4. Higher-Order Numerical Scheme for the Fractional Heat Equation with Neumann Boundary Conditions

4.1 Introduction

Compared with the numerical methods for diffusion equation with Direchlet boundary conditions, little [21, 45, 54, 61] has been done on the numerical methods for the equations with Neumann boundary conditions. Langlands and Henry [21] investigated the accuracy and stability of an implicit numerical scheme for solving a fractional diffusion equation with zero flux boundary condition. Zhao and Sun [61] obtained the box-type scheme for fractional sub-diffusion equation with Neumann boundary conditions.

The numerical schemes proposed in [61] have convergence of order $O(\tau^{2-\gamma} + h^4)$. Rena et al. [45] proposed a difference scheme combining the compact difference approach to the spatial discretization and L1 approximation for the Caputo fractional derivative and analyzed for the time fractional sub-diffusion equation with Neumann boundary conditions.
In this chapter, we consider the fourth-order compact scheme in space and modified trapezoidal rule obtained for fractional heat equation with Neumann boundary conditions to increase the order of convergence from $O(\tau^{2-\gamma} + h^{4})$ to $O(\tau^{2} + h^{4})$.

### 4.2 Fractional Heat Equation with Neumann Boundary Condition

Consider the following fractional heat equation

\[
\frac{\partial u(x,t)}{\partial t} = 0D^{1-\gamma}_{t} \left[ K_{\gamma} \frac{\partial^{2} u(x,t)}{\partial x^{2}} \right] + f(x,t), \quad 0 \leq x \leq L, \quad 0 < t \leq T, \tag{4.1}
\]

with initial condition

\[
u(x,0) = w(x), \quad 0 \leq x \leq L \tag{4.2}
\]

and boundary condition

\[
u_{x}(0,t) = \phi(t), \quad \nu_{x}(L,t) = \xi(t), \quad 0 < t \leq T, \tag{4.3}
\]

where $w(x), \phi(t)$ and $\xi(t)$ are known smooth functions. Using Lemma 2.1 and the relationship between Caputo fractional derivative and Riemann-Liouville fractional derivative (1.11), (4.1) can be can be rewritten as

\[
0D^{\gamma}_{t} u(x,t) = K_{\gamma} \frac{\partial^{2} u(x,t)}{\partial x^{2}} + g(x,t), \quad 0 \leq x \leq L, \quad 0 < t \leq T. \tag{4.4}
\]

where $g(x,t) = 0D^{\gamma-1}_{t} f(x,t), \ 0D^{\gamma-1}_{t} f(x,t)$ denotes Riemann-Liouville time fractional integral operator of order $1-\gamma$ of the function $f(x,t)$ defined by (1.17) and $0D^{\gamma}_{t} u(x,t)$ denotes Caputo time fractional derivative of order $\gamma$ of the function $u(x,t)$ defined by (1.21).
The exact solution \( \tilde{u} \) of initial and boundary value problem (4.2), (4.3) and (4.4) at the point \((x_j, t_k) \in \Omega_h \times \Omega_t\) is denoted by \( \tilde{u}_j^k \) and the corresponding solution vector is denoted by \( \tilde{u}^k = (\tilde{u}_0^k, \tilde{u}_1^k, ..., \tilde{u}_m^k)^T \). The exact solution of an approximating ordinary fractional differential equation and difference equation at the same point will be denoted by \( \tilde{U}_j(t_k) \) and \( \tilde{U}_j^k \) and the corresponding solution vectors are denoted by \( \tilde{U}(t_k) = (\tilde{U}_0(t_k), \tilde{U}_1(t_k), ..., \tilde{U}_m(t_k))^T \) and \( \tilde{U}^k = (\tilde{U}_0^k, \tilde{U}_1^k, ..., \tilde{U}_m^k)^T \) respectively.

### 4.3 Discretization in Space: Semi-Discrete Scheme

We use the fourth order compact difference schemes for the first and second derivatives:

\[
\begin{align*}
\left. \frac{\partial \tilde{u}(x, t)}{\partial x} \right|_{x=x_j} &= \frac{\delta_x}{2h(1+\frac{\delta_x^2}{6})} \tilde{u}(x_j, t) + O(h^4), \\
\left. \frac{\partial^2 \tilde{u}(x, t)}{\partial x^2} \right|_{x=x_j} &= \frac{\delta_x^2}{h^2(1+\frac{\delta_x^2}{12})} \tilde{u}(x_j, t) + O(h^4),
\end{align*}
\]

where \( \delta_x \tilde{u}(x_j, t) = \hat{u}(x_{j+1}, t) - \hat{u}(x_{j-1}, t) \), \( \delta_x^2 \tilde{u}(x_j, t) = \hat{u}(x_{j-1}, t) - 2\hat{u}(x_j, t) + \hat{u}(x_{j+1}, t) \). It follows from (4.3) and (4.4) that

\[
\begin{align*}
\hat{u}_x(0, t) &= \phi(t), \\
K_\gamma \hat{u}_{xx}(0, t) &= C_0^\gamma D_0^\gamma \phi(t) - g_x(0, t)
\end{align*}
\]

Also from (4.5) and (4.6) it follows that

\[
\hat{u}(x_{-1}, t) = \hat{u}(x_1, t) - 2h \left( \phi(t) + \frac{h^2}{6K_\gamma} \left[ C_0^\gamma D_0^\gamma \phi(t) - (g_x)(x_0, t) \right] \right) + O(h^5).
\]

The similar method which is used to obtain equation (4.7) is applied to derive the following equation at the point \((x_m, t) = (L, t)\). Therefore, Caputo derivative of \( \xi(t) \) should be taken.
\[ \tilde{u}(x_{m+1}, t) = \tilde{u}(x_{m-1}, t) + 2h \left( \xi(t) + \frac{h^2}{6K_\gamma} \left[ \mathcal{D}_t^{\gamma} \xi(t) - (g_x)(x_m, t) \right] \right) + O(h^5), \quad (4.8) \]

where \( x_{-1} = -h \) and \( x_{m+1} = L+h \) are fictitious points. Thus, we may approximate the initial boundary value problem (4.2)-(4.4) by the following semi-discrete scheme:

for \( 0 < t \leq T \),

\[ \mathcal{D}_t^{\gamma} \tilde{U}_j(t) = \frac{K_\gamma}{h^2} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}} \tilde{U}_j(t) + g(x_j, t), \quad 0 \leq j \leq m, \]

\[ \tilde{U}_{-1}(t) = \tilde{U}_1(t) - 2h\chi_1(t), \quad \tilde{U}_{m+1}(t) = \tilde{U}_{m-1}(t) + 2h\chi_2(t), \]

\[ \tilde{U}_j(0) = w(x_j), \quad 0 \leq j \leq m, \quad (4.9) \]

where

\[ \chi_1(t) = \phi(t) + \frac{h^2}{6K_\gamma} \left[ \mathcal{D}_t^{\gamma} \phi(t) - (g_x)(x_0, t) \right], \]

\[ \chi_2(t) = \xi(t) + \frac{h^2}{6K_\gamma} \left[ \mathcal{D}_t^{\gamma} \xi(t) - (g_x)(x_m, t) \right]. \]

### 4.4 Fully Discrete Scheme

The equivalent form of (4.9) for \( 0 < t \leq T \),

\[ \tilde{U}_j(t) = \tilde{U}_j(0) + J \mathcal{D}_t^{\gamma} \left( \tilde{U}_j(t) + g(x_j, t) \right), \quad 0 \leq j \leq m, \]

\[ \tilde{U}_{-1}(t) = \tilde{U}_1(t) - 2h\chi_1(t), \quad \tilde{U}_{m+1}(t) = \tilde{U}_{m-1}(t) + 2h\chi_2(t), \]

\[ \tilde{U}_j(0) = w(x_j), \quad 0 \leq j \leq m, \quad (4.10) \]

By using the modified trapezoidal rule for the Riemann-Liouville fractional integral then (4.10) is approximated by for \( 1 \leq k \leq n \),

\[ \tilde{U}_j^k = \tilde{U}_j^0 + \frac{\tau^\gamma}{\Gamma(2 + \gamma)} \left\{ \frac{K_\gamma}{h^2} \sum_{i=0}^{k} a_{k,i} \frac{\delta_x^2}{1 + \frac{\delta_x^2}{12}} \tilde{U}_j^i + \sum_{i=0}^{k} a_{k,i}g_j^i \right\}, 0 \leq j \leq m, \quad (4.11) \]

\[ \tilde{U}_{-1}^k = \tilde{U}_1^0 - 2h\chi_1(t_k), \quad (4.12) \]
\[ \tilde{U}^{k}_{m+1} = \tilde{U}^{k}_{m-1} + 2h\chi_2(t_k), \quad (4.13) \]

\[ \tilde{U}^0_{j} = w(x_j), 0 \leq j \leq m, \]

where \( a_{k,i} \) is defined as (3.1) and \( g^k_j = g(x_j, t_k) \). Substitute (4.12) and (4.13) in (4.11) when \( j = 0 \) and \( j = m \) respectively and arranging the terms, we have for \( 1 \leq k \leq n \),

\[ (10 + 24\mu_1)\tilde{U}^0_{0} + 2(1 - 12\mu_1)\tilde{U}^k_{1} = (10 - 24\mu_1a_{k,0})\tilde{U}^0_{0} + 12\mu_1 \sum_{i=1}^{k-1} a_{k,i}(2\tilde{U}^0_{i} - 2\tilde{U}^0_{0}) + \frac{\tau^\gamma}{\Gamma(\gamma + 2)} \sum_{i=0}^{k} a_{k,i}(g^i_0 - 10g^i_0 + g^i_1) \]

\[ + 2h(1 - 12\mu_1)\chi_1(t_k) - 2h(1 + 12\mu_1a_{k,0})\chi_1(t_0) - 24\mu_1 h \sum_{i=1}^{k} a_{k,i} \chi_1(t_i), \]

\[ (1 - 12\mu_1)\tilde{U}^k_{j-1} + (10 + 24\mu_1)\tilde{U}^k_{j} + (1 - 12\mu_1)\tilde{U}^k_{j+1} = (1 + 12\mu_1a_{k,0})\tilde{U}^0_{j-1} \]

\[ + (10 - 24\mu_1a_{k,0})\tilde{U}^0_{j} + (1 + 12\mu_1a_{k,0})\tilde{U}^0_{j+1} + 12\mu_1 \sum_{i=1}^{k-1} a_{k,i}(\tilde{U}^i_{j-1} - 2\tilde{U}^i_{j} + \tilde{U}^i_{j+1}), \]

\[ + \frac{\tau^\gamma}{\Gamma(\gamma + 2)} \sum_{i=0}^{k} a_{k,i}(g^i_{j-1} - 10g^i_{j} + g^i_{j+1}), 1 \leq j \leq m - 1, \]

\[ 2(1 - 12\mu_1)\tilde{U}^k_{m-1} + (10 + 24\mu_1)\tilde{U}^k_{m} = 2(1 + 12\mu_1a_{k,0})\tilde{U}^0_{m-1} \]

\[ + (10 - 24\mu_1a_{k,0})\tilde{U}^0_{m} + 12\mu_1 \sum_{i=1}^{k-1} a_{k,i}(2\tilde{U}^i_{m-1} - 2\tilde{U}^i_{m}) + \frac{\tau^\gamma}{\Gamma(\gamma + 2)} \sum_{i=0}^{k} a_{k,i}(g^i_{m-1} - 10g^i_{m}) \]

\[ + g^i_{m+1} - 2h(1 - 12\mu_1)\chi_2(t_k) + 2h(1 + 12\mu_1a_{k,0})\chi_2(t_0) + 24\mu_1 h \sum_{i=1}^{k} a_{k,i} \chi_2(t_i), \]

\[ \tilde{U}^0_{j} = w(x_j), 0 \leq j \leq m, \quad (4.14) \]

where \( \mu_1 = \frac{K_\gamma \tau^\gamma}{\Gamma(\gamma + 2)h^2} \). We need the values of \( g^j_{-1} \) and \( g^j_{m+1} \) and these may be obtained from the Taylor series expansion. Using the Taylor series expansion of \( g^j_{1} \) and \( g^j_{-1} \) about the point \((x_0, t_i)\), then \( g^j_{-1} + 10g^j_{0} + g^j_{1} \) can be written as \( 12g^j_{0} + h^2(g_{xx})_{0} + O(h^4) \). Similarly \( g^j_{m-1} + 10g^j_{m} + g^j_{m+1} \) can be written as \( 12g^j_{m} + h^2(g_{xx})_{m} + O(h^4) \) by using the Taylor series expansion of \( g^j_{m-1} \) and \( g^j_{m+1} \) about the point \((x_m, t_i)\).
Remark 4.1. Since $g(x,t)$ is known function, the exact value of $g(x,t)$ at the points $(x_{-1}, t_i) = (-h, t_i)$ and $(x_{m+1}, t_i) = (L + h, t_i)$ may be used.

Let us rewrite this system of equations (4.14) in the following matrix form:

$$
\tilde{A}\tilde{U}^k = \tilde{B}_k\tilde{U}^0 + 12\mu_1 \sum_{i=1}^{k-1} a_{k,i}\tilde{C}\tilde{U}^i + \tilde{G}^k, \quad 1 \leq k \leq n, \quad (4.15)
$$

where $\tilde{A}$, $\tilde{C}$ and $\tilde{B}_k (1 \leq k \leq n)$ are tridiagonal matrices and $\tilde{G}^k (1 \leq k \leq n)$ are column vectors in $R^{(m+1)}$ which are given by

$$
\tilde{A} = \begin{pmatrix}
10 + 24\mu_1 & 2(1 - 12\mu_1) \\
1 - 12\mu_1 & 10 + 24\mu_1 & 1 - 12\mu_1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & 1 - 12\mu_1 & 10 + 24\mu_1 & 1 - 12\mu_1 \\
& & & 2(1 - 12\mu_1) & 10 + 24\mu_1
\end{pmatrix}_{(m+1)\times(m+1)},
$$

$$
\tilde{C} = \begin{pmatrix}
-2 & 2 \\
1 & -2 & 1 \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -2 & 1 \\
& & & 2 & -2
\end{pmatrix}_{(m+1)\times(m+1)}.
$$
for $1 \leq k \leq n,$

\[
\tilde{B}_k = \begin{pmatrix}
10 - 24\mu_1 a_{k,0} & 2(1 + 12\mu_1 a_{k,0}) \\
1 + 12\mu_1 a_{k,0} & 10 - 24\mu_1 a_{k,0} & 1 + 12\mu_1 a_{k,0} \\
\vdots & \ddots & \ddots & \ddots \\
1 + 12\mu_1 a_{k,0} & 10 - 24\mu_1 a_{k,0} & 1 + 12\mu_1 a_{k,0} & 2(1 + 12\mu_1 a_{k,0}) & 10 - 24\mu_1 a_{k,0}
\end{pmatrix}_{(m+1) \times (m+1)}
\]

\[
\tilde{G}_k = \begin{pmatrix}
\frac{\tau \gamma}{\Gamma(2+\gamma)} \sum_{i=0}^{k} a_{k,i}(12g_i^0 + h^2(g_{xx})_i^0) + 2h(1 - 12\mu_1)\chi_1(t_k) - 2h(1 + 12\mu_1 a_{k,0})\chi_1(t_0) - 24h\mu_1 \sum_{i=0}^{k-1} a_{k,i}\chi_1(t_i) \\
\frac{\tau \gamma}{\Gamma(2+\gamma)} \sum_{i=0}^{k} a_{k,i}(g_i^0 + 10g_1^i + g_2^i) \\
\vdots \\
\frac{\tau \gamma}{\Gamma(2+\gamma)} \sum_{i=0}^{k} a_{k,i}(g_{m-2}^i + 10g_{m-1}^i + g_m^i) \\
\frac{\tau \gamma}{\Gamma(2+\gamma)} \sum_{i=0}^{k} a_{k,i}(12g_m^i + h^2(g_{xx})_m^i) - 2h(1 - 12\mu_1)\chi_2(t_k) + 2h(1 + 12\mu_1 a_{k,0})\chi_2(t_0) + 24h\mu_1 \sum_{i=0}^{k-1} a_{k,i}\chi_2(t_i)
\end{pmatrix}_{(m+1) \times 1}
\]
Theorem 4.1. The difference equations (4.14) has a unique solution.

Proof. For any \( \mu_1 = \frac{K \tau^\gamma}{\Gamma(2+\gamma)h^2} > 0 \), the coefficient matrix \( \tilde{A} \) for the difference equations is strictly diagonally dominant. Consequently the matrix \( \tilde{A} \) is nonsingular, thus are invertible. Hence completes the proof of the theorem. \( \square \)

4.5 Numerical Experiments

Example 4.1. [61] Fractional heat equation with Neumann boundary condition

\[
\frac{\partial u(x,t)}{\partial t} = 0D_t^{1-\gamma} \left[ \frac{\partial^2 u(x,t)}{\partial x^2} \right] + e^x \left[ (2 + \gamma)x^2(1-x)^2t^{1+\gamma} - \frac{\Gamma(3 + \gamma)}{\Gamma(2\gamma + 2)} t^{2\gamma+1}(2 - 8x + x^2 + 6x^3 + x^4) \right],
\]

\[0 \leq x \leq 1, \quad 0 < t \leq 1,\]

\[u(x,0) = 0, \quad u_x(0,t) = 0, \quad u_x(1,t) = 0.\]  \hspace{1cm} (4.16)

The exact solution of (4.16) is \( u(x,t) = e^x x^2(1-x)^2t^{2+\gamma} \). The error is defined as follows

\[E_\infty(h,\tau) = \max_{0 \leq j \leq m} \max_{1 \leq k \leq n} |u(x_j,t_k) - U_j^k|.\]

Denote

\[\text{order1} = \log_2 \left( \frac{E_\infty(h,\tau)}{E_\infty(h/2,\tau/2)} \right), \quad \text{order2} = \log_2 \left( \frac{E_\infty(h,\tau)}{E_\infty(h/2,\tau)} \right), \quad \text{order3} = \log_2 \left( \frac{E_\infty(h,\tau)}{E_\infty(h,\tau/2)} \right).\]
Figure 4.1: The numerical solution of the CT scheme defined by (4.14) at $\gamma = 0.5$. 

(a) $m = 4$ and $n = 4$

(b) $m = 8$ and $n = 8$

(c) $m = 16$ and $n = 16$
Figure 4.2: The numerical solution of the Box scheme at $\gamma = 0.5$. 

(a) $m = 4$ and $n = 4$

(b) $m = 8$ and $n = 8$

(c) $m = 16$ and $n = 16$
Convergence of the numerical solution of Box scheme and CT scheme defined by (4.14) for $\gamma = 0.5$ and Exact solution are given in Figure 4.1, Figure 4.2 and Figure 4.3 respectively.

Table 4.1: The maximum errors and convergence orders of Box and CT (4.14) schemes with different $\gamma$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$(h, \tau)$</th>
<th>$E_\infty(h, \tau)$ (BoxL1)</th>
<th>Order1 (BoxL1)</th>
<th>$E_\infty(h, \tau)$ (CT)</th>
<th>Order1 (CT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>(1/20, 1/200)</td>
<td>0.00006157</td>
<td>1.3179</td>
<td>0.0000298287</td>
<td>1.94813</td>
</tr>
<tr>
<td></td>
<td>(1/40, 1/400)</td>
<td>0.0000247</td>
<td>1.4527</td>
<td>7.3015×10^{-6}</td>
<td>1.95926</td>
</tr>
<tr>
<td></td>
<td>(1/80, 1/800)</td>
<td>9.022×10^{-6}</td>
<td>*</td>
<td>1.98788×10^{-6}</td>
<td>*</td>
</tr>
<tr>
<td>0.5</td>
<td>(1/20, 1/200)</td>
<td>0.0003877</td>
<td>1.3816</td>
<td>0.0000426633</td>
<td>1.98006</td>
</tr>
<tr>
<td></td>
<td>(1/40, 1/400)</td>
<td>0.0001488</td>
<td>1.4215</td>
<td>0.0000108143</td>
<td>1.98609</td>
</tr>
<tr>
<td></td>
<td>(1/80, 1/800)</td>
<td>0.00005554</td>
<td>*</td>
<td>2.72976×10^{-6}</td>
<td>*</td>
</tr>
<tr>
<td>0.7</td>
<td>(1/20, 1/200)</td>
<td>0.001184</td>
<td>1.2495</td>
<td>0.0000541512</td>
<td>1.99399</td>
</tr>
<tr>
<td></td>
<td>(1/40, 1/400)</td>
<td>0.0004978</td>
<td>1.2694</td>
<td>0.0000135943</td>
<td>1.99632</td>
</tr>
<tr>
<td></td>
<td>(1/80, 1/800)</td>
<td>0.00002037</td>
<td>*</td>
<td>3.40726×10^{-6}</td>
<td>*</td>
</tr>
</tbody>
</table>
In Table 4.1 we have compared the maximum errors as well as the convergence order for the difference scheme (Box scheme) in [61] and compact modified trapezoidal rule (4.14) in the case when $h$ and $\tau$ decrease simultaneously with different $\gamma$.

Table 4.2: When $h = 1/1000$, the maximum errors of Box and CT (4.14) schemes with $\gamma = 0.3$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$E_\infty(h, \tau)$(BoxL1)</th>
<th>Order3(BoxL1)</th>
<th>$E_\infty(h, \tau)$(CT)</th>
<th>Order3(CT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>0.00477651</td>
<td>1.49137</td>
<td>0.00240873</td>
<td>1.87643</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.00169888</td>
<td>1.55881</td>
<td>0.00656035</td>
<td>1.90646</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>0.00057654</td>
<td>*</td>
<td>0.00174995</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 4.3: When $h = 1/1000$, the maximum errors of Box and CT (4.14) schemes with $\gamma = 0.5$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$E_\infty(h, \tau)$(BoxL1)</th>
<th>Order3(BoxL1)</th>
<th>$E_\infty(h, \tau)$(CT)</th>
<th>Order3(CT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>0.0111597</td>
<td>1.34019</td>
<td>0.00382813</td>
<td>1.93026</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.00440774</td>
<td>1.39065</td>
<td>0.00100443</td>
<td>1.95345</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>0.00168109</td>
<td>*</td>
<td>0.00259343</td>
<td>*</td>
</tr>
</tbody>
</table>

Table 4.4: When $h = 1/1000$, the maximum errors of Box and CT (4.14) schemes with $\gamma = 0.7$.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$E_\infty(h, \tau)$(BoxL1)</th>
<th>Order3(BoxL1)</th>
<th>$E_\infty(h, \tau)$(CT)</th>
<th>Order3(CT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}$</td>
<td>0.0215705</td>
<td>1.2211</td>
<td>0.00515961</td>
<td>1.96387</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.00925279</td>
<td>1.23886</td>
<td>0.00132262</td>
<td>1.98036</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>0.00392049</td>
<td>*</td>
<td>0.000335187</td>
<td>*</td>
</tr>
</tbody>
</table>
In Table 4.2, 4.3, 4.4 we have compared the maximum errors as well as their convergence order for these difference schemes with $\gamma = 0.3$, $\gamma = 0.5$, and $\gamma = 0.7$ respectively when $h$ is sufficiently small. In Table 4.5, 4.6, 4.7 we have compared the maximum errors as well as their convergence order for these difference schemes with $\gamma = 0.3$, $\gamma = 0.5$, and $\gamma = 0.7$ respectively when $\tau$ is sufficiently small.