Chapter 2

On Contra $rw$-Continuous Functions

2.1 Introduction

Dontchev [15] introduced the notions of contra-continuity and strong $S$-closedness in topological spaces. He defined a function $f : X \to Y$ to be contra-continuous if the preimage of every open set of $Y$ is closed in $X$. In [15], he obtained very interesting and important results concerning contra-continuity, compactness, $S$-closedness and strong $S$-closedness. Recently a new weaker form of this class of functions called contra-semicontinuous function is introduced and investigated by Dontchev and Noiri [16]. They also introduced the notion of $RC$-continuity [16] between topological spaces which is weaker than contra-continuity and stronger than $B$-continuity [68]. In 1999, Jafari [25] introduced and investigated a new class of functions called contra-super-continuous functions which lies between classes of $RC$-continuous functions and contra-continuous functions.
In section 2 of this chapter is to introduce the notion of contra \( rw \)-continuous functions and to obtain fundamental properties of contra \( rw \)-continuous functions.

In section 3 of this chapter, the relationship between contra \( rw \)-continuity and other related functions are discussed.

### 2.2 Contra \( rw \)-Continuous Function and its Relationship With Other Functions

**Definition 2.2.1** A function \( f : X \to Y \) is called **contra \( rw \)-continuous** if \( f^{-1}(V) \) is \( rw \)-closed set in \( X \) for every open set \( V \) of \( Y \).

**Theorem 2.2.2** The following are equivalent for a function \( f : X \to Y \)

1. \( f \) is contra \( rw \)-continuous,
2. the inverse image of every closed set of \( Y \) is \( rw \)-open.

**Proof.** Let \( U \) be any closed set in \( Y \). Since \( Y \setminus U \) is open, then by (1) it follows that \( f^{-1}(Y \setminus U) = X \setminus f^{-1}(U) \) is \( rw \)-closed. This shows that, \( f^{-1}(U) \) is \( rw \)-open in \( X \). Converse is similar. \[ \blacksquare \]

**Theorem 2.2.3** Suppose that \( RWC(X) \) is closed under arbitrary intersections. Then the following are equivalent for a function \( f : X \to Y \):

1. \( f \) is contra \( rw \)-continuous,
2. the inverse image of every closed set of \( Y \) is \( rw \)-open,
3. for each \( x \in X \) and each closed set \( B \) in \( Y \) with \( f(x) \in B \), there exists a \( rw \)-open set \( A \) in \( X \) such that \( x \in A \) and \( f(A) \subset B \),

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Theorem 2.2.4
Suppose that
(4) \( f(rw-cl(A)) \subset ker(f(A)) \) for every subset \( A \) of \( X \),
(5) \( rw-cl(f^{-1}(B)) \subset f^{-1}(ker(B)) \) for every subset \( B \) of \( Y \).

Proof. (1) \( \Rightarrow \) (2): Obvious from Theorem 2.2.2.

(1) \( \Rightarrow \) (3) Let \( x \in X \) and \( B \) be a closed set in \( Y \) with \( f(x) \in B \). By (1),
it follows that \( f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \) is \( rw \)-closed and so \( f^{-1}(B) \) is \( rw \)-open.
Take \( A = f^{-1}(B) \), we obtain \( x \in A \) and \( f(A) \subset B \).

(3) \( \Rightarrow \) (2) Let \( B \) be a closed set in \( Y \) with \( x \in f^{-1}(B) \). Since \( f(x) \in B \), by
(3) there exists a \( rw \)-open set \( A \) in \( X \) containing \( x \) such that \( f(A) \subset B \). It
follows that \( x \in A \subset f^{-1}(B) \). Hence \( f^{-1}(B) \) is \( rw \)-open.

(2) \( \Rightarrow \) (1) Follows from the previous Theorem.

(2) \( \Rightarrow \) (4) Let \( A \) be any subset of \( X \). Let \( y \notin ker(f(A)) \). Then there exists a
closed set \( F \) containing \( y \) such that \( f(A) \cap F = \phi \). Hence we have \( A \cap f^{-1}(F) = \phi \).
Hence, we obtain \( f(rw-cl(A)) \cap F = \phi \) and \( rw-clA \cap f^{-1}(F) = \phi \) and \( y \notin f(rw-
cl(A)) \). Thus \( f(rw-cl(A)) \subset ker(f(A)) \).

(4) \( \Rightarrow \) (5) Let \( B \) be any subset of \( Y \). By (4), \( f(rw-cl(f^{-1}(B))) \subset ker(B) \) and
\( rw-cl(f^{-1}(B)) \subset f^{-1}(ker(B)) \)

(5) \( \Rightarrow \) (1) Let \( B \) be any open set of \( Y \). By (5), \( rw-cl(f^{-1}(B)) \subset f^{-1}(ker(B)) = f^{-1}(B) \) and \( rw-cl(f^{-1}(B)) = f^{-1}(B) \). We obtain that, \( f^{-1}(B) \) is \( rw \)-closed in
\( X \).

Theorem 2.2.4 Suppose that \( X \) and \( Y \) are spaces and \( RWO(X) \) is closed under
arbitrary union. If a function \( f : X \to Y \) is contra \( rw \)-continuous and \( Y \) is regular,
then \( f \) is \( rw \)-continuous.
Proof. Let \( x \) be an arbitrary point of \( X \) and \( V \) be an open set of \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exist an open set \( G \) in \( Y \) containing \( f(x) \) such that \( \text{cl}(G) \subseteq V \). Since \( f \) is contra \( rw \)-continuous, there exist \( U \in RWO(X) \) containing \( x \) such that \( f(U) \subseteq \text{cl}(G) \). Then \( f(U) \subseteq \text{cl}(G) \subseteq V \). Hence, \( f \) is \( rw \)-continuous.

Definition 2.2.5 A space \((X, T)\) is called \( rw-T_{\frac{1}{2}} \) if every \( rw \)-closed set is closed.

Theorem 2.2.6 Let \( f: (X, T) \rightarrow (Y, S) \) be a function. Suppose that \((X, T)\) is a \( rw-T_{\frac{1}{2}} \) space. Then the following are equivalent:

1. \( f \) is contra \( rw \)-continuous,
2. \( f \) is contra \( g \)-continuous,
3. \( f \) is contra continuous.

Proof. The proof is obvious.

Theorem 2.2.7 If a function \( f: X \rightarrow \prod Y_i \) is contra \( rw \)-continuous, then \( p_i \circ f: X \rightarrow Y_i \) is contra \( rw \)-continuous for each \( i \in I \), where \( p_i \) is the projection of \( \prod Y_i \) onto \( Y_i \).

Proof. Let \( V_i \) be any open set of \( Y_i \). Since \( p_i \) is continuous, \( p_i^{-1}(V_i) \) is open in \( \prod Y_i \). Since \( f \) is contra \( rw \)-continuous, \( f^{-1}(p_i^{-1}(V_i)) = (p_i \circ f)^{-1}(V_i) \in RW(X) \).

This shows that \( p_i \circ f \) is contra \( rw \)-continuous for each \( i \in I \).

Definition 2.2.8 A topological space \((X, T)\) is said to be locally \( rw \)-indiscrete if every \( rw \)-open set of \( X \) is closed in \( X \).
Proposition 2.2.9 Let \( f : (X, T) \to (Y, S) \) be a function. If \( f \) is contra \( rw \)-continuous and \( (X, T) \) is locally \( rw \)-indiscrete, then \( f \) is continuous.

Theorem 2.2.10 Let \( f : X \to Y \) and \( g : Y \to Z \) be functions. Then, the following properties hold.

1. If \( f \) is \( rw \)-irresolute and \( g \) is contra \( rw \)-continuous, then \( g \circ f : X \to Z \) is contra \( rw \)-continuous.

2. If \( f \) is contra \( rw \)-continuous and \( g \) is continuous, then \( g \circ f : X \to Z \) is \( rw \)-continuous.

3. If \( f \) is contra \( rw \)-continuous and \( g \) is \( RC \)-continuous, then \( g \circ f : X \to Z \) is \( rw \)-continuous.

4. If \( f \) is \( rw \)-continuous and \( g \) is contra continuous, then \( g \circ f : X \to Z \) is contra \( rw \)-continuous.

Theorem 2.2.11 Suppose that \( RWC(Y) \) is closed under arbitrary intersections. If \( f : X \to Y \) is a surjective \( rw \)-open function and \( g : Y \to Z \) is a function such that \( g \circ f : X \to Z \) is contra \( rw \)-continuous, then \( g \) is contra \( rw \)-continuous.

Proof. Suppose that \( x \) and \( y \) are two points in \( X \) and \( Y \) respectively, such that \( f(x) = y \). Let \( B \in C(Z, (g \circ f)(x)) \). Then there exits a \( rw \)-open set \( A \) in \( X \) containing \( x \) such that \( g(f(A)) \subset B \). Since \( f \) is \( rw \)-open, \( f(A) \) is a \( rw \)-open in \( Y \) containing \( y \) such that \( g(f(A)) \subset B \). This implies that \( g \) is contra \( rw \)-continuous.

Corollary 2.2.12 Let \( f : X \to Y \) be a surjective \( rw \)-irresolute and \( rw \)-open function and let \( g : Y \to Z \) be a function. Suppose that \( RWC(Y) \) is closed under
arbitrary intersection. Then \( g \circ f : X \to Z \) is contra \( rw \)-continuous if and only if \( g \) is contra \( rw \)-continuous.

**Proof.** Follows from Theorem 2.2.10 and 2.2.11.

### 2.3 Properties of Contra \( rw \)-Continuous Functions

**Definition 2.3.1** The \( rw \)-frontier of a subset \( A \) of a space \( X \) is given by \( rw-fr(A) = rw-cl(A) \cap rw-cl(X \setminus A) \).

**Theorem 2.3.2** Let the collection of all \( rw \)-closed sets of a space \((X, \tau)\) be closed under arbitrary intersections. The set of all points \( x \in X \) at which a function \( f : (X, \tau) \to (Y, \sigma) \) is not contra \( rw \)-continuous is identical with the union of \( rw \)-frontier of the inverse images of closed sets containing \( f(x) \).

**Proof.** (\( \Rightarrow \)) Suppose that \( f \) is not contra \( rw \)-continuous at \( x \in X \). Then there exists a closed set \( A \) of \( Y \) containing \( f(x) \) such that \( f(U) \) is not contained in \( A \) for every \( U \in RWO(X) \) containing \( x \). Then \( U \cap (X \setminus f^{-1}(A)) \neq \emptyset \) for every \( U \in RWO(X) \) containing \( x \) and hence \( x \in rw-cl(X \setminus f^{-1}(A)) \). On the other hand, we have \( x \in f^{-1}(A) \subset rw-cl(f^{-1}(A)) \) and hence \( x \in rw-fr(f^{-1}(A)) \).

(\( \Leftarrow \)) Suppose that \( f \) is contra \( rw \)-continuous at \( x \in X \), and let \( A \) be a closed set of \( Y \) containing \( f(x) \). Then there exists \( U \in RWO(X) \) containing \( x \) such that \( U \subset f^{-1}(A) \); hence \( x \in rw-int(f^{-1}(A)) \). Therefore, \( x \notin rw-fr(f^{-1}(A)) \) for each closed set \( A \) of \( Y \) containing \( f(x) \). This completes the proof.

**Corollary 2.3.3** Let \( RWC(X) \) be closed under arbitrary intersections. A function \( f : X \to Y \) is not contra \( rw \)-continuous at \( x \) if and only if \( x \in rw-fr(f^{-1}(F)) \)
for some $F \in C(Y, f(x))$.

**Definition 2.3.4** A space $X$ is said to be $rw-T_1$ if for each pair of distinct points $x$ and $y$ in $X$, there exists $rw$-open sets $U$ and $V$ containing $x$ and $y$, respectively, such that $y \notin U$ and $x \notin V$.

**Definition 2.3.5** A space $X$ is said to be $rw-T_2$ if for each pair of distinct points $x$ and $y$ in $X$, there exists $U \in RWO(X, x)$ and $V \in RWO(X, y)$ such that $U \cap V = \emptyset$.

**Theorem 2.3.6** Let $X$ and $Y$ be topological spaces. If

1. for each pair of distinct points $x$ and $y$ in $X$, there exists a function $f$ of $X$ into $Y$ such that $f(x) \neq f(y)$.
2. $Y$ is an Urysohn space and $f$ is contra $rw$-continuous at $x$ and $y$, then $X$ is $rw-T_2$.

**Proof.** Let $x$ and $y$ be distinct points in $X$. Then, there exists a Urysohn space $Y$ and a function $f : X \to Y$ such that $f(x) \neq f(y)$ and $f$ is contra $rw$-continuous at $x$ and $y$. Let $z = f(x)$ and $v = f(y)$. Then $z \neq v$. We have to prove $X$ is $rw-T_2$ space. Since $Y$ is Urysohn, there exist open sets $V$ and $W$ containing $z$ and $v$ respectively such that $cl(V) \cap cl(W) = \emptyset$. Since $f$ is contra $rw$-continuous at $x$ and $y$, then there exists $rw$-open sets $A$ and $B$ containing $x$ and $y$, respectively such that, $f(A) \subset cl(V)$ and $f(B) \subset cl(W)$. We have $A \cap B = \emptyset$. Since $cl(V) \cap cl(W) = \emptyset$, hence $X$ is $rw-T_2$. 

**Corollary 2.3.7** Let $f : X \to Y$ be a contra $rw$-continuous injection. If $Y$ is an Urysohn space, then it is $rw-T_2$. 

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Definition 2.3.8 A space $X$ is said to be $rw$-connected if $X$ is not the union of two disjoint nonempty $rw$-open sets.

Remark 2.3.9 Every $rw$-connected space is $g$-connected. The reverse of this implication is not true in general.

Example 2.3.10 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, Then $(X, \tau)$ is $g$-connected but it is not $rw$-connected.

Theorem 2.3.11 For a topological space $X$, the following properties are equivalent:

(1) $X$ is $rw$-connected,

(2) The only subsets of $X$ which are both $rw$-open and $rw$-closed are the empty set $\phi$ and $X$,

(3) Each contra $rw$-continuous function of $X$ into a discrete space $Y$ with at least two points is a constant function.

Proof. (1) $\Rightarrow$ (2) Suppose $A \subset X$ is a proper subset which is both $rw$-open and $rw$-closed. Then its complement $X\setminus A$ is also $rw$-open and $rw$-closed. Then $X = A \cup (X\setminus A)$ is a disjoint union of two nonempty $rw$-open sets which contradicts the fact that $X$ is $rw$-connected. Hence, $A = \phi$ or $X$.

(2) $\Rightarrow$ (1) Suppose $X = A \cup B$ where $A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$ and $A$ and $B$ are $rw$-open. Since $A = X\setminus B$, $A$ is $rw$-closed. But by hypothesis $A = \phi$, which is a contradiction. Hence (1) holds.

(2) $\Rightarrow$ (3) Let $f : X \to Y$ be a contra $rw$-continuous function where $Y$ is a discrete space with atleast two points. Then $f^{-1}(\{y\})$ is $rw$-closed and $rw$-open
for each $y \in Y$ and $X = \{f^{-1}\{\{y\}\} : y \in Y\}$. By hypothesis, $f^{-1}\{\{y\}\} = \phi$ or $X$. If $f^{-1}\{\{y\}\} = \phi$ for all $y \in Y$, $f$ is not a function. Also there cannot exist more than one $y \in Y$ such that $f^{-1}\{\{y\}\} = X$. Hence, there exists only one $y \in Y$ such that $f^{-1}\{\{y\}\} = X$ and $f^{-1}\{\{y_1\}\} = \phi$ where $y \neq y_1 \in Y$. This shows that $f$ is a constant function.

$(3) \Rightarrow (2)$ Let $P$ be both $rw$-open and $rw$-closed in $X$. Suppose $P \neq \phi$. Let $f : X \to Y$ be a contra $rw$-continuous function defined by $f(P) = \{a\}$ and $f(X \setminus P) = \{b\}$ where $a \neq b$ and $a, b \in Y$. By hypothesis, $f$ is constant. Therefore, $P = X$.

**Theorem 2.3.12** If $f$ is a contra $rw$-continuous function from a $rw$-connected space $X$ onto any space $Y$, then $Y$ is not a discrete space.

**Proof.** Suppose that $Y$ is discrete. Let $A$ be a proper nonempty clopen subset of $Y$. Then $f^{-1}(A)$ is a proper nonempty $rw$-clopen subset of $X$, which is a contradiction to the fact that $X$ is $rw$-connected.

**Theorem 2.3.13** A space $X$ is $rw$-connected if every contra $rw$-continuous function from a space $X$ into any $T_0$-space $Y$ is constant.

**Proof.** Suppose that $X$ is not $rw$-connected and that every contra $rw$-continuous function from $X$ into $Y$ is constant. Since $X$ is not $rw$-connected, there exists a proper nonempty $rw$-clopen subset $A$ of $X$. Let $Y = \{a, b\}$ and $\tau = \{Y, \phi, \{a\}, \{b\}\}$ be a topology for $Y$. Let $f : X \to Y$ be a function such that $f(A) = \{a\}$ and $f(X \setminus A) = \{b\}$. Then $f$ is non-constant and contra $rw$-continuous such that $Y$ is $T_0$, which is a contradiction. Hence, $X$ must be $rw$-connected.
Theorem 2.3.14  If \( f : X \to Y \) is a contra \( \text{rw} \)-continuous surjection and \( X \) is \( \text{rw} \)-connected, then \( Y \) is connected.

Proof. Suppose that \( Y \) is not a connected space. There exists nonempty disjoint open sets \( V_1 \) and \( V_2 \) such that \( Y = V_1 \cup V_2 \). Therefore, \( V_1 \) and \( V_2 \) are clopen in \( Y \). Since \( f \) is contra \( \text{rw} \)-continuous, \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are nonempty disjoint and \( X = f^{-1}(V_1) \cup f^{-1}(V_2) \). This shows that \( X \) is not \( \text{rw} \)-connected. This contradicts that \( Y \) is not connected. Hence, \( Y \) is connected.

Theorem 2.3.15  Let \( p : X \times Y \to X \) be a projection. If \( A \) is \( \text{rw} \)-closed subset of \( X \), then \( p^{-1}(A) = A \times Y \) is \( \text{rw} \)-closed subset of \( X \times Y \).

Proof. Let \( A \times Y \subset U \) and \( U \) be regular semiopen subset of \( X \times Y \). Then \( U = V \times Y \) for some regular semi open set of \( X \). Since \( A \) is \( \text{rw} \)-closed in \( X \), \( cl(A) \subset V \) and so \( cl(A) \times Y \subset V \times Y = U \), i.e., \( cl(A \times Y) \subset U \). Hence, \( A \times Y \) is \( \text{rw} \)-closed subset of \( X \times Y \).

Proposition 2.3.16  If \( f : X \to Y \) is a \( \text{rw} \)- irresolute surjection and \( X \) is \( \text{rw} \)-connected, then \( Y \) is \( \text{rw} \)-connected.

Proposition 2.3.17  If the product space of two nonempty topological spaces is \( \text{rw} \)-connected, then each factor space is \( \text{rw} \)-connected.

Proof. Let \( X \times Y \) be the product space of the nonempty spaces \( X \) and \( Y \) and \( X \times Y \) be \( \text{rw} \)-connected. The projection \( p : X \times Y \to X \) is \( \text{rw} \)- irresolute and then \( p(X \times Y) = X \) is \( \text{rw} \)-connected. The proof for the space \( Y \) is similar to the case of \( X \).