} \sum_{i=0}^m \left( p^\alpha_{j,i} + q^\alpha_{j,i} \right) u(x_i, t),
\]

(5.5)
\[ p_{j,i}^\alpha = \begin{cases} 
  a_{j-1,i}^\alpha - 2a_{j,i}^\alpha + a_{j+1,i}^\alpha & i \leq j - 1, \\
  -2a_{j,i}^\alpha + a_{j+1,i}^\alpha & i = j, \\
  a_{j+1,i}^\alpha & i = j + 1, \\
  0 & i > j + 1. 
\end{cases} \]

\[ a_{j,i}^\alpha = \begin{cases} 
  (j - 1)^{3-\alpha} - j^{2-\alpha}(j - 3 + \alpha) & i = 0 \ & i \neq j, \\
  (j - i + 1)^{3-\alpha} - 2(j - i)^{3-\alpha} + (j - i - 1)^{3-\alpha} & 1 \leq i \leq j - 1, \\
  1 & i = j. 
\end{cases} \]

\[ q_{j,i}^\alpha = \begin{cases} 
  0 & i < j - 1, \\
  b_{j-1,i}^\alpha & i = j - 1, \\
  b_{j-1,i}^\alpha - 2b_{j,i}^\alpha & i = j, \\
  b_{j-1,i}^\alpha - 2b_{j,i}^\alpha + b_{j+1,i}^\alpha & j + 1 \leq i \leq m. 
\end{cases} \]

\[ b_{j,i}^\alpha = \begin{cases} 
  1 & i = j, \\
  (i - j + 1)^{3-\alpha} - 2(i - j)^{3-\alpha} + (i - j - 1)^{3-\alpha} & j + 1 \leq i \leq m - 1, \\
  (m - j - 1)^{3-\alpha} - (m - j)^{2-\alpha}(m - j - 3 + \alpha) & i = m \ & i \neq j. 
\end{cases} \]

**Lemma 5.1. [6]** The coefficients \( g_{j,i}^\alpha, \ \alpha \in (1, 2] \) defined in (5.5) satisfy

(i) \( g_{i,i}^\alpha < 0, \ g_{j,i}^\alpha > 0 (j \neq i), \)

(ii) \( \sum_{i=0}^{m} g_{j,i}^\alpha < 0 \) and \( -g_{i,i}^\alpha > \sum_{i=0,i \neq j}^{m} g_{j,i}^\alpha. \)
5.4 Numerical Scheme

The exact solution \( u \) of the initial and boundary value problem (5.2)-(5.4) at the point \((x_j, t_k) \in \Omega_h \times \Omega_T\) is denoted by \( u_j^k \). The exact solution of an approximating fractional ordinary differential equation and difference equation at the same point will be denoted by \( U_j(t_k) \) and \( U_j^k \) respectively.

By using the approximation operator (5.5), we may approximate the initial and boundary value problem (5.2)-(5.4) by the following semi-discrete scheme:

\[
\begin{align*}
\mathcal{C}_0 D_t^\gamma U_j(t) &= c(x_j, t) \delta_{x,x} U_j(t) + g(x_j, t), \quad 1 \leq j \leq m - 1, \quad 0 < t \leq T, \\
U_0(t) &= U_m(t) = 0, \quad 0 \leq t \leq T, \\
U_j(0) &= w(x_j), \quad 0 \leq j \leq m.
\end{align*}
\] (5.6)

In [9, 14, 19], the existence and uniqueness of (5.6) are given. The equivalent form of (5.6) is also given by

\[
\begin{align*}
U_j(t) &= U_j(0) + \mathcal{D}_t^{-\gamma} (c(x_j, t) \delta_{x,x} U_j(t) + g(x_j, t)), \quad 1 \leq j \leq m - 1, \quad 0 < t \leq T, \\
U_0(t) &= U_m(t) = 0, \quad 0 \leq t \leq T, \\
U_j(0) &= w(x_j), \quad 0 \leq j \leq m,
\end{align*}
\] (5.7)

where \( \mathcal{D}_t^{-\gamma} \) denotes the Riemann-Liouville time fractional integral of order \( \gamma \) defined by [10, 41],

\[
\mathcal{D}_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma-1} f(\tau) d\tau.
\]
By using the modified trapezoidal method [10] for the Riemann-Liouville fractional integral, for \(1 \leq k \leq n\), (5.7) is approximated by

\[
U^k_j = U^0_j + \frac{\tau^\gamma}{\Gamma(2 + \gamma)} \left\{ \sum_{i=0}^{k} a_{k,i} c^i_j \delta_{\alpha,x} U^i_j + \sum_{i=0}^{k} a_{k,i} g^i_j \right\},
\]

\[
U^k_0 = U^k_m = 0,
\]

\[
U^0_j = w(x_j), \quad 0 \leq j \leq m,
\]

where \(g^k_j = g(x_j, t_k)\), \(c^k_j = c(x_j, t_k)\) and

\[
a_{k,i} = \begin{cases} 
(k - 1)^\gamma + 1 - (k - \gamma - 1)k^{\gamma} & \text{if } i = 0, \\
(k - i + 1)^\gamma + 1 - 2(k - i)^{\gamma} + (k - i - 1)^{\gamma} & \text{if } 1 \leq i \leq k - 1, \\
1 & \text{if } i = k.
\end{cases}
\]

Arranging the terms, we have

\[
\left(1 - \frac{\tau^\gamma}{\Gamma(2 + \gamma)} c^k_j \delta_{\alpha,x}\right) U^k_j = \left(1 + \frac{\tau^\gamma}{\Gamma(2 + \gamma)} a_{k,0} c^0_j \delta_{\alpha,x}\right) U^0_j + \frac{\tau^\gamma}{\Gamma(2 + \gamma)} \left(\sum_{i=1}^{k-1} a_{k,i} c^i_j \delta_{\alpha,x} U^i_j + \sum_{i=0}^{k} a_{k,i} g^i_j \right), \quad 1 \leq j \leq m - 1,
\]

\[
U^k_0 = U^k_m = 0,
\]

\[
U^0_j = w(x_j), \quad 0 \leq j \leq m.
\]

We rewrite this system (5.9) in the following matrix form, for \(1 \leq k \leq n\),

\[
(I_{m-1} - \mu_2 A_k) U^k = (I_{m-1} + \mu_2 a_{k,0} A_0) U^0 + \mu_2 \sum_{i=1}^{k-1} a_{k,i} A_i U^i + \hat{C}^k_{x_1}, \quad (5.10)
\]

where \(\mu_2 = -\frac{\kappa_0^\gamma}{\Gamma(2 + \gamma) \Gamma(4 - \alpha) \hat{h}^\alpha}\), \(I_{m-1}\) is the identity matrix of order \((m - 1)\), for \(1 \leq k \leq n\), \(A_k\) denotes \((m - 1) \times (m - 1)\) matrix with the \((i, j)\)th entry \(c^i_j g^\alpha_{i,j}\) and...
finally $\hat{G}_k$ and $U^k$ are the column matrices of order $(m - 1)$ with the $(i, 1)^{th}$ entries $rac{\tau_\gamma}{\Gamma(2+\gamma)} \sum_{l=0}^{k} a_{k,l} g^l_i$ and $U^k_i$ respectively.

**Theorem 5.1.** The difference equations (5.9) has a unique solution.

**Proof.** Since $\mu_2 > 0$, for any $\mu_2$, by Lemma 5.1, the coefficient matrix $(I_{m-1} - \mu_2 A_k)$ for the difference equations are strictly diagonally dominant. Consequently the matrix $(I_{m-1} - \mu_2 A_k)$ is non singular, thus are invertible. Hence completes the proof of the theorem. \qed

### 5.5 Numerical Experiments

**Example 5.1.** Consider the system (5.1)-(5.3) on the finite domain $0 < x < 1$, $0 < t \leq 1$, with the coefficient $c(x,t) = x^\alpha t^{1-\gamma}$, the initial condition $u(x,0) = 0$ and the forcing functions

\[
f(x,t) = (2 + \gamma)t^{1+\gamma}x^2(x-1)^2 + \frac{6t^{2+\gamma}x^\alpha}{\Gamma(3 + \gamma)\cos(\alpha \pi/2)} \left[ \frac{x^{2-\alpha} + (1-x)^{2-\alpha}}{\Gamma(3 - \alpha)} - 6\frac{x^{3-\alpha} + (1-x)^{3-\alpha}}{\Gamma(4 - \alpha)} + 12\frac{x^{4-\alpha} + (1-x)^{4-\alpha}}{\Gamma(5 - \alpha)} \right],
\]

\[
g(x,t) = \partial\partial_t^{-1} f(x,t) = \frac{1}{2} \Gamma(3 + \gamma)t^2 x^2(x-1)^2 + \frac{t^3 x^\alpha}{\cos(\alpha \pi/2)} \left[ \frac{x^{2-\alpha} + (1-x)^{2-\alpha}}{\Gamma(3 - \alpha)} - 6\frac{x^{3-\alpha} + (1-x)^{3-\alpha}}{\Gamma(4 - \alpha)} + 12\frac{x^{4-\alpha} + (1-x)^{4-\alpha}}{\Gamma(5 - \alpha)} \right].
\]

The exact solution of this fractional heat equation is $u(x,t) = t^{2+\gamma}x^2(1-x)^2$.

The error is defined as follows:

\[
\overline{E}_\infty(h, \tau) = \max_{0 \leq j \leq m} |u(x_j, t_n) - U^n_j|.
\]

Denote

\[
\text{Order} = \log_2 \left( \frac{\overline{E}_\infty(h, \tau)}{\overline{E}_\infty(h/2, \tau/2)} \right).
\]
Table 5.1: The maximum errors and convergence orders of \( SIT \) and \( SIL1 \) schemes when \( \gamma = 0.1 \; \alpha = 1.9 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \tau )</th>
<th>( E_\infty (h, \tau) )(SIT)</th>
<th>Order(SIT)</th>
<th>( E_\infty (h, \tau) )(SIL1)</th>
<th>Order(SIL1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
<td>0.000416692</td>
<td>1.97112</td>
<td>0.00041966</td>
<td>1.9684</td>
</tr>
<tr>
<td>( \frac{1}{40} )</td>
<td>( \frac{1}{40} )</td>
<td>0.000106279</td>
<td>1.97348</td>
<td>0.000107238</td>
<td>1.97062</td>
</tr>
<tr>
<td>( \frac{1}{80} )</td>
<td>( \frac{1}{80} )</td>
<td>0.0000270626</td>
<td>1.97559</td>
<td>0.000027361</td>
<td>1.97235</td>
</tr>
<tr>
<td>( \frac{1}{160} )</td>
<td>( \frac{1}{160} )</td>
<td>( 6.88109 \times 10^{-6} )</td>
<td>*</td>
<td>( 6.9726 \times 10^{-6} )</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 5.2: The maximum errors and convergence orders of \( SIT \) and \( SIL1 \) schemes when \( \gamma = 0.5 \; \alpha = 1.9 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \tau )</th>
<th>( E_\infty (h, \tau) )(SIT)</th>
<th>Order(SIT)</th>
<th>( E_\infty (h, \tau) )(SIL1)</th>
<th>Order(SIL1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
<td>0.000361536</td>
<td>1.97236</td>
<td>0.00052505</td>
<td>1.76859</td>
</tr>
<tr>
<td>( \frac{1}{40} )</td>
<td>( \frac{1}{40} )</td>
<td>0.0000921321</td>
<td>1.97517</td>
<td>0.0001541</td>
<td>1.73099</td>
</tr>
<tr>
<td>( \frac{1}{80} )</td>
<td>( \frac{1}{80} )</td>
<td>0.0000234329</td>
<td>1.97808</td>
<td>0.000046422</td>
<td>1.69085</td>
</tr>
<tr>
<td>( \frac{1}{160} )</td>
<td>( \frac{1}{160} )</td>
<td>( 5.94791 \times 10^{-6} )</td>
<td>*</td>
<td>0.000014379</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 5.3: The maximum errors and convergence orders of \( SIT \) and \( SIL1 \) schemes when \( \gamma = 0.9 \; \alpha = 1.9 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \tau )</th>
<th>( E_\infty (h, \tau) )(SIT)</th>
<th>Order(SIT)</th>
<th>( E_\infty (h, \tau) )(SIL1)</th>
<th>Order(SIL1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{20} )</td>
<td>( \frac{1}{20} )</td>
<td>0.000297077</td>
<td>1.97311</td>
<td>0.0017987</td>
<td>1.18291</td>
</tr>
<tr>
<td>( \frac{1}{40} )</td>
<td>( \frac{1}{40} )</td>
<td>0.0000756665</td>
<td>1.97925</td>
<td>0.00079226</td>
<td>1.14452</td>
</tr>
<tr>
<td>( \frac{1}{80} )</td>
<td>( \frac{1}{80} )</td>
<td>0.0000191907</td>
<td>1.981</td>
<td>0.00035837</td>
<td>1.1245</td>
</tr>
<tr>
<td>( \frac{1}{160} )</td>
<td>( \frac{1}{160} )</td>
<td>( 4.86129 \times 10^{-6} )</td>
<td>*</td>
<td>0.00016437</td>
<td>*</td>
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</table>
Table 5.4: The maximum errors and convergence orders of SIT and SIL1 schemes when $\gamma = 0.1$ $\alpha = 1.2$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\tau$</th>
<th>$E_\infty(h, \tau)$(SIT)</th>
<th>Order(SIT)</th>
<th>$E_\infty(h, \tau)$(SIL1)</th>
<th>Order(SIL1)</th>
</tr>
</thead>
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<tr>
<td>$\frac{1}{20}$</td>
<td>$\frac{1}{20}$</td>
<td>0.000380407</td>
<td>1.66054</td>
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<td>$\frac{1}{40}$</td>
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<td>0.000120331</td>
<td>1.84426</td>
<td>0.00012018</td>
<td>1.84378</td>
</tr>
<tr>
<td>$\frac{1}{80}$</td>
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<td>0.00003512</td>
<td>1.9234</td>
<td>0.000033481</td>
<td>1.92307</td>
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<tr>
<td>$\frac{1}{160}$</td>
<td>$\frac{1}{160}$</td>
<td>$8.83485 \times 10^{-6}$</td>
<td>*</td>
<td>$8.8287 \times 10^{-6}$</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 5.5: The maximum errors and convergence orders of SIT and SIL1 schemes when $\gamma = 0.5$ $\alpha = 1.2$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\tau$</th>
<th>$E_\infty(h, \tau)$(SIT)</th>
<th>Order(SIT)</th>
<th>$E_\infty(h, \tau)$(SIL1)</th>
<th>Order(SIL1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{20}$</td>
<td>$\frac{1}{20}$</td>
<td>0.000377983</td>
<td>1.66225</td>
<td>0.00051756</td>
<td>1.74058</td>
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<tr>
<td>$\frac{1}{40}$</td>
<td>$\frac{1}{40}$</td>
<td>0.000119422</td>
<td>1.84241</td>
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<td>1.6908</td>
</tr>
<tr>
<td>$\frac{1}{80}$</td>
<td>$\frac{1}{80}$</td>
<td>0.0000333015</td>
<td>1.92123</td>
<td>0.000047975</td>
<td>1.65026</td>
</tr>
<tr>
<td>$\frac{1}{160}$</td>
<td>$\frac{1}{160}$</td>
<td>$8.79255 \times 10^{-6}$</td>
<td>*</td>
<td>0.000015284</td>
<td>*</td>
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</table>

Table 5.6: The maximum errors and convergence orders of SIT and SIL1 schemes when $\gamma = 0.9$ $\alpha = 1.2$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\tau$</th>
<th>$E_\infty(h, \tau)$(SIT)</th>
<th>Order(SIT)</th>
<th>$E_\infty(h, \tau)$(SIL1)</th>
<th>Order(SIL1)</th>
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<tr>
<td>$\frac{1}{20}$</td>
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<td>0.000368941</td>
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<td>0.00020662</td>
<td>1.15995</td>
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<td>0.000117138</td>
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<td>1.1312</td>
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<td>0.0000328297</td>
<td>1.91546</td>
<td>0.00042215</td>
<td>1.11635</td>
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<tr>
<td>$\frac{1}{160}$</td>
<td>$\frac{1}{160}$</td>
<td>$8.70275 \times 10^{-6}$</td>
<td>*</td>
<td>0.00019472</td>
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Table 5.7: When $h = 1/1000$, maximum errors and convergence orders of scheme (5.10)

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$(\gamma, \alpha) = (0.3, 1.3)$</th>
<th>$E_\infty(h, \tau)$</th>
<th>Order</th>
<th>$(\gamma, \alpha) = (0.5, 1.5)$</th>
<th>$E_\infty(h, \tau)$</th>
<th>Order</th>
<th>$(\gamma, \alpha) = (0.9, 1.9)$</th>
<th>$E_\infty(h, \tau)$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.000298015</td>
<td>1.91112</td>
<td>1.000448867</td>
<td>1.96072</td>
<td>0.000718754</td>
<td>2.00126</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>0.0000792381</td>
<td>1.92576</td>
<td>0.000115314</td>
<td>1.96806</td>
<td>0.000179532</td>
<td>1.99743</td>
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<td>$\frac{1}{16}$</td>
<td>0.0000208556</td>
<td>1.92302</td>
<td>0.0000294738</td>
<td>1.96322</td>
<td>0.0000449631</td>
<td>1.9901</td>
<td></td>
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</tr>
<tr>
<td>$\frac{1}{32}$</td>
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<td>$7.5587 \times 10^{-6}$</td>
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<td>*</td>
<td></td>
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</tbody>
</table>

Table 5.8: When $\tau = 1/1000$, maximum errors and convergence orders of scheme (5.10)

<table>
<thead>
<tr>
<th>$h$</th>
<th>$(\gamma, \alpha) = (0.3, 1.3)$</th>
<th>$E_\infty(h, \tau)$</th>
<th>Order</th>
<th>$(\gamma, \alpha) = (0.5, 1.5)$</th>
<th>$E_\infty(h, \tau)$</th>
<th>Order</th>
<th>$(\gamma, \alpha) = (0.7, 1.7)$</th>
<th>$E_\infty(h, \tau)$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.00844918</td>
<td>2.0137</td>
<td>0.00790282</td>
<td>1.97023</td>
<td>0.0066279</td>
<td>1.90735</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>0.00209234</td>
<td>2.00232</td>
<td>0.00201689</td>
<td>1.97652</td>
<td>0.00176688</td>
<td>1.93613</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{16}$</td>
<td>0.000522245</td>
<td>*</td>
<td>0.000512495</td>
<td>*</td>
<td>0.000461714</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.1: Comparison of numerical and exact solution when $\gamma = 0.5$ when $t = 0.5$ and $\alpha = 1.5$.
In Table 5.1-5.6, we have compared the errors as well as their convergence order for the difference scheme (SIL1) in [6] and spline interpolation modified trapezoidal rule (SIT) (5.10) in the case when \( h \) and \( \tau \) decrease simultaneously for different \( \gamma \) and \( \alpha \). In Table 5.7, we have obtained the maximum errors as well as their convergence order for the difference schemes with \( (\gamma, \alpha) = (0.3, 1.3) \), \( (\gamma, \alpha) = (0.5, 1.5) \) and \( (\gamma, \alpha) = (0.7, 1.7) \) when \( h \) is sufficiently small. In Table 5.8, we have obtained the maximum errors as well as their convergence order for the difference scheme SIT with \( (\gamma, \alpha) = (0.3, 1.3) \), \( (\gamma, \alpha) = (0.5, 1.5) \) and \( (\gamma, \alpha) = (0.7, 1.7) \), when \( \tau \) is sufficiently small. Comparison of numerical and exact solutions for \( \gamma = 0.5 \), \( t = 0.5 \) and \( \alpha = 1.5 \) when \( m = n = 20 \) and \( m = n = 40 \) is given in Figure 5.1(a) and Figure 5.1(b) respectively. For different \( \gamma \), exact solution is plotted when \( t = 0.5 \) in Figure 5.2.