Chapter 3

Decomposition of complete bipartite multigraphs into paths and cycles having $k$ edges

In this chapter, we investigate the existence of a $\{P_{k+1}, C_k\}_{\{p,q\}}$–decomposition of $K_{m,n}(\lambda)$. Since, we study only the existence of a $\{P_{k+1}, C_k\}_{\{p,q\}}$–decomposition of $K_{m,n}(\lambda)$, we abbreviate the notation for such decomposition as $(k; p, q)$–decomposition of $K_{m,n}(\lambda)$. The obvious necessary condition for such existence is $k(p + q) = |E(K_{m,n}(\lambda))|$. As we consider only cases where all vertices are of even degree, the case $p \neq 1$ is also obviously necessary, since the presence of a single path in the decomposition would give two vertices of odd degree and the resulting
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

The graph is not cycle decomposable. Call the situation with \(k(p + q) = |E(K_{m,n}(\lambda))|\), all vertex degrees are even, and \(p \neq 1\) the good case. We prove that in the good case \(K_{m,n}(\lambda)\) has a \((k; p, q)\)-decomposition when \(\lambda \equiv 0 \pmod{2}\), \(m, n \geq \frac{k}{2}\), \(m + n > k\), and \(k(p + q) = 2mn\) for \(k \equiv 0 \pmod{2}\).

Further, we show that if \(K_{m,n}(\lambda), \lambda \geq 3, k \equiv 0 \pmod{4}\) (resp., \(k \equiv 2 \pmod{4}\)) has a \((k; p, q)\)-decomposition in the good case with \(k/2 \leq m, n \leq k\), (resp., \(k/2 \leq m, n \leq 3k/2\)) then such decomposition of \(K_{m,n}(\lambda)\) exists in the good case when \(\lambda \geq 3; m, n \geq k\) (resp., \(m, n \geq 3k/2\)).

To prove our results, we state the following:

By considering the underlying graph of \(K_{m,n}^*\), we have the following from Theorem 2.2:

**Theorem 3.1.** The graph \(K_{m,n}(2)\) has a \(C_k\)-decomposition if and only if \(m \geq \frac{k}{2}, n \geq \frac{k}{2}\), and \(k\) divides \(2mn\).

\[3.1\] \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

In this section, we investigate the existence of a \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k \equiv 0 \pmod{2}\).

**Remark 3.2.** Let \(k \in \mathbb{N}\). If \(G\) and \(H\) have a \((k; p, q)\)-decomposition, then \(G \oplus H\) has such decomposition.

**Lemma 3.3.** Let \(p, q\) be nonnegative integers and \(\{k, m, n\} \in \mathbb{N}\) such that \(k \equiv 0 \pmod{2}\) and \(m + n > k\). The graph \(K_{m,n}(2)\) has a \((k; p, q)\)-decomposition if and only if \(m, n \geq k/2, k(p + q) = 2mn\), and \(p \neq 1\).
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

**Proof. Necessity.** Conditions \(m, n \geq k/2\), \(k(p + q) = 2mn\), and \(p \neq 1\) are trivial.

**Sufficiency.** Let \(k \equiv 0 \pmod{2}\). In order to have a \(C_k\)-decomposition in \(K_{m,n}(2)\), we can always find \(u, v\) such that \(k = 2uv\), \(m = ru\), \(n = sv\), \(r \geq v\), and \(s \geq u\), where \(r\) and \(s\) are positive integers. We denote vertices of the partite sets of \(K_{ru,sv}\) by \(x_i, 0 \leq i \leq ru - 1\) and \(y_j, 0 \leq j \leq sv - 1\). By Theorem 3.1, the graph \(K_{ru,sv}(2)\) has a \(C_{2uv}\)-decomposition as follows:

\[
C_{\lambda\mu} = \left( \cdots \left( \cdots x_{(\mu+i)u+jy(\lambda+j)v+i} \cdots \right)_{0 \leq i \leq v-1} \right)_{0 \leq j \leq u-1},
\]

where the indices of \(x\) are taken with modulo \(ru\) and those of \(y\) with modulo \(sv\). Now we construct the required number of \(P_{k+1}\) from the \(C_k\)-decomposition given above, in two cases.

**Case (1). when \(p\) is even.**

For a fixed \(\mu\) and \(0 \leq \lambda \leq s - 1\), we can have \(C_{\lambda\mu}\) and \(C_{(\lambda+1)\mu}\) as above. Since \(x_{\mu\lambda v} \in E(C_{\lambda\mu})\), \(x_{\mu\lambda v}y_{(\lambda+u+1)v-1} \in E(C_{(\lambda+1)\mu})\), \(y_{\lambda v} \notin V(C_{(\lambda+1)\mu})\), and \(y_{(\lambda+u+1)v-1} \notin V(C_{\lambda\mu})\), we have two edge-disjoint paths of length \(k\), say \(P_{\lambda\mu}\) and \(P_{(\lambda+1)\mu}\) from \(C_{\lambda\mu}\) and \(C_{(\lambda+1)\mu}\) as follows (see Figure 3.1):

\[
P_{\lambda\mu} = (C_{\lambda\mu} - x_{\mu\lambda v}) \cup x_{\mu\lambda v}y_{(\lambda+u+1)v-1} \quad \text{and}
\]

\[
P_{(\lambda+1)\mu} = (C_{(\lambda+1)\mu} - x_{\mu\lambda v}y_{(\lambda+u+1)v-1}) \cup x_{\mu\lambda v}y_{\lambda v}.
\]

Similarly, we can find a pair of paths of length \(k\) from the pair of cycles \(C_{\lambda\mu}\) and \(C_{(\lambda+1)\mu}\), where \(\lambda = 0, 2, \ldots, s - 2\) or \(s - 1\) and \(0 \leq \mu \leq r - 1\). Hence we have the desired paths.
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

![Figure 3.1](image)

**Figure 3.1:** \(C_{\lambda\mu} \oplus C_{(\lambda+1)\mu} = \mathbb{P}_{\lambda\mu} \oplus \mathbb{P}_{(\lambda+1)\mu}\)

Now for a fixed \(\lambda\) and \(0 \leq \mu \leq r-1\), we can have \(C_{\lambda\mu}\) and \(C_{(\lambda+1)\mu}\) as above. Since \(x_{\mu p}y_{(\lambda+p)q-1} \in E(C_{\lambda\mu})\), \(x_{(\mu+q+1)p-1}y_{(\lambda+p)q-1} \in E(C_{(\lambda+1)\mu})\), \(x_{\mu p} \notin V(C_{(\lambda+1)\mu})\), and \(x_{(\mu+q+1)p-1} \notin V(C_{\lambda\mu})\), we have two edge-disjoint paths of length \(k\), say \(\mathbb{P}_{\lambda\mu}\) and \(\mathbb{P}_{(\lambda+1)\mu}\) from \(C_{\lambda\mu}\) and \(C_{(\lambda+1)\mu}\) as follows (see Figure 3.2):

![Figure 3.2](image)

**Figure 3.2:** \(C_{\lambda\mu} \oplus C_{(\lambda+1)\mu} = \mathbb{P}_{\lambda\mu} \oplus \mathbb{P}_{(\lambda+1)\mu}\)
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

\[
\mathbb{P}_{\lambda \mu} = \left( C_{\lambda \mu} - x_{\mu p} y_{(\lambda+p)q-1} \right) \cup x_{(\mu+q+1)p-1} y_{(\lambda+p)q-1} \text{ and }
\]
\[
\mathbb{P}_{\lambda(\mu+1)} = \left( C_{\lambda(\mu+1)} - x_{(\mu+q+1)p-1} y_{(\lambda+p)q-1} \right) \cup x_{\mu p} y_{(\lambda+p)q-1}.
\]

Similarly, we can find a pair of paths of length \(k\) from the pair of cycles \(C_{\lambda \mu}\) and \(C_{\lambda(\mu+1)}\), where \(\mu = 0, 2, \ldots, r-2\) or \(r-1\). Hence we have the desired paths.

**Case (2). when \(p\) is odd.**

Fixing \(q = \gcd(n, k/2)\), we have \(p = k/2q\), \(s = n/q\). Since \(k\) divides \(2mn\), i.e. \(2pq\) divides \(2mn\) and \(q\) divides \(n\), implies \(r = m/p\).

**Subcase (a). when \((v+2)u - 1 \leq m\) and \(q + 2 \leq r\).**

Since \(r \geq 3\) and \(s \geq 1\), we can have \(C_{00}\), \(C_{01}\), and \(C_{02}\) (see Figure 6.6). By applying a procedure similar to Construction 2.4, we have three edge-disjoint paths of length \(k\), say \(\mathbb{P}_{00}\), \(\mathbb{P}_{01}\), and \(\mathbb{P}_{02}\) from \(C_{00}\), \(C_{01}\), and \(C_{02}\) as follows (see Figure 3.4):

\[
\mathbb{P}_{00} = \left( C_{00} - x_{0yuv-1} \right) \cup x_{(v+1)u-1} y_{uv-1},
\]
\[
\mathbb{P}_{01} = \left( C_{01} - x_{(v+1)u-1} y_{uv-1} \right) \cup x_{(v+2)u-1} y_{uv-1}, \text{ and}
\]
\[
\mathbb{P}_{02} = \left( C_{02} - x_{(v+2)u-1} y_{uv-1} \right) \cup x_{0yuv-1}.
\]

By applying a procedure similar to Case (1), the remaining pairs of cycles \(C_{\lambda \mu} \oplus C_{\lambda(\mu+1)}\), \((\lambda, \mu)\), \((\lambda, \mu + 1) \neq (0, 0), (0, 1), (0, 2)\) decomposes into pairs of paths. Hence we have the desired decomposition.
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

Figure 3.3: \(C_{00} \oplus C_{01} \oplus C_{02}\)

Figure 3.4: \(P_{00} \oplus P_{01} \oplus P_{02}\)
Subcase (b). when \((u + 2)v - 1 \leq n\) and \(p + 2 \leq s\).

Since \(r \geq 1\) and \(s \geq 3\), we can have \(C_{00}, C_{10}, \text{ and } C_{20}\) (see Figure 3.5). By applying a procedure similar to Construction 2.4, we have three edge-disjoint paths of length \(k\), say \(P_{00}, P_{10}, \text{ and } P_{20}\) from \(C_{00}, C_{10}, \text{ and } C_{20}\) as follows (see Figure 3.6):

\[
P_{00} = (C_{00} - x_0y_{uv-1}) \cup x_0y_{(u+1)v-1},
\]
\[
P_{10} = (C_{10} - x_0y_{(u+1)v-1}) \cup x_0y_{(u+2)v-1}, \quad \text{and}
\]
\[
P_{20} = (C_{20} - x_0y_{(u+2)v-1}) \cup x_0y_{uv-1}.
\]

By applying a procedure similar to Case (1), the remaining pairs of cycles \(C_{\lambda \mu} \oplus C_{(\lambda+1)\mu} (\lambda, \mu), (\lambda + 1, \mu) \neq (0, 0), (1, 0), (2, 0)\) decomposes into pairs of paths. Hence we have a desired decomposition.
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

Subcase (c). when \((v + 1)u - 1 \leq m\), \((u + 1)v - 1 \leq n\), and \(q + 1 \leq r\), \(m\) or \(n \neq k/2\)

Since \(r, s \geq 2\), we can have \(C_{00}\), \(C_{10}\), and \(C_{11}\). By applying a procedure similar to Case (1), we have two edge-disjoint paths of length \(k\), say \(P_{10}\) and \(P_{11}\) from \(C_{10}\) and \(C_{11}\) as follows:

\[
P_{10} = (C_{10} - x_0y(u+1)v - 1) \cup x(v+1)u - 1y(u+1)v - 1\quad \text{and}
\]
\[
P_{11} = (C_{11} - x(v+1)u - 1y(u+1)v - 1) \cup x_0y(u+1)v - 1.
\]

Now consider \(C_{00}\) and \(P_{11}\) (see Figure 3.7), since \(x_0y_{uv-1} \in E(C_{00})\), \(x(v+1)u - 2y_{uv-1} \in E(P_{11})\), \(x(v+1)u - 2 \notin V(C_{00})\), and \(x_0 \in V(P_{11})\), we have two edge-disjoint paths of length \(k\), say \(P_{00}\) and \(\hat{P}_{11}\) from \(C_{00}\) and \(P_{11}\) as follows (see Figure 3.8):

\[
P_{00} = (C_{00} - x_0y_{uv-1}) \cup x(v+1)u - 2y_{uv-1}\quad \text{and}
\]
\[
\hat{P}_{11} = (P_{11} - x(v+1)u - 2y_{uv-1}) \cup x_0y_{uv-1}.
\]
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

By applying a procedure similar to Case (1), the remaining pairs of cycles both \(C_{\lambda\mu} \oplus C_{(\lambda+1)\mu}\) and \(C_{\lambda\mu} \oplus C_{\lambda(\mu+1)}\), \((\lambda, \mu), (\lambda + 1, \mu) (\lambda, \mu + 1) \neq (0, 0), (0, 1), (1, 1)\) decomposes into pairs of paths. Hence the graph \(K_{m,n}(2)\) has the desired decomposition.

**Subcase (d). when \(m = k/2 + 1\) and \(n = k/2\)**

When \(m = k/2 + 1\) and \(n = k/2\), we have \(s = p = 1\) and \(r = q + 1\). Since \(\lambda = 2\) and \(0 \leq \mu \leq r - 1\), we can have \(C_{00}\) and \(C_{01}\) (see Figure 3.9). By applying a procedure similar to Case (1), we have two edge-disjoint paths of length \(k\), say \(P_{00}\) and \(P_{01}\) from \(C_{00}\) and \(C_{01}\) as follows (see Figure 6.7(a)):

\[
P_{00} = (C_{00} - x_0y_2a-3) \cup x_2a-2y_2a-3 \quad \text{and}
\]
\[
P_{01} = (C_{01} - x_2a-2y_2a-3) \cup x_0y_2a-3.
\]
3.1. \((k; p, q)\)–decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

Figure 3.9: \(C_{00} \oplus C_{01}\)

Figure 3.10: \(P_{00} \oplus P_{01}\)

Figure 3.11: \(C_{00} \oplus C_{01}\)
3.1. \((k; p, q)\)-decomposition of \(K_{m,n}(\lambda)\), when \(k\) and \(\lambda\) both even.

Let \(a = r + 1/2\) be a positive integer. Now we consider \(P_{00}\) and \(C_{0a}\) (see Figure 6.7(b)). Since \(x_{2a-1}y_{a-2} \in E(C_{a0})\), \(x_{a-1}y_{a-2} \in E(P_{00})\), and \(x_{a-1} \notin V(C_{a0})\) we have two edge-disjoint paths of length \(k\), say \(P_{0a}\) and \(\hat{P}_{00}\) from \(C_{0a}\) and \(P_{00}\) as follows (see Figure 3.12):

![Figure 3.12: \(P_{00} \oplus P_{01}\)](image)

By applying a procedure similar to Case (1), the remaining pairs of cycles \(C_{0\mu}\) and \(C_{0(\mu+1)}\), \(2 \leq \mu \neq a \leq r - 1\) decomposes into pairs of paths. Hence the graph \(K_{m,n}(2)\) has the desired decomposition.

\[\text{Theorem 3.4.} \text{ Let } p, q \text{ be nonnegative integers and } \{k, m, n, \lambda\} \in \mathbb{N} \text{ such that } k \equiv \lambda \equiv 0 \pmod{2}, m + n > k \geq 4, \text{ and } k \text{ divides } 2mn. \text{ If } m, n \geq k/2, k(p + q) = \lambda mn, \text{ and } p \neq 1 \text{ then the graph } K_{m,n}(\lambda) \text{ has a } (k; p, q)-\text{decomposition.}\]

\[\text{Proof.} \text{ When } \lambda \geq 2, \text{ we can write } K_{m,n}(\lambda) = (\lambda/2)K_{m,n}(2). \text{ By Lemma 3.3 and Remark 3.2, the graph } (\lambda/2)K_{m,n}(2) \text{ has a } (k; p, q)-\text{decomposition. Hence the graph } K_{m,n}(\lambda) \text{ has the desired decomposition.} \]
Remark 3.5.

1. Let \( k, m, n \) be positive even integers such that \( k \geq 4 \). If the graph \( K_{m,n}(\lambda) \) has a \((k; p, q)\)-decomposition, then for every positive integer \( x \), the graph \( K_{m,n}(x\lambda) \) has a \((k; p, q)\)-decomposition.

2. Let \( k, m, n \) be positive even integers such that \( k \geq 4 \). If the graph \( K_{m,n}(\lambda) \) has a \((k; p, q)\)-decomposition, then for all positive integers \( r \) and \( s \), the graph \( K_{rm,sn}(\lambda) \) has a \((k; p, q)\)-decomposition.

3. Let \( k, n_1, n_2, \ldots, n_m \) be positive even integers such that \( k \geq 4 \). If the graph \( K_{n_i,n_j}(\lambda) \), for \( 1 \leq i \neq j \leq m \) has a \((k; p, q)\)-decomposition, then the graph \( K_{n_1,n_2,\ldots,n_m}(\lambda) \) has a \((k; p, q)\)-decomposition.

### 3.2 \((k; p, q)\)-decomposition of \( K_{m,n}(\lambda) \), when \( \lambda \geq 3 \).

In this section, we investigate the existence of a \((k; p, q)\)-decomposition of \( K_{m,n}(\lambda) \), when \( \lambda \geq 3 \) and \( \lambda m \equiv \lambda n \equiv k \equiv 0 \) (mod 2).

**Theorem 3.6.** Let \( \{k, m, n, \lambda\} \in \mathbb{N} \) and \( i, j \) be nonnegative integers such that \( \lambda \geq 3 \), \( \lambda m \equiv \lambda n \equiv 0 \) (mod 2), and \( k \equiv 0 \) (mod 4). If \( K_{\frac{k}{2}+i,\frac{k}{2}+j}(\lambda) \), \( 0 \leq i, j \leq k/2 \) has a \((k; p, q)\)-decomposition, then the graph \( K_{m,n}(\lambda) \), where \( m, n \geq k \), has a \((k; p, q)\)-decomposition.

**Proof.** By the hypothesis, let \( m = tk + x \) and \( n = sk + y \), where \( t \) and \( s \) are positive integers, \( x \) and \( y \) are nonnegative integers such that \( 0 \leq x, y < k \).
When $x = y = 0$, we can write $K_{m,n}(\lambda) = K_{tk,sk}(\lambda) = \lambda tsK_{k,k}$. When $x = y = k/2$, we can write

$$K_{m,n}(\lambda) = K_{(t-1)k + \frac{m}{2}, (s-1)k + \frac{n}{2}}(\lambda)$$

$$= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,\frac{m}{2}}(\lambda) \oplus K_{\frac{m}{2}, (s-1)k}(\lambda) \oplus K_{\frac{m}{2}, \frac{n}{2}}(\lambda)$$

$$= ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{\frac{m}{2}, k} + (s-1)\lambda K_{\frac{n}{2}, k} \oplus \lambda K_{\frac{m}{2}, \frac{n}{2}}.$$

Since $k \equiv 0 \pmod{4}$, by Theorem 2.18 the graphs $K_{k,k}$, $K_{\frac{m}{2}, k}$, and $K_{\frac{n}{2}, k}$ have a $(k;p,q)$-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case (1).** $x = 0$ and $0 < y < k$

When $0 < y < k/2$, we can write $K_{m,n}(\lambda) = K_{tk,(s-1)k + \frac{y}{2} + \frac{y}{2} + y}(\lambda) = K_{tk,(s-1)k + \frac{y}{2}}(\lambda) \oplus K_{tk,y + \frac{y}{2}}(\lambda) = (t\lambda)K_{tk,(s-1)k + \frac{y}{2}} \oplus tK_{tk,y + \frac{y}{2}}(\lambda) = (t(s-1)\lambda)K_{k,k} \oplus (t\lambda)K_{\frac{m}{2}} \oplus tK_{k,y + \frac{y}{2}}(\lambda)$. By Theorem 2.18, the graphs $K_{k,k}$, $K_{\frac{m}{2}, k}$ both have a $(k;p,q)$-decomposition and by the hypothesis, the graph $K_{k,y + \frac{y}{2}}(\lambda)$ has a $(k;p,q)$-decomposition.

When $k/2 \leq y < k$, we can write $K_{m,n}(\lambda) = K_{tk,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) = (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda)$. By Theorem 2.18, the graph $K_{k,k}$ has a $(k;p,q)$-decomposition and by the hypothesis, the graph $K_{k,y}(\lambda)$ has a $(k;p,q)$-decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case (2).** $k/2 < x < k$ and $k/2 \leq y < k$

We can write

$$K_{m,n}(\lambda) = K_{tk+x,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{x,sk}(\lambda) \oplus K_{x,y}(\lambda)$$

$$= (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda) \oplus sK_{x,k}(\lambda) \oplus K_{x,y}(\lambda)$$
and $\lambda mn/k = \lambda (tk + x)(sk + y)/k = \lambda (tsk + sx + ty) + \lambda xy/k$. By Theorem 2.18, the graph $K_{k,k}$ has a $(k; p, q)$–decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{x,k}(\lambda)$ both have a $(k; p, q)$–decomposition. Since $k$ divides $\lambda mn$, we have $k$ divides $\lambda xy$ and also $k/2 \leq x, y < k$, then by the hypothesis, $K_{x,y}(\lambda)$ has a $(k; p, q)$–decomposition. Hence, by Remark 3.2, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case (3).** $0 < x, y \leq k/2$

We can write

$$K_{m,n}(\lambda) = K_{(t-1)k+(k+x), (s-1)k+(k+y)}(\lambda)$$

$$= K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,k+y}(\lambda) \oplus K_{k+x,(s-1)k}(\lambda) \oplus K_{k+x,k+y}(\lambda)$$

$$= (t-1)(s-1)K_{k,k}(\lambda) \oplus (t-1)K_{k,k+y}(\lambda) \oplus (s-1)K_{k+x,k}(\lambda)$$

$$\oplus K_{k/2,k+y}(\lambda) \oplus K_{k/2+k,y}(\lambda)$$

$$= \lambda(t-1)(s-1)K_{k,k} \oplus (t-1)K_{k,k/2}(\lambda) \oplus (t-1)K_{k,k/2+y}(\lambda)$$

$$\oplus (s-1)K_{k/2,k}(\lambda) \oplus (s-1)K_{k/2+k,y}(\lambda)$$

$$\oplus K_{k/2+k,2k}(\lambda) \oplus K_{k/2+k,y2}(\lambda),$$

and $\lambda mn/k = \lambda (tk + x)(sk + y)/k = \lambda k(t-1)(s-1) + \lambda(t-1)(k+y) + \lambda(k+x)(s-1) + \lambda(k+x+y) + (\lambda xy)/k$. By Theorem 2.18, the graphs $K_{k,k}$ and $K_{k/2,k}$ both have a $(k; p, q)$–decomposition and by the hypothesis, the graphs $K_{k,k/2+y}(\lambda)$, $K_{k/2+k,y}(\lambda)$, both have a $(k; p, q)$–decomposition. Since $k$ divides $\lambda mn$ and $k \equiv 0 \pmod{4}$, we have $k$ divides $\lambda(k/2 + x)(k/2 + y)$, $2$ divides $\lambda x$, and $2$ divides $\lambda y$ and $k/2 \leq (k/2 + x), (k/2 + y) \leq k$. Then by the hypothesis, the graphs $K_{k/2+k,y}(\lambda)$, $K_{k/2+k/2+y}(\lambda)$, and $K_{k/2+k/2+y}(\lambda)$ have a $(k; p, q)$–decomposition. The graph $K_{k/2,k+y}(\lambda)$ can be viewed as
3.2. $(k; p, q)$—decomposition of $K_{m,n}(\lambda)$, when $\lambda \geq 3$.  

$K_{k/2,k/2}(\lambda) \oplus K_{k/2,k/2+y}(\lambda) = \lambda K_{k/2,k/2} \oplus K_{k/2,k/2+y}(\lambda)$. By Theorem 2.2, the graph $K_{k/2,k/2}$ has a $C_k$—decomposition and by the hypothesis, the graph $K_{k/2,k/2+y}(\lambda)$ has a $(k; p, q)$—decomposition. Now for any pair of cycles of length $k$, one from the graph $\lambda K_{k/2,k/2}$, say $C_\alpha$ and the other from the graph $K_{k/2,k/2+y}(\lambda)$, say $C_\beta$, we have a common vertex in $C_\alpha \oplus C_\beta$, say $v$, such that at least one neighbor of $v$ from each cycle does not belong to the other cycle. Then by the Construction 2.4 we have two edge-disjoint paths of length $k$ from $C_\alpha$ and $C_\beta$, say $C_{\alpha'}$ and $C_{\beta'}$, respectively. 

By applying a similar procedure to the remaining pairs of cycles, we have edge-disjoint pairs of paths. Hence the graph $K_{k/2,k+y}(\lambda)$ has a $(k; p, q)$—decomposition. Therefore, by Remark 3.2, the graph $K_{m,n}(\lambda)$ has the desired decomposition. 

Case (4). $0 < x \leq k/2$ and $k/2 < y < k$

We can write

$$K_{m,n}(\lambda) = K_{(t-1)k+(k+x),sk+y}(\lambda)$$

$$= K_{(t-1)k,sk}(\lambda) \oplus K_{(t-1)k,y}(\lambda) \oplus K_{k+x,sk}(\lambda) \oplus K_{k+x,y}(\lambda)$$

$$= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k+x,k}(\lambda) \oplus K_{k+x,y}(\lambda)$$

$$= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k/2,k}(\lambda) \oplus sK_{k/2+x,k}(\lambda)$$

$$\oplus K_{k/2,y}(\lambda) \oplus K_{k/2+x,y}(\lambda),$$

and $\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda((t-1)sk + (t-1)y + sk/2 + s(k/2 + x)) + \lambda(k+x)y/k$. By Theorem 2.18, the graphs $K_{k,k}$ and $K_{k/2,k}$ both have a $(k; p, q)$—decomposition. Since $k$ divides $\lambda mn$, we have 2 divides $\lambda y$, $k$ divides $xy\lambda$ and also $k/2 \leq (k/2 + x), y \leq k$, then by the hypothesis, the graphs $K_{k,y}(\lambda)$, $K_{k/2,x,k}(\lambda)$, and $K_{k/2+x,y}(\lambda)$ have a
(k; p, q)−decomposition. Hence, by Remark 3.2, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

\[\square\]

**Theorem 3.7.** Let \(\{k, m, n, \lambda\} \in \mathbb{N}\) and \(i, j\) be nonnegative integers such that \(\lambda \geq 3, \lambda m \equiv \lambda n \equiv 0 \mod 2,\) and \(k \equiv 2 \mod 4\). If $K_{\frac{k}{2}+i, \frac{k}{2}+j}(\lambda), 0 \leq i, j \leq k$ has a $(k; p, q)$−decomposition, then the graph $K_{m,n}(\lambda)$, where $m, n \geq 3k/2$, has a $(k; p, q)$−decomposition.

**Proof.** By the hypothesis, let \(m = tk + x\) and \(n = sk + y\), where \(t\) and \(s\) are positive integers, \(x\) and \(y\) are nonnegative integers such that \(0 \leq x, y < k\).

When \(x = y = k/2\), we can write

$$ K_{m,n}(\lambda) = K_{(t-1)k + \frac{k}{2}, (s-1)k + \frac{k}{2}}(\lambda) $$

$$ = K_{(t-1)k, (s-1)k}(\lambda) \oplus K_{\frac{k}{2}, (s-1)k}(\lambda) \oplus K_{\frac{k}{2}, \frac{k}{2}}(\lambda) $$

$$ = ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)\lambda K_{k, \frac{k}{2}} \oplus (s-1)\lambda K_{\frac{k}{2}, k} \oplus \lambda K_{\frac{k}{2}, \frac{k}{2}}. $$

By Theorem 2.18, the graph $K_{k,k}$, has a $(k; p, q)$−decomposition and by the hypothesis, the graphs $K_{k, \frac{k}{2}}$, and $K_{\frac{k}{2}, \frac{k}{2}}$ both have a $(k; p, q)$−decomposition. Hence the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case (1).** \(0 \leq x, y < k/2\)

When \(0 \leq x, y < k/2\), we have \(t, s \geq 2\). We can write

$$ K_{m,n}(\lambda) = K_{(t-1)k+(k+x),(s-1)k+(k+y)}(\lambda) $$

$$ = K_{(t-1)k,(s-1)k}(\lambda) \oplus K_{(t-1)k,k+y}(\lambda) \oplus K_{k+x,(s-1)k}(\lambda) \oplus K_{k+x,k+y}(\lambda) $$

$$ = ((t-1)(s-1)\lambda)K_{k,k} \oplus (t-1)K_{k,k+y}(\lambda) \oplus (s-1)K_{k,k+x}(\lambda) \oplus K_{k+x,k+y}(\lambda), $$
and $\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda((t - 1)(s - 1)k + (s - 1)(k + x) + (t - 1)(k + y)) + \lambda(k + x)(k + y)/k$. By Theorem 2.18, the graph $K_{k,k}$ has a $(k;p,q)$--decomposition and by the hypothesis, the graphs $K_{k,k+y}(\lambda)$ and $K_{k+x,k}(\lambda)$ both have a $(k;p,q)$--decomposition. Since $k$ divides $\lambda mn$, we have $k$ divides $\lambda(k+x)(k+y)$ and also $k/2 \leq (k+x), (k+y) \leq 3k/2$, then by the hypothesis, the graph $K_{k+x,k+y}(\lambda)$ has a $(k;p,q)$--decomposition. Hence, by Remark 3.2, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case (2).** $k/2 \leq x < k$ and $k/2 < y < k$

We can write $K_{m,n}(\lambda) = K_{tk+x,sk+y}(\lambda) = K_{tk,sk}(\lambda) \oplus K_{tk,y}(\lambda) \oplus K_{x,sk}(\lambda) \oplus K_{x,y}(\lambda) = (ts\lambda)K_{k,k} \oplus tK_{k,y}(\lambda) \oplus sK_{x,k}(\lambda) \oplus K_{x,y}(\lambda)$, and $\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda(ts k + sk + ty) + \lambda xy/k$. By Theorem 2.18, the graph $K_{k,k}$ has a $(k;p,q)$--decomposition and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{x,k}(\lambda)$ both have a $(k;p,q)$--decomposition. Since $k$ divides $\lambda mn$, we have $k$ divides $\lambda xy$ and also $k/2 \leq x, y < k$, then by the hypothesis, the graph $K_{x,y}(\lambda)$ has a $(k;p,q)$--decomposition. Hence, by Remark 3.2, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

**Case (3).** $0 \leq x < k/2$ and $k/2 \leq y < k$

When $0 \leq x < k/2$ and $k/2 \leq y < k$, we have $t \geq 2$ and $s \geq 1$. We can write

\[
K_{m,n}(\lambda) = K_{(t-1)k+(k+x),sk+y}(\lambda) \\
= K_{(t-1)k,sk}(\lambda) \oplus K_{(t-1)k+y}(\lambda) \oplus K_{k+x,sk}(\lambda) \oplus K_{k+x,y}(\lambda) \\
= ((t-1)s\lambda)K_{k,k} \oplus (t-1)K_{k,y}(\lambda) \oplus sK_{k+x,k}(\lambda) \oplus K_{k+x,y}(\lambda),
\]

and $\lambda mn/k = \lambda(tk + x)(sk + y)/k = \lambda((t-1)sk + s(k + x) + (t-1)y) + \lambda(k + x)y/k$. By Theorem 2.18, the graph $K_{k,k}$ has a $(k;p,q)$--decomposition
and by the hypothesis, the graphs $K_{k,y}(\lambda)$ and $K_{k+x,k}(\lambda)$ both have a $(k;p,q)$-decomposition. Since $k$ divides $\lambda mn$, we have $k$ divides $\lambda(k + x)y$ and also $k/2 \leq (k + x), y \leq 3k/2$, then by the hypothesis, the graph $K_{k+x,y}(\lambda)$ has a $(k;p,q)$-decomposition. Hence, by Remark 3.2, the graph $K_{m,n}(\lambda)$ has the desired decomposition.

Note: The results of this chapter has been communicated (revised version) to *Discussiones Mathematicae Graph Theory*.