CHAPTER-6

FUZZY ROUGH $BG$-BOUNDARY SPACES

The concepts of rough groups and rough subgroups were introduced by R. Biswas and S. Nanda [14]. The concepts of fuzzy groups and fuzzy topological groups were introduced and studied by D. Foster [27]. The concept of boundary of a fuzzy set was introduced by R. H. Warren [81]. In this chapter, the concepts of fuzzy rough topological groups and fuzzy rough $G$ structure spaces are introduced and studied. In this connection, the concept of fuzzy rough $BG$ boundary space is introduced. Interesting characterization is established.

6.1 FUZZY ROUGH $G$ STRUCTURE SPACE AND FUZZY ROUGH $BG$-BOUNDARY

In this section, the concepts of fuzzy rough topological groups, fuzzy rough $G$ structure spaces and fuzzy rough $G$ boundary are introduced and studied.

Definition 6.1.1. Let $X$ be a rough group. A fuzzy rough set $G = (G_L, G_U)$ on $X$ is said to be a fuzzy rough group if and only if it satisfies the following
conditions:

(i) \( G_L(x, y) \geq \min\{G_L(x), G_L(y)\} \) and \( G_U(x, y) \geq \min\{G_U(x), G_U(y)\} \) for all \( x, y \in X \).

(ii) \( G_L(x^{-1}) \geq G_L(x) \) and \( G_U(x^{-1}) \geq G_U(x) \) for all \( x \in X \).

**Definition 6.1.2.** Let \( A \) be a fuzzy rough set in \( X \) and \( T \) be a fuzzy rough topology on \( X \). Then the fuzzy rough subspace topology on \( A \) is the family of fuzzy rough subsets of \( A \) which are the intersections with \( A \) of fuzzy rough open sets in \( X \). The fuzzy rough subspace topology is denoted by \( T_A \), and the pair \((A, T_A)\) is called a fuzzy rough subspace of \((X, T)\).

**Definition 6.1.3.** Let \((A, T_A)\) and \((B, S_B)\) be any two fuzzy rough subspaces of fuzzy rough topological spaces \((X, T)\), \((Y, S)\) respectively. A function \( f : (A, T_A) \to (B, S_B) \) is said to be a relatively fuzzy rough continuous function if and only if for each fuzzy rough open \( \hat{V} = V \cap B \) in \( S_B \), the intersection \( f^{-1}(\hat{V}) \cap A \) is fuzzy rough open in \( T_A \).

**Definition 6.1.4.** Let \( X \) be a rough set. Then \( X \) is said to be a rough group if \( X_L \) and \( X_U \) are groups.

**Definition 6.1.5.** Let \( X \) be a rough group and \( T \) be a fuzzy rough topology on \( X \). Let \( G \) be any fuzzy rough group in \( X \) and let \( G \) be endowed with the fuzzy rough subspace topology \( T_G \). Then \( G \) is a fuzzy rough topological group in \( X \) if and only if it satisfies the following two conditions:
(i) The mapping $\alpha : (x, y) \rightarrow xy$ of $(G, T_G) \times (G, T_G)$ into $(G, T_G)$ is relatively fuzzy rough continuous.

(ii) The mapping $\beta : x \rightarrow x^{-1}$ of $(G, T_G)$ into $(G, T_G)$ is relatively fuzzy rough continuous.

**Definition 6.1.6.** A non empty set $X$ is a family $\mathcal{G}$ of fuzzy rough topological groups in $X$ satisfies the following conditions:

(i) $\tilde{0}, \tilde{1} \in \mathcal{G}$.

(ii) If $A, B \in \mathcal{G}$, then $A \cap B \in \mathcal{G}$.

(iii) If $A_j \in \mathcal{G}$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \mathcal{G}$.

then $\mathcal{G}$ is said to be a fuzzy rough topological group structure on $X$ and the pair $(X, \mathcal{G})$ is said to be a fuzzy rough topological group (in short, fuzzy rough $\mathcal{G}$) structure space. Any member of fuzzy rough $\mathcal{G}$ structure space is called a fuzzy rough open group. The complement of fuzzy rough open group is a fuzzy rough closed group.

**Definition 6.1.7.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A = (A_L, A_U)$ be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{G}$ interior of $A$ is defined by

$$FR\mathcal{G}int(A) = \bigcup\{B : B \text{ is a fuzzy rough open group and } B \subseteq A\}.$$
**Definition 6.1.8.** Let \((X, \mathcal{G})\) be a fuzzy rough \(\mathcal{G}\) structure space. Let \(A = (A_L, A_U)\) be any fuzzy rough topological group. Then the fuzzy rough \(\mathcal{G}\) closure of \(A\) is defined by

\[
FR\mathcal{G}cl(A) = \cap\{B : B \text{ is a fuzzy rough closed group and } B \supseteq A\}.
\]

**Definition 6.1.9.** Let \((X, \mathcal{G})\) be a fuzzy rough \(\mathcal{G}\) structure space. Let \(A\) be any fuzzy rough topological group. Then \(A\) is said to be a fuzzy rough \(t\)-open group if \(FR\mathcal{G}int(A) = FR\mathcal{G}int(FR\mathcal{G}cl(A))\)

**Definition 6.1.10.** Let \((X, \mathcal{G})\) be a fuzzy rough \(\mathcal{G}\) structure space. Let \(A\) be any fuzzy rough topological group. Then \(A\) is said to be a fuzzy rough \(B\)-open group if \(A = B \cap C\) where \(B\) is a fuzzy rough open group and \(C\) is a fuzzy rough \(t\)-open group. The complement of fuzzy rough \(B\)-open group is a fuzzy rough \(B\)-closed group.

**Definition 6.1.11.** Let \((X, \mathcal{G})\) be a fuzzy rough \(\mathcal{G}\) structure space. Let \(A = (A_L, A_U)\) be any fuzzy rough topological group. Then the fuzzy rough \(B\mathcal{G}\) interior of \(A\) is defined by

\[
FRB\mathcal{G}int(A) = \cup\{B : B \text{ is a fuzzy rough } B\text{-open group in } X \text{ and } B \subseteq A\}.
\]

**Definition 6.1.12.** Let \((X, \mathcal{G})\) be a fuzzy rough \(\mathcal{G}\) structure space. Let \(A = (A_L, A_U)\) be any fuzzy rough topological group. Then the fuzzy rough \(B\mathcal{G}\) closure of \(A\) is defined by

\[
FRB\mathcal{G}cl(A) = \cap\{B : B \text{ is a fuzzy rough } B\text{-closed group in } X \text{ and } B \supseteq A\}.
\]
**Proposition 6.1.1.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A$ be any fuzzy rough topological group. Then the following conditions hold:

(i) $F RB\mathcal{G}int(A) \subseteq A \subseteq F RB\mathcal{G}cl(A)$.

(ii) $(F RB\mathcal{G}int(A))' = F RB\mathcal{G}cl(A')$.

(iii) $(F RB\mathcal{G}cl(A))' = F RB\mathcal{G}int(A')$.

**Proof.** The proof follows from Definition 6.1.11. and Definition 6.1.12.

**Definition 6.1.13.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A$ be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{G}$-boundary of $A$, is denoted and defined as

$$FR\mathcal{G}bd(A) = FR\mathcal{G}cl(A) \cap FR\mathcal{G}cl(A').$$

**Definition 6.1.14.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A$ be any fuzzy rough topological group. Then the fuzzy rough $\mathcal{B}\mathcal{G}$-boundary of $A$, is denoted and defined as

$$FR\mathcal{B}\mathcal{G}bd(A) = FR\mathcal{B}\mathcal{G}cl(A) \cap FR\mathcal{B}\mathcal{G}cl(A').$$

**Proposition 6.1.2.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $A$ and $B$ be any two fuzzy rough topological groups. Then the following conditions hold:

(i) $F RB\mathcal{G}bd(A) = F RB\mathcal{G}bd(A')$. 

141
(ii) If \(A\) is a fuzzy rough closed group, then \(FR\overline{BG}bd(A) \subseteq A\).

(iii) If \(A\) is a fuzzy rough open group, then \(FR\overline{BG}bd(A) \subseteq A'\).

(iv) Let \(A \subseteq B\) and \(B\) be any fuzzy rough closed group (resp., \(A\) be any fuzzy rough open group). Then \(FR\overline{BG}bd(A) \subseteq B\) (resp., \(FR\overline{BG}bd(A) \subseteq B'\)).

(v) \((FR\overline{BG}bd(A))' = FR\overline{BG}int(A) \cup FR\overline{BG}int(A')\).

**Proof.** (i) \[ FR\overline{BG}bd(A) = FR\overline{BG}cl(A) \cap FR\overline{BG}cl(A') \]
\[ = FR\overline{BG}cl(A') \cap FR\overline{BG}cl(A) \]
\[ = FR\overline{BG}cl(A') \cap FR\overline{BG}cl(A')' \]
\[ = FR\overline{BG}bd(A'). \]

(ii) \[ FR\overline{BG}bd(A) = FR\overline{BG}cl(A) \cap FR\overline{BG}cl(A') \]
\[ \subseteq FR\overline{BG}cl(A) \]
\[ \subseteq A. \]

Hence, \(FR\overline{BG}bd(A) \subseteq A\).

(iii) Let \(A\) be any fuzzy rough \(B\)-open group. Then, \(A'\) is fuzzy rough \(B\)-closed group. By (ii), \(FR\overline{BG}bd(A') \subseteq A'\) and by (i), \(FR\overline{BG}bd(A) \subseteq A'\).

(iv) Since \(A \subseteq B\) implies that \(FR\overline{BG}cl(A) \subseteq FR\overline{BG}cl(B)\), we have
\[ FR\overline{BG}bd(A) = FR\overline{BG}cl(A) \cap FR\overline{BG}cl(A') \]
\[ \subseteq FR\overline{BG}cl(B) \cap FR\overline{BG}cl(A') \]
\[ \subseteq FR\overline{BG}cl(B) \]
\[ = B, \text{ since } B \text{ is a } B\text{-closed group.} \]

142
\[(v) \quad (FRBG_{bd}(A))' = (FRBG_{cl}(A) \cap FRBG_{cl}(A'))'
= (FRBG_{cl}(A))' \cup (FRBG_{cl}(A'))'
= FRBG_{int}(A') \cup FRBG_{int}(A).\]

**Definition 6.1.15.** Let \(A\) and \(B\) be any two fuzzy rough topological groups. Then \(A - B\) is defined by \(A - B = A \cap B'\).

**Proposition 6.1.3.** Let \((X, \mathcal{G})\) be a fuzzy rough \(\mathcal{G}\) structure space. Let \(A\) be any fuzzy rough topological group. Then the following conditions hold:

(i) \(FRBG_{bd}(A) = FRBG_{cl}(A) - FRBG_{int}(A)\).

(ii) \(FRBG_{bd}(FRBG_{int}(A)) \subseteq FRBG_{bd}(A)\).

(iii) \(FRBG_{bd}(FRBG_{cl}(A)) \subseteq FRBG_{bd}(A)\).

(iv) \(FRBG_{int}(A) \subset A - FRBG_{bd}(A)\).

**Proof.** (i) Since \((FRBG_{cl}(A'))' = FRBG_{int}(A)\). Therefore,

\[
FRBG_{bd}(A) = FRBG_{cl}(A) \cap FRBG_{cl}(A')
= FRBG_{cl}(A) - (FRBG_{cl}(A'))'
= FRBG_{cl}(A) - FRBG_{int}(A).
\]

Thus, \(FRBG_{bd}(A) = FRBG_{cl}(A) - FRBG_{int}(A)\). Hence (i).

(ii) \(FRBG_{bd}(FRBG_{int}(A)) = FRBG_{cl}(FRBG_{int}(A)) - FRBG_{int}(FRBG_{int}(A))
= FRBG_{cl}(FRBG_{int}(A)) - FRBG_{int}(A)\)
⊆ \( FRBG_{cl}(A) - FRBG_{int}(A) \)

\[ = FRBG_{bd}(A). \]

**(iii)** \( FRBG_{bd}(FRBG_{cl}(A)) \)

\[ = FRBG_{cl}(FRBG_{cl}(A)) - FRBG_{int}(FRBG_{cl}(A)) \]

\[ = FRBG_{cl}(A) - FRBG_{int}(FRBG_{cl}(A)) \]

\[ ⊆ FRBG_{cl}(A) - FRBG_{int}(A) \]

\[ = FRBG_{bd}(A). \]

**(iv)** \( A - FRBG_{bd}(A) = A \cap (FRBG_{bd}(A))' \)

\[ = A \cap (FRBG_{cl}(A) \cap FRBG_{cl}(A'))' \]

\[ = A \cap (FRBG_{int}(A') \cup FRBG_{int}(A)) \]

\[ = (A \cap FRBG_{int}(A')) \cup (A \cap FRBG_{int}(A)) \]

\[ = (A \cap FRBG_{int}(A')) \cup FRBG_{int}(A) \]

\[ ⊇ FRBG_{int}(A). \]

**Proposition 6.1.4.** Let \((X, \mathcal{G})\) be a fuzzy rough \(\mathcal{G}\) structure space. Let \(A\) and \(B\) be any two fuzzy rough topological groups. Then, \(FRBG_{bd}(A \cup B) \subseteq FRBG_{bd}(A) \cup FRBG_{bd}(B)\).

**Proof.** \(FRBG_{bd}(A \cup B) = FRBG_{cl}(A \cup B) \cap FRBG_{cl}(A \cup B)'\)

\[ \subseteq (FRBG_{cl}(A) \cup FRBG_{cl}(B)) \cap (FRBG_{cl}(A') \cap FRBG_{cl}(B')) \]

\[ = [FRBG_{cl}(A) \cap (FRBG_{cl}(A') \cap FRBG_{cl}(B'))] \cup \]

\[ [FRBG_{cl}(B) \cap (FRBG_{cl}(A') \cap FRBG_{cl}(B'))] \]

\[ = (FRBG_{bd}(A) \cap FRBG_{cl}(B')) \cup (FRBG_{bd}(B) \cap FRBG_{cl}(A')) \]

\[ \subseteq FRBG_{bd}(A) \cup FRBG_{bd}(B). \]
Proposition 6.1.5. Let \((X, G)\) be a fuzzy rough \(G\) structure space. Let \(A\) and \(B\) be any two fuzzy rough topological groups. Then, 

\[ FRBGbd(A \cap B) \subseteq FRBGbd(A) \cup FRBGbd(B). \]

**Proof.** 

\[
FRBGbd(A \cap B) \\
= FRBGcl(A \cap B) \cap FRBGcl(A \cap B)' \\
\subseteq (FRBGcl(A) \cap FRBGcl(B)) \cap (FRBGcl(A') \cup FRBGcl(B')) \\
= [(FRBGcl(A) \cap FRBGcl(B)) \cap FRBGcl(A')] \cup \\
[(FRBGcl(A) \cap FRBGcl(B)) \cap FRBGcl(B')]] \\
= (FRBGbd(A) \cap FRBGcl(B)) \cup (FRBGbd(B) \cap FRBGcl(A)) \\
\subseteq FRBGbd(A) \cup FRBGbd(B).
\]

Proposition 6.1.6. Let \((X, G)\) be a fuzzy rough \(G\) structure space. Let \(A\) be any fuzzy rough topological group. Then the following conditions hold:

(i) \(FRBGbd(FRBGbd(A)) \subseteq FRBGbd(A)\).

(ii) \(FRBGbd(FRBGbd(FRBGbd(A))) \subseteq FRBGbd(FRBGbd(A))\).

**Proof.**

(i) \(FRBGbd(FRBGbd(A))\)

\[
= FRBGcl(FRBGbd(A)) \cap FRBGcl(FRBGbd(A))' \\
\subseteq FRBGcl(FRBGbd(A)) \\
= FRBGbd(A).
\]
Definition 6.1.16. Let \( A = (A_L, A_U) \) be a fuzzy rough topological group of \( X \) and \( B = (B_L, B_U) \) be a fuzzy rough topological group of \( Y \), then the fuzzy rough topological group \( A \times B = (A_L \times B_L, A_U \times B_U) \) of \( X \times Y \) is defined by

\[
(A_L \times B_L)(x, y) = \min\{A_L(x), B_L(y)\} \quad \text{for every} \quad (x, y) \in X_L \times Y_L
\]

\[
(A_U \times B_U)(x, y) = \min\{A_U(x), B_U(y)\} \quad \text{for every} \quad (x, y) \in X_U \times Y_U.
\]

Note 6.1.1. Let \( A \) and \( B \) be any two fuzzy rough topological groups in \( X \) and \( Y \) then, \( (A \times B)' = (1_L - A_L \times B_L, 1_U - A_U \times B_U) \).

Proposition 6.1.7. If \( A = (A_L, A_U) \) is a fuzzy rough topological group of \( X \) and \( B = (B_L, B_U) \) is a fuzzy rough topological group of \( Y \), then \( (A \times B)' = A' \times 1 \cup 1 \times B' \).

Proof. Since,

\[
A_L \times B_L(x, y) = \min(A_L(x), B_L(y)), \quad \text{for every} \quad (x, y) \in X_L \times Y_L
\]

\[
1_L - A_L \times B_L(x, y) = \max(1 - A_L(x), 1 - B_L(y))
\]

\[
= \max(A'_U(x), B'_U(y))
\]

\[
= \max((A'_U \times 1_U)(x, y), (1_U \times B'_U)(x, y))
\]

\[
1_L - A_L \times B_L = A'_U \times 1_U \cup 1_U \times B'_U
\]
and similarly $1_U - A_U \times B_U = A'_L \times 1_L \cup 1_L \times B'_L$. This implies that,

$$(A \times B)' = A' \times \bar{1} \cup \bar{1} \times B'.$$

**Definition 6.1.17.** Let $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ be any two fuzzy rough structure spaces. The fuzzy rough product $\mathcal{G}$ structure space of $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ is the cartesian product $(X, \mathcal{G}_1) \times (Y, \mathcal{G}_2)$ of sets $(X, \mathcal{G}_1)$ and $(Y, \mathcal{G}_2)$ together with the fuzzy rough structure $\mathcal{G}_1 \times \mathcal{G}_2$ generated by the family $\{ p_1^{-1}(A) , p_2^{-1}(B) | A \in \mathcal{G}_1, B \in \mathcal{G}_2, \text{ where } p_1 \text{ and } p_2 \text{ are projections of } (X, \mathcal{G}_1) \times (Y, \mathcal{G}_2) \text{ onto } (X, \mathcal{G}_1) \text{ and } (Y, \mathcal{G}_2), \text{ respectively}\}$. 

**Proposition 6.1.8.** Let $A = (A_L, A_U)$ be a fuzzy rough $B$-closed group of a fuzzy rough $\mathcal{G}_1$ structure space $X$ and $B = (B_L, B_U)$ be a fuzzy rough $B$-closed group of a fuzzy rough $\mathcal{G}_2$ structure space $Y$. Then $A \times B$ is a fuzzy rough $B$-closed group of the fuzzy rough product $\mathcal{G}$ structure space $X \times Y$.

**Proof.** Let $A$ and $B$ be any fuzzy rough topological groups in $X$ and $Y$. By Proposition 6.1.7, $\bar{1} - A \times B = A' \times \bar{1} \cup \bar{1} \times B'$. Since $A' \times \bar{1}$ and $\bar{1} \times B'$ are fuzzy rough $B$-open groups in $X$ and $Y$ respectively. $A' \times \bar{1} \cup \bar{1} \times B'$ is a fuzzy rough $B$-open group of $X \times Y$. Hence, $\bar{1} - A \times B$ is a fuzzy rough $B$-open group of $X \times Y$. Consequently, $A \times B$ is a fuzzy rough $B$-closed group of $X \times Y$.

**Proposition 6.1.9.** If $A = (A_L, A_U)$ is a fuzzy rough topological group of a fuzzy rough $\mathcal{G}_1$ structure space $X$ and $B = (B_L, B_U)$ is a fuzzy rough topological group of a fuzzy rough $\mathcal{G}_2$ structure space $Y$, then $FRBG_1cl(A) \times FRBG_2cl(B)$. 

147
\[ FRBG_2 cl(B) \supseteq FRBG cl(A \times B). \] Also, we have \[ FRBG_1 int(A) \times FRBG_2 int(B) \supseteq FRBG int(A \times B). \]

**Proof.** Since \( A \subseteq FRBG_1 cl(A) \) and \( B \subseteq FRBG_2 cl(B) \), \( A \times B \subseteq FRBG_1 cl(A) \times FRBG_2 cl(B) \).

By Proposition 6.1.8, \( FRBG cl(A \times B) \subseteq FRBG cl(FRBG_1 cl(A) \times FRBG_2 cl(B)) \).

**Definition 6.1.18.** A fuzzy rough \( \mathcal{G}_1 \) structure space \((X, \mathcal{G}_1)\) is fuzzy rough \( \mathcal{B} \)-product related to another fuzzy rough \( \mathcal{G}_2 \) structure space \((Y, \mathcal{G}_2)\) if for any fuzzy rough topological group \( C = (C_L, C_U) \) of \( X \) and \( D = (D_L, D_U) \) of \( Y \) whenever \( A' \not\subseteq C \) and \( B' \not\subseteq D \) implies that \((A' \times \bar{1}) \cup (\bar{1} \times B') \supseteq C \times D\), where \( A = (A_L, A_U) \) is a fuzzy rough \( \mathcal{B} \)-open group of \( X \) and \( B = (B_L, B_U) \) is a fuzzy rough \( \mathcal{B} \)-open group of \( Y \), there exist \( A_1 \in \mathcal{G}_1 \) and \( B_1 \in \mathcal{G}_2 \) such that \( A_1' \supseteq C \) or \( B_1' \supseteq D \) and \((A' \times \bar{1}) \cup (\bar{1} \times B') = (A_1' \times \bar{1}) \cup (\bar{1} \times B_1')\).

**Proposition 6.1.10.** Let \((X, \mathcal{G}_1)\) and \((Y, \mathcal{G}_2)\) be any two fuzzy rough structure spaces such that \((X, \mathcal{G}_1)\) is \( \mathcal{B} \)-product related to \((Y, \mathcal{G}_2)\). Then, for a fuzzy rough topological group \( A = (A_L, A_U) \) of \( X \) and a fuzzy rough topological group \( B = (B_L, B_U) \) of \( Y \),

(i) \[ FRBG cl(A \times B) = FRBG_1 cl(A) \times FRBG_2 cl(B), \]

(ii) \[ FRBG int(A \times B) = FRBG_1 int(A) \times FRBG_2 int(B). \]

**Proof.** (i) For fuzzy rough topological groups \( A_i = (A_{L_i}, A_{U_i}) \)'s of \( X \) and \( B_j = (B_{L_j}, B_{U_j}) \)'s of \( Y \), we first note that,
(i) \( \inf \{ A_i, B_j \} = \min(\inf(A_i), \inf(B_j)) \), 

(ii) \( \inf \{ A_i \times \tilde{1} \} = \inf(A_i) \times \tilde{1} \), 

(iii) \( \inf \{ \tilde{1} \times B_j \} = \tilde{1} \times \inf(B_j) \).

By Proposition 6.1.9, it follows that 

\[ FRBG_{1cl}(A) \times FRBG_{2cl}(B) \supseteq FRBG_{cl}(A \times B). \tag{6.1.1} \]

It is sufficient to show that 

\[ FRBG_{cl}(A \times B) \supseteq FRBG_{1cl}(A) \times FRBG_{2cl}(B). \]

Let \( A_i \) be a fuzzy rough \( B \)-open group in \( G_1 \) and \( B_j \) be a fuzzy rough \( B \)-open group in \( G_2 \). Then, 

\[
FRBG_{cl}(A \times B) = \inf \{ (A_i \times B_j)' \mid (A_i \times B_j)' \supseteq A \times B \} \\
= \inf \{ A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \mid A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \supseteq A \times B \} \\
= \inf \{ A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \mid A'_i \supseteq A \text{ or } B'_j \supseteq B \} \\
= \min(\inf \{ A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \mid A'_i \supseteq A \}, \inf \{ A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \mid B'_j \supseteq B \}).
\]

Since 

\[
\inf \{ A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \mid A'_i \supseteq A \} \supseteq \inf \{ A'_i \times \tilde{1} \mid A'_i \supseteq A \} \\
= \inf \{ A'_i \mid A'_i \supseteq A \} \times \tilde{1} \\
= FRBG_{1cl}(A) \times \tilde{1}
\]

and 

\[
\inf \{ A'_i \times \tilde{1} \cup \tilde{1} \times B'_j \mid B'_j \supseteq B \} \supseteq \inf \{ \tilde{1} \times B'_j \mid B'_j \supseteq B \}
\]
\[= \hat{1} \times \inf \{B' \mid B' \supseteq B\} \]
\[= \hat{1} \times FRBG_{2cl}(B).\]

We have, \(FRBG_{cl}(A \times B) \supseteq \min(FRBG_{1cl}(A') \times \hat{1}, \hat{1} \times FRBG_{2cl}(B')) = FRBG_{1cl}(A) \times FRBG_{2cl}(B).\)

\[FRBG_{cl}(A \times B) \supseteq FRBG_{1cl}(A) \times FRBG_{2cl}(B) \quad (6.1.2)\]

From (6.1.1) and (6.1.2),

\[FRBG_{cl}(A \times B) = FRBG_{1cl}(A) \times FRBG_{2cl}(B).\]

(ii) The proof is similar to that of (i) and the Proposition 6.1.9.

**Proposition 6.1.11.** Let \(A, B, C\) and \(D\) be fuzzy rough topological groups in \(X\). Then \((A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)\).

**Proof.**

\[((A_L \cap B_L) \times (C_L \cap D_L))(x,y) = min((A_L \cap B_L)(x), (C \cap D)(y))\]
\[= min(min(A_L(x), B_L(x)), min(C_L(y), D_L(y)))\]
\[= min(min(A_L(x), D_L(y)), min(B_L(x), C_L(y)))\]
\[= min((A_L \times D_L)(x,y), (B_L \times C_L)(x,y))\]
\[= ((A_L \times D_L) \cap (B_L \times C_L))(x,y)\]

for all \((x,y) \in X_L \times X_L\).

Similarly,

\[((A_U \cap B_U) \times (C_U \cap D_U))(x,y) = ((A_U \times D_U) \cap (B_U \times C_U))(x,y) \text{ for all } (x,y) \in X_U \times X_U.\]
Hence, \((A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C)\).

**Proposition 6.1.12.** Let \((X, \mathcal{G}_i) (i=1,2,\ldots,n)\) be a family of fuzzy rough product related structures spaces. If each \(A_i\) is a fuzzy rough topological groups in \(X_i\), then

\[
FRBG_{i, bd}\prod_{i=1}^{n}(A_i) = [FRBG_{1, bd}A_1 \times F RBG_{2, cl}(A_2) \times \ldots \times F RBG_{n, cl}A_n]]
\]

\[
\cup [FRBG_{1, cl}A_1 \times F RBG_{2, bd}(A_2) \times \ldots \times F RBG_{n, cl}A_n]]
\]

\[
\cup \ldots \cup [FRBG_{1, cl}A_1 \times F RBG_{2, cl}(A_2) \times \ldots \times F RBG_{n, bd}A_n)].
\]

**Proof.** We use Propositions 6.1.3 (i), 6.1.10 and 6.1.11 to prove this. It suffices to prove this for \(n=2\). Consider

\[
FRBG_{n, bd}(A_1 \times A_2)
\]

\[
= FRBG_{n, cl}(A_1 \times A_2) - FRBG_{n, int}(A_1 \times A_2)
\]

\[
= (FRBG_{1, cl}(A_1) \times F RBG_{2, cl}(A_2)) - (FRBG_{1, int}(A_1) \times F RBG_{2, int}(A_2))
\]

\[
= (FRBG_{1, cl}(A_1) \times F RBG_{2, cl}(A_2)) - (FRBG_{1, int}(A_1) \cap F RBG_{1, cl}(A_1))
\]

\[
\times (F RBG_{2, int}(A_2) \cap F RBG_{2, cl}(A_2))
\]

\[
= (FRBG_{1, cl}(A_1) \times F RBG_{2, cl}(A_2)) - (FRBG_{1, int}(A_1) \times F RBG_{2, cl}(A_2))
\]

\[
\cap (FRBG_{1, cl}(A_1) \times F RBG_{2, int}(A_2)) \ (by \ Proposition \ 4.10)
\]

\[
= [(FRBG_{1, cl}(A_1) \times F RBG_{2, cl}(A_2)) - (FRBG_{1, int}(A_1) \times F RBG_{2, cl}(A_2))]
\]

\[
\cup [(FRBG_{1, cl}(A_1) \times F RBG_{2, cl}(A_2)) - (FRBG_{1, cl}(A_1) \times F RBG_{2, int}(A_2))]
\]

151
\begin{align*}
&= [(FRBG_1 cl(A_1) - FRBG_1 int(A_1)) \times FRBG_2 cl(A_2)] \\
&\quad \cup [FRBG_1 cl(A_1) \times (FRBG_2 cl(A_2) - FRBG_2 int(A_2))] \\
&= (FRBG_1 bd(A_1) \times FRBG_2 cl(A_2)) \cup (FRBG_1 cl(A_1) \times FRBG_2 bd(A_2)).
\end{align*}

**Definition 6.1.19.** Let \((X, G_1)\) and \((Y, G_2)\) be any two fuzzy rough structure spaces. A function \(f : (X, G_1) \to (Y, G_2)\) is said to be fuzzy rough \(BG\)-continuous if and only if for each fuzzy rough open group \(W\) in \(G_2\) the inverse image \(f^{-1}(W)\) is a fuzzy rough \(B\)-open group in \(G_1\).

**Proposition 6.1.13.** Let \((X, G_1)\) and \((Y, G_2)\) be any two fuzzy rough structure spaces. Let \(f : (X, G_1) \to (Y, G_2)\) be a fuzzy rough \(BG\)-continuous function. Then,

\[ FRBGbd(f^{-1}(A)) \subseteq f^{-1}(FRGbd(A)). \]

**Proof.** Let \(f\) be a fuzzy rough \(BG\)-continuous function. Let \(A\) be any fuzzy rough topological group in \((Y, G_2)\). Then, \(FRGcl(A)\) is a fuzzy rough \(G\)-closed group in \((Y, G_2)\), which implies that \(f^{-1}(FRGcl(A))\) is a fuzzy rough \(BG\)-closed group in \((X, G_1)\). Therefore,

\[ FRBGbd(f^{-1}(A)) = FRBGcl(f^{-1}(A)) \cap FRBGcl(f^{-1}(A)') \]
\[ \subseteq FRBGcl(f^{-1}(FRGcl(A))) \cap FRBGcl(f^{-1}(FRGcl(A'))) \]
\[ = f^{-1}(FRGcl(A)) \cap f^{-1}(FRGcl(A')) \]
\[ = f^{-1}(FRGbd(A)). \]
Therefore, \( FRB\mathcal{G}bd(f^{-1}(A)) \subseteq f^{-1}(FRB\mathcal{G}bd(A)) \).

**Definition 6.1.20.** Let \((X, \mathcal{G}_1)\) and \((Y, \mathcal{G}_2)\) be any two fuzzy rough structure spaces. A function \( f : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2) \) is said to be fuzzy rough \( \mathcal{G} \)-irresolute if and only if for each fuzzy rough \( \mathcal{B} \)-open group \( W \) in \( \mathcal{G}_2 \) the inverse image \( f^{-1}(W) \) is a fuzzy rough \( \mathcal{B} \)-open group in \( \mathcal{G}_1 \).

**Proposition 6.1.14.** Let \((X, \mathcal{G}_1)\) and \((Y, \mathcal{G}_2)\) be any two fuzzy rough structure spaces. Let \( f : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2) \) be a fuzzy rough \( \mathcal{G} \)-irresolute function. Then,

\[
FRB\mathcal{G}bd(f^{-1}(A)) \subseteq f^{-1}(FRB\mathcal{G}bd(A)).
\]

**Proof.** Let \( f \) be a fuzzy rough \( \mathcal{G} \)-irresolute function. Let \( A \) be any fuzzy rough topological group in \((Y, \mathcal{G}_2)\). Then, \( FRB\mathcal{G}cl(A) \) is a fuzzy rough \( \mathcal{G} \)-closed group in \((Y, \mathcal{G}_2)\), which implies that \( f^{-1}(FRB\mathcal{G}cl(A)) \) is a fuzzy rough \( \mathcal{G} \)-closed group in \((X, \mathcal{G}_1)\). Therefore,

\[
FRB\mathcal{G}bd(f^{-1}(A)) = FRB\mathcal{G}cl(f^{-1}(A)) \cap FRB\mathcal{G}cl(f^{-1}(A))' \\
\subseteq FRB\mathcal{G}cl(f^{-1}(FRB\mathcal{G}cl(A))) \cap FRB\mathcal{G}cl(f^{-1}(FRB\mathcal{G}cl(A'))) \\
= f^{-1}(FRB\mathcal{G}cl(A)) \cap f^{-1}(FRB\mathcal{G}cl(A')) \\
= f^{-1}(FRB\mathcal{G}cl(A) \cap FRB\mathcal{G}cl(A')) \\
= f^{-1}(FRB\mathcal{G}bd(A)).
\]

Therefore, \( FRB\mathcal{G}bd(f^{-1}(A)) \subseteq f^{-1}(FRB\mathcal{G}bd(A)) \).
6.2 CHARACTERIZATION OF FUZZY ROUGH $B\mathcal{G}$-BOUNDARY SPACES

In this section, the concept of fuzzy rough $B\mathcal{G}$-boundary space is introduced. In this connection, the characterization of fuzzy rough $B\mathcal{G}$-boundary space is established.

**Definition 6.2.1.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $F_{RBG}bd(A)$ be the fuzzy rough $B\mathcal{G}$-boundary of $A$. Then the fuzzy rough $B\mathcal{G}$-interior of $F_{RBG}bd(A)$ is defined by

$$F_{RBG}^\circ(F_{RBG}bd(A)) = \bigcup \{ B : B \text{ is a fuzzy rough } B\text{-open group and } B \subseteq F_{RBG}bd(A) \}.$$ 

**Definition 6.2.2.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $F_{RBG}bd(A)$ be the fuzzy rough $B\mathcal{G}$-boundary of $A$. Then the fuzzy rough $B\mathcal{G}$-closure of $F_{RBG}bd(A)$ is defined by

$$F_{RBG}^\circ(F_{RBG}bd(A)) = \bigcap \{ B : B \text{ is a fuzzy rough } B\text{-closed group and } B \supseteq F_{RBG}bd(A) \}.$$ 

**Proposition 6.2.1.** Let $(X, \mathcal{G})$ be a fuzzy rough $\mathcal{G}$ structure space. Let $F_{RBG}bd(A)$ be the fuzzy rough $B\mathcal{G}$-boundary of $A$. Then the following conditions hold.

(i) $F_{RBG}^\circ(F_{RBG}bd(A)) \subseteq F_{RBG}bd(A) \subseteq F_{RBG}^\circ(F_{RBG}bd(A))$.

(ii) $(F_{RBG}^\circ(F_{RBG}bd(A)))' = F_{RBG}^\circ(F_{RBG}bd(A)')$.

(iii) $(F_{RBG}^\circ(F_{RBG}bd(A)))' = F_{RBG}^\circ(F_{RBG}bd(A)')$. 

154
Proof: The proof follows from Definition 6.2.1 and Definition 6.2.2.

**Definition 6.2.3.** Let \((X, G)\) be a fuzzy rough \(G\) structure space. Then \((X, G)\) is said to be a fuzzy rough \(BG\)-boundary space if the fuzzy rough \(BG\)-closure of fuzzy rough \(BG\)-boundary of each fuzzy rough open group is a fuzzy rough \(B\)-open group. That is, \(FRBG^{-}(FRBG_{bd}(A))\) is fuzzy rough \(B\)-open group for every \(A \in G\).

**Proposition 6.2.2.** Let \((X, G)\) be a fuzzy rough \(G\) structure space. Then the following statements are equivalent:

(i) \((X, G)\) is a fuzzy rough \(BG\)-boundary space.

(ii) Let \(FRBG_{bd}(A)\) be fuzzy rough \(BG\)-boundary of \(A\). Then \(FRBG^{\circ}(FRBG_{bd}(A))\) is a fuzzy rough \(B\)-closed group.

(iii) For each \(FRBG_{bd}(A)\),

\[
FRBG^{-}(FRBG_{bd}(A)) + FRBG^{-}(FRBG^{-}(FRBG_{bd}(A)))' = \tilde{1}.
\]

(iv) For every pair of fuzzy rough \(BG\)-boundary sets \(FRBG_{bd}(A)\) and \(FRBG_{bd}(B)\) with \(FRBG^{-}(FRBG_{bd}(A)) + FRBG_{bd}(B) = \tilde{1}\), we have \(FRBG^{-}(FRBG_{bd}(A)) + FRG^{-}(FRBG_{bd}(B)) = \tilde{1}\).

Proof. (i)\(\Rightarrow\)(ii): Let \(FRBG_{bd}(A)\) be the fuzzy rough \(BG\) boundary of \(A\). Then, \((FRBG_{bd}(A))'\) is a fuzzy rough boundary complement of \(FRBG_{bd}(A)\). Now,

\[
FRBG^{-}(FRBG_{bd}(A))' = (FRBG^{\circ}(FRBG_{bd}(A)))'.
\]
By (i), \( FRBG^{-}(FRBGbd(A))' \) is a fuzzy rough \( B \)-open group, which implies that \( FRBG^{\circ}(FRBGbd(A)) \) is a fuzzy rough \( B \)-closed group.

(ii) \( \Rightarrow \) (iii): Let \( FRBGbd(A) \) be the fuzzy rough \( BG \)-boundary of \( A \). Then,
\[
FRBG^{-}(FRBGbd(A)) + FRBG^{-}(FRG^{-}(FRBGbd(A)))'
= FRBG^{-}(FRBGbd(A)) + FRBG^{-}(FRG^{\circ}(FRBGbd(A)))'
= FRBG^{-}(FRBGbd(A)) + (FRBG^{-}(FRBGbd(A)))'
= \tilde{1}.
\]
Since \( FRBGbd(A) \) is a fuzzy rough \( BG \)-boundary of \( A \), (\( FRBGbd(A) \)') is a fuzzy rough \( BG \)-boundary of \( FRBGbd(A) \). Hence by (ii), \( FRBG^{\circ}(FRBGbd(A))' \) is fuzzy rough \( B \)-closed group. Therefore, by (6.2.1)
\[
FRBG^{-}(FRBGbd(A)) + FRBG^{-}(FRBGbd(A)) = \tilde{1}.
\]

(iii) \( \Rightarrow \) (iv): Let \( FRBGbd(A) \) and \( FRBGbd(B) \) be any two fuzzy rough \( BG \)-boundary of \( A \) and \( B \) respectively, such that
\[
FRBG^{-}(FRBGbd(A)) + FRBG^{-}(FRBGbd(B)) = \tilde{1}.
\]
Then by (iii),
\[
\tilde{1} = FRBG^{-}(FRBGbd(A)) + FRBG^{-}(FRG^{-}(FRBGbd(A)))'
= FRBG^{-}(FRBGbd(A)) + FRBG^{-}(FRBGbd(B)).
\]
Therefore, \( FRBG^{-}(FRBGbd(A)) + FRBG^{-}(FRBGbd(B)) = \tilde{1} \).

(iv) \( \Rightarrow \) (i): Let \( FRBGbd(A) \) be a fuzzy rough \( BG \)-boundary of \( A \). Put \( FRBGbd(B) = (FRBG^{-}(FRBGbd(A)))' = \tilde{1} - FRBG^{-}(FRBGbd(A)) \). Then,
\[
FRBG^-(FRBG\text{bd}(A)) + FRBG\text{bd}(B) = \tilde{1}. \text{ Therefore by (iv), } FRBG^-(FRBG\text{bd}(A)) + FRBG^-(FRBG\text{bd}(B)) = \tilde{1}. \text{ This implies that, } FRBG^-(FRBG\text{bd}(A)) \text{ is a fuzzy rough } B\text{-open group and so } (X, G) \text{ is a fuzzy rough } BG\text{-boundary spaces.}
\]

**Remark 6.2.1.** Every fuzzy rough \( B \)-open group is a fuzzy rough \( C \)-open group.

**Proposition 6.2.3.** If \((X, G)\) is a fuzzy rough \( BG \)-boundary space then every fuzzy rough \( BG \)-closure of \( BG \)-boundary of each fuzzy rough open group is a fuzzy rough \( C \)-open group.

**Proof.** The proof follows from Definition 6.2.3 and Remark 6.2.1.