Chapter 7

New Approaches for Generalized Closed Sets and Maps in Bitopological Spaces

7.1 Introduction

The triple \((X, \tau_1, \tau_2)\) where \(X\) is a set and \(\tau_1\) and \(\tau_2\) are topologies on \(X\) is called a bitopological spaces. Kelly [33] initiated a systematic study of the concept of bitopological spaces in 1963. He generalized the topological concepts to bitopological setting. Following the work of Kelly [33] on the bitopological spaces, various authors, like Arya and Nour [4], Di Maio and Noiri [14], Fukutake [19], Nagaveni [52], Sampath Kumar [61], Reilly [60], Popa [59], Maki [41], Arockiarani [3], Gnanambal [22] and Sheik John [62] have contributed various concepts of topology by considering bitopological spaces. Benchalli and Wali [70] have worked on generalized closed sets, their generalizations and related concepts in general topology.
In section 2 of this chapter, \((i, j)\)-\(Bg\)-closed sets in bitopological space have been introduced and studied. Among many other results it is observed that every \((i, j)\)-\(w\)-closed set is \((i, j)\)-\(Bg\)-closed set which implies \((i, j)\)-\(rg\)-closed set but not conversely.

In section 3 of this chapter, we have introduced \((i, j)\)-\(Bg\)-open sets in bitopological space and study some of their properties. In section 4 of this chapter, we shall use the \((i, j)\)-\(Bg\)-closed subsets of bitopological space \((X, \tau_1, \tau_2)\) to define a new closure operator "\((i, j)\)-\(Bg\)-cl" and thus new topology \(\tau_{Bg}(i, j)\) on the space and shall examine some of the properties of this new topology.

In section 5 of this chapter, a new class of maps called \(D_B(i, j)\)-\(\sigma_k\)-continuous maps in bitopological spaces are introduced and investigated. During this process, some of their properties are obtained. It is found that every \(C(i, j)\)-\(\sigma_k\)-continuous map is \(D_B(i, j)\)-\(\sigma_k\)-continuous which implies \(D_r(i, j)\)-\(\sigma_K\)-continuous but not conversely. Also, we have introduced the concept of \(Bg\)-bi-continuity, \(Bg\)-\(s\)-bi-continuity and pairwise \(Bg\)-irresolute in bitopological spaces and study some of the properties.

Throughout this chapter \((X, \tau_1, \tau_2)\), \((Y, \sigma_1, \sigma_2)\) and \((Z, \eta_1, \eta_2)\) denote nonempty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and fixed integers \(i, j, k, e, m, n \in \{1, 2\}\).

### 7.2 \((\tau_i, \tau_j)\)-\(Bg\)-Closed Sets

In this section we define \((\tau_i, \tau_j)\)-\(Bg\)-closed set, and obtain some of their basic properties.
Definition 7.2.1 Let \( i, j \in \{1, 2\} \) be fixed integers. In a bitopological space \((X, \tau_1, \tau_2)\), a subset \( A \subseteq X \) is said to be \((\tau_i, \tau_j)\)-\textbf{B-generalized closed} (briefly, \((i, j)\)-\textbf{Bg-closed}) set if \( \tau_j\text{-}Bcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \in \tau_i \) (\( \tau_i \)-open).

We denote the family of all \((i, j)\)-\textbf{Bg-closed} sets in a bitopological space \((X, \tau_1, \tau_2)\) by \( D_B(\tau_i, \tau_j) \) or \( D_B(i, j) \).

Remark 7.2.2 By setting \( \tau_1 = \tau_2 \) in Definition 7.2.1, an \((i, j)\)-\textbf{Bg-closed} set reduces to a \textbf{Bg-closed} set in \( X \).

Theorem 7.2.3 If \( A \) is \((i, j)\)-\textbf{w-closed} subset of \((X, \tau_1, \tau_2)\), then \( A \) is \((i, j)\)-\textbf{Bg-closed}.

Proof. Let \( A \) be a \((i, j)\)-\textbf{w-closed} subset of \((X, \tau_1, \tau_2)\). Let \( U \in O(X, \tau_i) \) be such that \( A \subseteq U \). Since \( O(X, \tau_i) \subseteq SO(X, \tau_i) \), we have \( U \in SO(X, \tau_i) \). Then by hypothesis, \( \tau_j\text{-}cl(A) \subseteq U \). This implies \( \tau_j\text{-}Bcl(A) \subseteq U \). Therefore \( A \) is a \((i, j)\)-\textbf{Bg-closed}.

The converse of this Theorem 7.2.3 need not be true, as seen from the following Example.

Example 7.2.4 Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\} \), a non open set \( B = \{a, b\} \). Then the subsets \( \{c\}, \{b, c\}, \{c, d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\} \) and \( \{a, b, d\} \) are \((1, 2)\)-\textbf{Bg-closed} sets, but not \((1, 2)\)-\textbf{w-closed} sets in the bitopological space \((X, \tau_1, \tau_2)\).

Remark 7.2.5 \((i, j)\)-\textbf{Bg-closed} sets and \((i, j)\)-\textbf{rw-closed} sets are independent as seen from the following Examples.
Example 7.2.6  Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$, a non open set $B = \{a, b\}$. Then the subset $\{a, b\}$ is $(1, 2)$-rw-closed sets, but not $(1, 2)$-Bg-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

Example 7.2.7  Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$, a non open set $B = \{a, b\}$. Then the subsets $\{c\}, \{d\}, \{b, c\}, \{c, d\}, \{a, c\}, \{b, d\}$ and $\{a, c, d\}$ are $(1, 2)$-Bg-closed sets but not $(1, 2)$-rw-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

Theorem 7.2.8  If $A$ is $(i, j)$-Bg-closed subset of $(X, \tau_1, \tau_2)$, then $A$ is $(i, j)$-rg-closed.

Proof.  Let $A$ be $(i, j)$-Bg-closed subset of $(X, \tau_1, \tau_2)$. Let $G \in RO(X, \tau_i)$ be such that $A \subseteq U$. Since $RO(X, \tau_i) \subseteq O(X, \tau_i)$, we have $U \in O(X, \tau_i)$. Then by hypothesis, $\tau_j-Bcl(A) \subseteq U$. This implies $\tau_j-cl(A) \subseteq U$. Therefore $A$ is $(i, j)$-rg-closed.  

The converse of this Theorem 7.2.8 need not be true, as seen from the following Example.

Example 7.2.9  Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$, a non open set $B = \{b, c\}$. Then the subset $\{a, b\}$ are $(1, 2)$-rg-closed sets, but not $(1, 2)$-Bg-closed sets in the bitopological space $(X, \tau_1, \tau_2)$.

Theorem 7.2.10  If $A$ is $\tau_j$-closed subset of a bitopological space $(X, \tau_1, \tau_2)$, then the set $A$ is $(i, j)$-Bg-closed.
Proof. Let \( U \in O(X, \tau_i) \ i = \{1, 2\} \) be such that \( A \subseteq U \). Then by hypothesis, \( \tau_j \cdot Bcl(A) = A \), which implies \( \tau_j \cdot Bcl(A) \subseteq U \). Therefore \( A \) is \((i, j)\)-Bg-closed.

The converse of this Theorem 7.2.10 need not be true, as seen from the following Example.

**Example 7.2.11** Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b, c\}\} \), a non open set \( B = \{b, c\} \). Then the subsets \( \{c\}, \{b, c\}, \{a, d\}, \{a, c\}, \{b, d\}, \{a, b, c\} \) and \( \{a, b, d\} \) are \((1, 2)\)-Bg-closed sets, but not \( \tau_2 \)-closed set in the bitopological space \((X, \tau_1, \tau_2)\).

**Theorem 7.2.12** If \( A \) is \((i, j)\)-g-closed subset of a bitopological space \((X, \tau_1, \tau_2)\), then the set \( A \) is \((i, j)\)-Bg-closed.

Proof. Let \( A \) be \((i, j)\)-g-closed subset of \((X, \tau_1, \tau_2)\). Let \( U \in \tau_i \) be such that \( A \subseteq U \). Then by hypothesis, \( \tau_j \cdot cl(A) \subseteq U \). Also \( \tau_j \cdot Bcl(A) \subseteq \tau_j \cdot cl(A) \) which implies \( \tau_j \cdot Bcl(A) \subseteq U \). Therefore \( A \) is \((i, j)\)-Bg-closed. ■

The converse of this Theorem 7.2.12 need not be true, as seen from the following Example.

**Example 7.2.13** Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \), a non open set \( B = \{b, c\} \). Then the subsets \( \{a\} \) and \( \{b\} \) are \((1, 2)\)-Bg-closed sets, but not \((1, 2)\)-g-closed sets in the bitopological space \((X, \tau_1, \tau_2)\).

**Theorem 7.2.14** If \( A \) is \((i, j)\)-Bg-closed subset of a bitopological space \((X, \tau_1, \tau_2)\), then the set \( A \) is \((i, j)\)-gpr-closed.
Proof. Let $A$ be $(i, j)$-$Bg$-closed subset of $(X, \tau_1, \tau_2)$. Let $U \in RO(X, \tau_i)$ be such that $A \subseteq U$. Since $RO(X, \tau_i) \subseteq O(X, \tau_i)$, we have $G \in O(X, \tau_i)$. Then by hypothesis, $\tau_j - Bcl(A) \subseteq U$. Also $\tau_j - pcl(A) \subseteq \tau_j - Bcl(A)$ which implies $\tau_j - pcl(A) \subseteq U$. Therefore $A$ is $(i, j)$-$gpr$-closed.

The converse of this Theorem 7.2.14 need not be true, as seen from the following Example.

Example 7.2.15 Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$, a non open set $B = \{b, c\}$. Then the subsets $\{a, b\}$ is $(1, 2)$-$gpr$-closed set but not $(1, 2)$-$Bg$-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

Theorem 7.2.16 If $A$ is $(i, j)$-$wg$-closed subset of a bitopological space $(X, \tau_1, \tau_2)$, then the set $A$ is $(i, j)$-$Bg$-closed.

Proof. Let $A$ be $(i, j)$-$wg$-closed subset of $(X, \tau_1, \tau_2)$. Let $U \in O(X, \tau_i)$ be such that $A \subseteq U$. Then by hypothesis, $\tau_j - cl(\tau_i - int(A)) \subseteq U$. Also $\tau_j - Bcl(A) \subseteq \tau_j - cl(\tau_i - int(A))$ which implies $\tau_j - Bcl(A) \subseteq U$. Therefore $A$ is $(i, j)$-$Bg$-closed.

The converse of this Theorem 7.2.16 need not be true, as seen from the following Example.

Example 7.2.17 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, a non open set $B = \{b, c\}$. Then the subset $\{a\}$ is $(1, 2)$-$Bg$-closed set but not $(1, 2)$-$wg$-closed set in the bitopological space $(X, \tau_1, \tau_2)$. 

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**Theorem 7.2.18** If \( A \) is \((i, j)\)-\(gp\)-closed subset of a bitopological space \((X, \tau_1, \tau_2)\), then the set \( A \) is \((i, j)\)-\(Bg\)-closed.

**Proof.** Let \( A \) be a \((i, j)\)-\(gp\)-closed subset of \((X, \tau_1, \tau_2)\). Let \( U \in O(X, \tau_i) \) be such that \( A \subseteq U \). Then by hypothesis, \( \tau_j\text{-}pcl(A) \subseteq U \). Also \( \tau_j\text{-}Bcl(A) \subseteq \tau_j\text{-}pcl(A) \) which implies \( \tau_j\text{-}Bcl(A) \subseteq U \). Therefore \( A \) is \((i, j)\)-\(Bg\)-closed. \( \blacksquare \)

The converse of this Theorem 7.2.18 need not be true, as seen from the following Example.

**Example 7.2.19** Let \( X = \{a, b, c\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \), a non open set \( B = \{b, c\} \). Then the subsets \( \{a\} \) and \( \{b\} \) are \((1, 2)\)-\(gp\)-closed sets but not \((1, 2)\)-\(Bg\)-closed set in the bitopological space \((X, \tau_1, \tau_2)\).

**Theorem 7.2.20** If \( A \) is \((i, j)\)-\(g^*\)-closed subset of a bitopological space \((X, \tau_1, \tau_2)\), then the set \( A \) is \((i, j)\)-\(Bg\)-closed.

**Proof.** Let \( A \) be a \((i, j)\)-\(g^*\)-closed subset of \((X, \tau_1, \tau_2)\). Let \( U \in O(X, \tau_i) \) be such that \( A \subseteq U \). Since \( O(X, \tau_i) \subseteq GO(X, \tau_i) \), we have \( U \in GO(X, \tau_i) \). Then by hypothesis, \( \tau_j\text{-}cl(A) \subseteq U \). Also \( \tau_j\text{-}Bcl(A) \subseteq \tau_j\text{-}cl(A) \) which implies \( \tau_j\text{-}Bcl(A) \subseteq U \). Therefore \( A \) is \((i, j)\)-\(Bg\)-closed. \( \blacksquare \)

The converse of this Theorem 7.2.20 need not be true, as seen from the following Example.

**Example 7.2.21** Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \), a non open set \( B = \{b, c\} \). Then the subsets \( \{c\}, \{b, c\}, \{a, d\}, \{a, c\}, \{b, d\}, \{a, b, c\} \) and \( \{a, b, d\} \) are \((1, 2)\)-\(Bg\)-closed sets but not \((1, 2)\)-\(g^*\)-closed set in the bitopological space \((X, \tau_1, \tau_2)\).
Remark 7.2.22 From the above discussions and known results we have the following diagram of implications

\[ A \rightarrow B \] we mean \( A \) implies \( B \) but not conversely and \( A \leftrightarrow B \) means \( A \) and \( B \) are independent of each other.

![Diagram](fig-4)

Theorem 7.2.23 If \( A, B \in D_B(i, j) \), then \( A \cup B \in D_B(i, j) \).

Proof. Let \( U \in O(X, \tau_i) \) be such that \( A \cup B \subseteq U \). Then \( A \subseteq U \) and \( B \subseteq U \). Since \( A, B \in D_B(i, j) \), we have \( \tau_j - Bcl(A) \subseteq U \) and \( \tau_j - Bcl(B) \subseteq U \). That is \( \tau_j - Bcl(A) \cup \tau_j - Bcl(B) \subseteq U \). Also \( \tau_j - Bcl(A) \cup \tau_j - Bcl(B) = \tau_j - Bcl(A \cup B) \) and so \( \tau_j - Bcl(A \cup B) \subseteq U \). Therefore \( A \cup B \in D_B(i, j) \).

Remark 7.2.24 The intersection of two \((i, j)\)-\(Bg\)-closed sets is generally not an \((i, j)\)-\(Bg\)-closed set as seen from the following Example.

Example 7.2.25 Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b, c\}\} \), a non open set \( B = \{b, c\} \). Then the subsets \( \{a, d\} \) and \( \{a, c\} \) are \((1, 2)\)-\(Bg\)-closed sets, but \( \{a, d\} \cap \{a, c\} = \{a\} \) is not \((1, 2)\)-\(Bg\)-closed set in the bitopological space \((X, \tau_1, \tau_2)\).

Remark 7.2.26 The family \( D_B(1, 2) \) is generally not equal to the family \( D_B(2, 1) \) as seen from the following Example.
Example 7.2.27 Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b, c\}\}$, a non-open set $B = \{b, c\}$. Then $D_B(1, 2) = P(X) - \{\{a\}, \{b\}\}$ and $D_B(2, 1) = P(X) - \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. Therefore $D_B(1, 2) \neq D_B(2, 1)$.

Theorem 7.2.28 If $\tau_1 \subseteq \tau_2$ and $O(X, \tau_1) \subseteq O(X, \tau_2)$ in $(X, \tau_1, \tau_2)$, then $D_B(\tau_1, \tau_2) \supseteq D_B(\tau_2, \tau_1)$.

Proof. Let $A \in D_B(\tau_2, \tau_1)$. That is $A$ is an $(2, 1)$-$Bg$-closed set. To prove that $A \in D_B(\tau_1, \tau_2)$. Let $G \in O(X, \tau_1)$ be such that $A \subseteq G$. Since $O(X, \tau_1) \subseteq O(X, \tau_2)$, we have $G \in O(X, \tau_2)$. As $A$ is $(2, 1)$-$Bg$-closed set, we have $\tau_1$-$Bcl(A) \subseteq G$. Since $\tau_1 \subseteq \tau_2$, we have $\tau_2$-$Bcl(A) \subseteq \tau_1$-$Bcl(A)$ and it follows that $\tau_2$-$Bcl(A) \subseteq G$. Hence $A$ is a $(1, 2)$-$Bg$-closed set. That is $A \in D_B(\tau_1, \tau_2)$. Therefore $D_B(\tau_1, \tau_2) \supseteq D_B(\tau_2, \tau_1)$.

Theorem 7.2.29 If $A$ is $(i, j)$-$Bg$-closed, then $\tau_j$-$Bcl(A) - A$ contains no nonempty $\tau_i$-$B$-open set.

Proof. Let $A$ be an $(i, j)$-$Bg$-closed set. Suppose $F$ is a non-empty $\tau_i$-$B$-open set contained in $\tau_j$-$Bcl(A) - A$. Now $F \subseteq X - A$ which implies $A \subseteq F^c$. Also $F^c$ is a $\tau_i$-open. Since $A$ is an $(i, j)$-$Bg$-closed set, we have $\tau_j$-$Bcl(A) \subseteq F^c$. Consequently $F \subseteq \tau_j$-$Bcl(A) \cap (\tau_j$-$Bcl(A))^c = \phi$, which is a contradiction. Hence $\tau_j$-$Bcl(A) - A$ does not contain any non-empty $\tau_i$-$B$-open set.

The converse of this Theorem 7.2.29 does not hold as seen from the following Example.

Example 7.2.30 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ a non-open set $B = \{b, c\}$. If $A = \{a, b\}$, then $\tau_2$-$Bcl(A) - A = X - \{a, b\} = \{c\}$ does not contain any non-empty $\tau_i$-$B$-open set. But $A$ is not $(1, 2)$-$Bg$-closed set in the bitopological space $(X, \tau_1, \tau_2)$.

Corollary 7.2.31 If $A$ is $(i, j)$-$Bg$-closed in $(X, \tau_1, \tau_2)$, then $A$ is $\tau_j$-closed if and only if $\tau_j$-$Bcl(A) - A$ is a $\tau_i$-$B$-open set.

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Proof. Suppose \( A \) is \( \tau_j \)-closed. Then \( \tau_j-Bcl(A) = A \) and so \( \tau_j-Bcl(A) - A = \phi \), which is \( \tau_i-B \)-open set.

Conversely, suppose \( \tau_j-Bcl(A) - A \) is \( \tau_i-B \)-open. Since \( A \) is \( (i, j) \)-\( Bg \)-closed, by Theorem 7.2.29, \( \tau_j-Bcl(A) - A \) does not contain any non-empty \( \tau_i-B \)-open set. Therefore \( \tau_j-Bcl(A) - A = \phi \). That is \( \tau_j-Bcl(A) = A \) and hence \( A \) is \( \tau_j \)-closed.

\[ \square \]

**Theorem 7.2.32** If \( A \) is an \( (i, j) \)-\( Bg \)-closed set and \( \tau_i \subseteq BO(X, \tau_i) \), then \( \tau_j-Bcl(\{x\}) \cap A \neq \phi \) for each \( x \in \tau_j-Bcl(A) \).

Proof. Let \( A \) be an \( (i, j) \)-\( Bg \)-closed set and \( \tau_i \subseteq BO(X, \tau_i) \). Suppose \( \tau_i-Bcl(\{x\}) \cap A = \phi \) for some \( x \in \tau_j-Bcl(A) \), then \( A \subseteq (\tau_i-Bcl(\{x\}))^c \). Now \( (\tau_i-Bcl(\{x\}))^c \in \tau_i \subseteq BO(X, \tau_i) \), by hypothesis. That is \( (\tau_i-Bcl(\{x\}))^c \) is a \( \tau_i-Bg \)-open. Since \( A \) is a \( (i, j) \)-\( Bg \)-closed, we have \( \tau_j-Bcl(A) \subseteq (\tau_i-Bcl(\{x\}))^c \). This shows that \( x \notin \tau_j-Bcl(A) \). This contradicts the assumption.

\[ \square \]

**Theorem 7.2.33** If \( U \) is an \( (i, j) \)-\( Bg \)-closed set and \( U \subseteq V \subseteq \tau_j-Bcl(U) \), then \( V \) is an \( (i, j) \)-\( Bg \)-closed set.

Proof. Let \( G \) be a \( \tau_i-B \)-open set such that \( V \subseteq G \). As \( U \) is \( (i, j) \)-\( Bg \)-closed set and \( U \subseteq G \), we have \( \tau_j-Bcl(U) \subseteq G \). Now \( V \subseteq \tau_j-Bcl(U) \) which implies, \( \tau_j-Bcl(V) \subseteq \tau_j-Bcl(\{\tau_j-Bcl(U)\}) = \tau_j-Bcl(U) \subseteq G \). Thus \( \tau_j-Bcl(U) \subseteq G \). Therefore \( V \) is \( (i, j) \)-\( Bg \)-closed set.

\[ \square \]

**Theorem 7.2.34** In a bitopological space \( (X, \tau_1, \tau_2) \), if \( BO(X, \tau_i) = \{X, \phi\} \), then every subset of \( (X, \tau_1, \tau_2) \) is an \( (i, j) \)-\( Bg \)-closed.

Proof. Let \( BO(X, \tau_i) = \{X, \phi\} \) in a bitopological space \( (X, \tau_1, \tau_2) \). Let \( A \) be any subset of \( X \). To prove that \( A \) is an \( (i, j) \)-\( Bg \)-closed, suppose \( A = \phi \), then \( A \) is an \( (i, j) \)-\( Bg \)-closed. Suppose \( A \neq \phi \), then \( X \) is the only \( \tau_i-B \)-open set and \( \tau_j-Bcl(A) \subset X \). Hence \( A \) is an \( (i, j) \)-\( Bg \)-closed set.

\[ \square \]

The converse of the above Theorem 7.2.34 need not be true in general as seen from the following Example.
Example 7.2.35 Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \emptyset\} \). Then every subset of \( X \) is \((2, 1)\)-\(Bg\)-closed set but \( BO(X, \tau_1) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) with non open set \( B = \{b, c\} \).

Theorem 7.2.36 If \( A \) is a \( \tau_1 \)-open and \((i, j)\)-\(g\)-closed, then \( A \) is \((i, j)\)-\(Bg\)-closed.

Proof. Let \( G \) be a \( \tau_i \)-open set such that \( A \subseteq G \). Now \( A \subseteq A \), \( A \) is \( \tau_i \)-open and \((i, j)\)-\(g\)-closed, we have \( \tau_j - cl(A) \subseteq A \). Then \( \tau_j - Bcl(A) \subseteq A \). That is \( \tau_j - Bcl(A) \subseteq G \). Therefore \( A \) is \((i, j)\)-\(Bg\)-closed.

7.3 \((\tau_i, \tau_j)\)-\(Bg\)-Open Sets

In this section, we introduce \((\tau_i, \tau_j)\)-\(Bg\)-open sets in bitopological spaces and study some of their properties.

Definition 7.3.1 Let \( i, j \in \{1, 2\} \) be fixed integers. In a bitopological space \((X, \tau_1, \tau_2)\), a subset \( A \subset X \) is said to be \((\tau_i, \tau_j)\)-\(B\)-generalized open (briefly, \((i, j)\)-\(Bg\)-open) set if \( A^c \) is \((i, j)\)-\(B\)-generalized closed (briefly, \((i, j)\)-\(Bg\) closed).

Theorem 7.3.2 In a bitopological space \((X, \tau_1, \tau_2)\), we have the following

\[\begin{align*}
\text{i) } & \text{ Every } (i, j)\text{-}w\text{-open set is } (i, j)\text{-}Bg\text{-open but not conversely.} \\
\text{ii) } & \text{ Every } (i, j)\text{-}Bg\text{-open set is } (i, j)\text{-}gpr\text{-open but not conversely.} \\
\text{iii) } & \text{ Every } (i, j)\text{-}gpr\text{-open set is } (i, j)\text{-}Bg\text{-open but not conversely.} \\
\text{iv) } & \text{ Every } wg\text{-open set is } (i, j)\text{-}Bg\text{-open but not conversely.} \\
\text{v) } & \text{ Every } Bg\text{-open set is } (i, j)\text{-}rg\text{-open but not conversely.} \\
\text{vi) } & \text{ Every } Bg\text{-open set is } (i, j)\text{-}Bg\text{-open but not conversely.}
\end{align*}\]

Proof. The proof follows from the Theorems 7.2.3, 7.2.14, 7.2.18, 7.2.16, 7.2.8 and 7.2.12.
Theorem 7.3.3  If $A$ and $B$ are $(i, j)$-$Bg$-open sets, then $A \cap B$ is $(i, j)$-$Bg$-open.

Proof. The proof follows from the Theorem 7.2.23. □

Remark 7.3.4  The union of two $(i, j)$-$Bg$-open sets is generally not an $(i, j)$-$Bg$-open set as seen from the following Example.

Example 7.3.5  Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$, a non-open set $B = \{b, c\}$. Then the subsets $\{c\}$ and $\{d\}$ are $(1, 2)$-$Bg$-open sets, but $\{c\} \cup \{d\} = \{c, d\}$ is not $(1, 2)$-$Bg$-open set in the bitopological space $(X, \tau_1, \tau_2)$.

7.4  $(\tau_i, \tau_j)$-$Bg$-Closure in Bitopological Spaces

Definition 7.4.1  Let $(X, \tau_1, \tau_2)$ be a bitopological space and $i, j \in \{1, 2\}$ be fixed integers. For each subset $E$ of $X$, we define $(\tau_i, \tau_j)$-$Bg$-cl$(E) = \bigcap\{A : E \subseteq A \in (i, j)$-$Bg$-closed $\}$ (briefly, $(i, j)$-$Bg$-cl$(E)$).

Theorem 7.4.2  If $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$. Then
(i) $(i, j)$-$Bg$-cl$(X) = X$ and $(i, j)$-$Bg$-cl$(\emptyset) = \emptyset$.
(ii) $A \subseteq (i, j)$-$Bg$-cl$(A)$.
(iii) If $B$ is any $(i, j)$-$Bg$-closed set containing $A$, then $(i, j)$-$Bg$-cl$(A) \subseteq B$.

Proof. Follows from the Definition 7.4.1. □

Theorem 7.4.3  Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ and $i, j \in \{1, 2\}$ be fixed integers. If $A \subseteq B$, then $(i, j)$-$Bg$-cl$(A) \subseteq (i, j)$-$Bg$-cl$(B)$.

Proof. Let $A \subseteq B$. By Definition 7.4.1, $(i, j)$-$Bg$-cl$(B) = \cap\{F : B \subseteq F \in D_B(i, j)\}$. If $B \subseteq F \in D_B(i, j)$, since $A \subseteq B$, $A \subseteq B \subseteq F \in D_B(i, j)$, we have $(i, j)$-$Bg$-cl$(A) \subseteq F$. Therefore $(i, j)$-$Bg$-cl$(A) \subseteq \cap\{F : B \subseteq F \in D_B(i, j)\} = (i, j)$-$Bg$-cl$(B)$. That is $(i, j)$-$Bg$-cl$(A) \subseteq (i, j)$-$Bg$-cl$(B)$. □
Theorem 7.4.4  Let $A$ be a subset of $(X, \tau_1, \tau_2)$. If $\tau_1 \subseteq \tau_2$, then $(1, 2)\cdot Bg-cl(A) \subseteq (2, 1)\cdot Bg-cl(A)$.

Proof. By Definition 7.4.1, it follows that $(1, 2)\cdot Bg-cl(A) = \cap \{F : A \subseteq F \in D_B(1, 2)\}$. Since $\tau_1 \subseteq \tau_2$ by Theorem 7.2.28 $D_B(2, 1) \subseteq D_B(1, 2)$. Therefore $\cap \{F : A \subseteq F \in D_B(2, 1)\} \subseteq \cap \{F : A \subseteq F \in D_B(2, 1)\}$. That is $(1, 2)\cdot Bg-cl(A) \subseteq \cap \{F : A \subseteq F \in D_B(2, 1)\} = (2, 1)\cdot Bg-cl(A)$. Hence $(1, 2)\cdot Bg-cl(A) \subseteq (2, 1)\cdot Bg-cl(A)$. ■

Theorem 7.4.5  Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $i, j \in \{1, 2\}$ be fixed integers, then $A \subseteq (i, j)\cdot Bcl(A) \subseteq \tau_j \cdot cl(A)$.

Proof. By Definition 7.4.1, it follows that $A \subseteq (i, j)\cdot Bg-cl(A)$. Now to prove that $(i, j)\cdot Bg-cl(A) \subseteq \tau_j \cdot cl(A)$. By Definition of $B$-closure, $\tau_j \cdot Bcl(A) = \cap \{F \subseteq X : A \subseteq F \text{ and } F \text{ is } \tau_j \cdot B\text{-closed} \}$. If $A \subseteq F$ and $F$ is $\tau_j \cdot B$-closed, then $F$ is $(i, j)\cdot B$-closed, as every $\tau_j \cdot B$-closed set is $(i, j)\cdot Bg$-closed. Therefore $(i, j)\cdot B-cl(A) \subseteq \cap \{F \subseteq X : A \subseteq F \text{ and } F \text{ is } \tau_j \cdot B\text{-closed} \} = \tau_j \cdot Bcl(A)$. Hence $(i, j)\cdot Bg-cl(A) \subseteq \tau_j \cdot cl(A)$. ■

Remark 7.4.6 Containment relation in the above Theorem 7.4.5 may be proper as seen from the following Example.

Example 7.4.7 Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}\}$, a non open set $B = \{b, c\}$. Then $\tau_2$-closed sets are $X, \phi, \{b, c\}$ and $(1, 2)\cdot Bg$-closed sets are $X, \phi, \{c\}, \{a, c\}, \{b, c\}$. Take $A = \{a\}$. Then $\tau_2 \cdot cl(A) = X$ and $(1, 2)\cdot Bg-cl(A) = \{a, c\}$. Now $A \subset (1, 2)\cdot Bg-cl(A)$, but $A \neq (1, 2)\cdot Bg-cl(A)$. Also $(1, 2)\cdot Bg-cl(A) \subset \tau_2 \cdot cl(A)$, but $(i, j)\cdot Bg-cl(A) \neq \tau_j \cdot cl(A)$.

Theorem 7.4.8 Let $A$ be a subset of $(X, \tau_1, \tau_2)$ and $i, j \in \{1, 2\}$ be fixed integers. If $A$ is $(i, j)\cdot Bg$-closed, then $(i, j)\cdot Bg-cl(A) = A$.  

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**Proof.** Let \( A \) be \((i, j)\)-\( Bg \)-closed subset of \((X, \tau_1, \tau_2)\). We know that \( A \subseteq (i, j)\)-\( Bg \)-cl\((A)\). Also \( A \subseteq A \) and \( A \) is a \((i, j)\)-\( Bg \)-closed. By the Theorem 7.4.2(iii), \((i, j)\)-\( Bg \)-cl\((A)\) \(\subseteq A\). Hence \((i, j)\)-\( Bg \)-cl\((A)\) = \( A \).

**Remark 7.4.9** The converse of the above Theorem 7.4.8 need not be true as seen from the following Example.

**Example 7.4.10** Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b, c\}\} \), a non open set \( B = \{b, c\} \). Then \((1, 2)\)-\( Bg \)-closed sets are \( P(X) - \{\{a\}, \{b\}, \{a, b\}\} \). Take \( A = \{a\} \). Now \((1, 2)\)-\( Bg \)-cl\((A)\) = \( \{a\} \), but \( \{a\} \) is not \((1, 2)\)-\( Bg \)-closed set.

**Theorem 7.4.11** The operator \((i, j)\)-\( Bg \)-closure in Definition 7.4.1, is the Kuratowski closure operator on \( X \).

**Proof.** (i) \((i, j)\)-\( Bg \)-cl\((\phi)\) = \( \phi \), by Theorem 7.4.2(i).

(ii) \( E \subseteq (i, j)\)-\( Bg \)-cl\((E)\) for any subset \( E \) of \( X \) by Theorem 7.4.2(ii).

(iii) Suppose \( E \) and \( F \) are two subsets of \((X, \tau_1, \tau_2)\). It follows from Theorem 7.4.3, that \((i, j)\)-\( Bg \)-cl\((F)\) \(\subseteq (i, j)\)-\( Bg \)-cl\((E \cup F)\) and that \((i, j)\)-\( Bg \)-cl\((E)\) \(\subseteq (i, j)\)-\( Bg \)-cl\((E \cup F)\). Hence we have \((i, j)\)-\( Bg \)-cl\((E)\cup(i, j)\)-\( Bg \)-cl\((F)\) \(\subseteq (i, j)\)-\( Bg \)-cl\((E \cup F)\).

Now if \( x \) does not belongs to \((i, j)\)-\( Bg \)-cl\((E)\cup(i, j)\)-\( Bg \)-cl\((F)\), then \( x \notin (i, j)\)-\( Bg \)-cl\((E)\) and \( x \notin (i, j)\)-\( Bg \)-cl\((F)\), it follows that there exist \( A, B \in Bg(i, j) \) such that \( E \subseteq A, x \notin A \) and \( F \subseteq B, x \notin B \). Hence \( E \cup F \subseteq A \cup B \), \( x \notin A \cup B \). Since \( A \cup B \) is \((i, j)\)-\( Bg \)-closed, by Theorem 7.2.14, \( x \) does not belong to \((i, j)\)-\( Bg \)-cl\((E \cup F)\). Then we have \((i, j)\)-\( Bg \)-cl\((E \cup F)\) \(\subseteq (i, j)\)-\( Bg \)-cl\((E)\cup(i, j)\)-\( Bg \)-cl\((F)\). From the above discussion we have \((i, j)\)-\( Bg \)-cl\((E \cup F)\) = \((i, j)\)-\( Bg \)-cl\((E)\cup(i, j)\)-\( Bg \)-cl\((F)\).

(iv) Let \( E \) be any subset of \((X, \tau_1, \tau_2)\). By the Definition of \((i, j)\)-\( Bg \)-closure, \((i, j)\)-\( Bg \)-cl\((E)\) = \( \cap\{A \subseteq X : E \subseteq A \in D_B(i, j)\} \). If \( E \subseteq A \in D_B(i, j) \), then \((i, j)\)-\( Bg \)-cl\((E)\) \(\subseteq A\). Since \( A \) is \((i, j)\)-\( Bg \)-closed set containing \((i, j)\)-\( Bg \)-cl\((E)\), by Theorem 7.4.2 (iii), \((i, j)\)-\( Bg \)-cl\((E)\) \(\subseteq A\), Hence
Proof. Let \((i, j)\) be two fixed integers. Then
\[
\text{the topology on } X \text{ generated by } (i, j) - Bg\text{-closure in the usual manner. That is } \\
\tau_{Bg}(i, j) = \{ E \subset X : (i, j) - Bg\text{-cl}(E) = E^c \}.
\]

Conversely, \((i, j) - Bg\text{-cl}(E) \subset (i, j) - Bg\text{-cl}((i, j) - Bg\text{-cl}(E))\) is true by Theorem 7.4.2(iii). Then we have \((i, j) - Bg\text{-cl}(E) = (i, j) - Bg\text{-cl}((i, j) - Bg\text{-cl}(E))\). Hence \((i, j) - Bg\text{-closure is a Kuratowski closure operator on } X.\n
From the above Theorem 7.4.11, \((i, j) - Bg\text{-closure defines the new topology on } X.\n
**Definition 7.4.12** Let \(i, j \in \{1, 2\}\) be two fixed integers. Let \(\tau_{Bg}(i, j)\) be the topology on \(X\) generated by \((i, j) - Bg\text{-closure in the usual manner. That is } \tau_{Bg}(i, j) = \{ E \subset X : (i, j) - Bg\text{-cl}(E^c) = E^c \}.\n
**Theorem 7.4.13** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(i, j \in \{1, 2\}\) be two fixed integers. Then \(\tau_j \subseteq \tau_{Bg}(i, j).\n
**Proof.** Let \(G \in \tau_j\), it follows that \(G^c\) is \(\tau_j\text{-closed. By Theorem 7.2.10, } G^c\) is \((i, j) - Bg\text{-closed. Therefore } (i, j) - Bg\text{-cl}(G^c) = G^c\), by Theorem 7.4.8 That is \(G \in \tau_{Bg}(i, j)\) and hence \(\tau_j \subseteq \tau_{Bg}(i, j).\) \n
**Remark 7.4.14** Containment relation in the above Theorem 7.4.13 may be proper as seen from the following Example.

**Example 7.4.15** Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\), a non open set \(B = \{b, c\}\). Then \((1, 2) - Bg\text{-closed sets are } P(X) - \{\{a\}, \{b\}, \{a, b\}\}\) and \(\tau_{Bg}(1, 2) = P(X).\) Clearly \(\tau_2 \subseteq \tau_{Bg}(1, 2)\), but \(\tau_2 \neq \tau_{Bg}(1, 2).\)

**Theorem 7.4.16** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(i, j \in \{1, 2\}\) be two fixed integers. If a subset \(E\) of \(X\) is \((i, j) - Bg\text{-closed, then } E\) is \(\tau_{Bg}(i, j)\text{-closed.}\)

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Proof. Let a subset $E$ of $X$ be $(i, j)$-$Bg$-closed. By Theorem 7.4.8, $(i, j)$-$Bg$-$cl(E) = E$. That is $(i, j)$-$Bg$-$cl\{(E^c)^c\} = (E^c)^c$, it follows that $E^c \in \tau_{Bg}(i, j)$. Therefore, $E$ is $\tau_{Bg}(i, j)$-closed.

Remark 7.4.17 The converse of the above Theorem 7.4.16 need not be true as seen from the following Example.

Example 7.4.18 For $(X, \tau_1, \tau_2)$ of Example 7.4.15, the subset $A = \{a\}$ is $\tau_{Bg}(1, 2)$-closed, but not $(1, 2)$-$Bg$-closed.

Theorem 7.4.19 If $\tau_1 \subseteq \tau_2$ and $BO(X, \tau_1) \subseteq BO(X, \tau_2)$ in $(X, \tau_1, \tau_2)$, then $\tau_{Bg}(2, 1) \subseteq \tau_{Bg}(1, 2)$.

Proof. Let $G \in \tau_{Bg}(2, 1)$, then $(2, 1)$-$Bg$-$cl(G^c) = G^c$. To prove that $G \in \tau_{Bg}(1, 2)$. That is to prove $(1, 2)$-$Bg$-$cl(G^c) = G^c$. Now $(1, 2)$-$Bg$-$cl(G^c) = \cap \{F \subseteq X : G^c \subseteq F \in D_{Bg}(1, 2)\}$. Since $\tau_1 \subseteq \tau_2$ and $BO(X, \tau_1) \subseteq BO(X, \tau_2)$, by Theorem 7.2.16, $D_{Bg}(2, 1) \subseteq D_{Bg}(1, 2)$. Thus $\cap \{F \subseteq X : G^c \subseteq F \in D_{Bg}(1, 2)\} \subseteq \cap \{F \subseteq X : G^c \subseteq F \in D_{Bg}(2, 1)\}$. That is $(1, 2)$-$Bg$-$cl(G^c) \subseteq (2, 1)$-$Bg$-$cl(G^c) = G^c$, and so $(1, 2)$-$Bg$-$cl(G^c) \subseteq G^c$.

Conversely $G^c \subseteq (1, 2)$-$Bg$-$cl(G^c)$ is true by the Theorem 7.4.2(ii). Then we have $(1, 2)$-$Bg$-$cl(G^c) = G^c$. That is $G \in \tau_{Bg}(1, 2)$ and hence $\tau_{Bg}(2, 1) \subseteq \tau_{Bg}(1, 2)$.

7.5 $D_B(i, j)$-$\sigma_k$-Continuous and $gB$-$bi$-Continuous Maps

In this section a new class of maps called $D_B(i, j)$-$\sigma_k$-continuous maps in bitopological spaces are introduced and investigated. During this process, some of their properties are obtained. It is found that every $C(i, j)$-$\sigma_k$-continuous map is
$D_B(i, j)\cdot \sigma_k$-continuous which implies $D_r(i, j)\cdot \sigma_k$-continuous. Also, we introduced the concept of $Bg\cdot bi$-continuity and $Bg\cdot s\cdot bi$-continuity in bitopological spaces and study some of their properties.

**Definition 7.5.1** A map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $D_B(i, j)\cdot \sigma_k$-continuous if the inverse image of every $\sigma_k$-closed set is an $(i, j)$-$Bg$-closed set in $(X, \tau_1, \tau_2)$.

**Remark 7.5.2** If $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2 = \sigma$ in Definition 7.5.1, then the $D_B(i, j)\cdot \sigma_k$-continuity of maps coincides with $gB$-continuity of maps in topological spaces.

**Theorem 7.5.3** If a map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $\tau_j\cdot \sigma_k$-continuous, then it is $D_B(i, j)\cdot \sigma_k$-continuous.

**Proof.** Let $V$ be a $\sigma_k$-closed set. Since $f$ is $\tau_j\cdot \sigma_k$-continuous, $f^{-1}(V)$ is $\tau_j$-closed. By Theorem 7.2.10, $f^{-1}(V)$ is $(i, j)$-$Bg$-closed in $(X, \tau_1, \tau_2)$. Therefore $f$ is $D_B(i, j)\cdot \sigma_k$-continuous.

The converse of this Theorem 7.5.3 need not be true as seen from the following Example.

**Example 7.5.4** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{p\}\}$ and $\sigma_2 = \{Y, \phi, \{q\}\}$. Define a map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = f(b) = f(d) = p$, $f(c) = q$. Then $f$ is $D_B(1, 2)\cdot \sigma_1$-continuous but it is not $\tau_2\cdot \sigma_1$-continuous, since for the $\sigma_1$-closed set $\{q\}$, $f^{-1}(\{q\}) = \{c\}$ which is not $\tau_2$-closed.

**Theorem 7.5.5** If a map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $C(i, j)\cdot \sigma_k$-continuous, then it is $D_B(i, j)\cdot \sigma_k$-continuous.

**Proof.** Let $V$ be a $\sigma_k$-closed set. Since $f$ is $C(i, j)\cdot \sigma_k$-continuous, $f^{-1}(V)$ is $(i, j)$-$w$-closed. By Theorem 7.2.3, $f^{-1}(V)$ is $(i, j)$-$Bg$-closed in $(X, \tau_1, \tau_2)$. Therefore $f$ is $D_B(i, j)\cdot \sigma_k$-continuous.
The converse of this Theorem 7.5.5 need not be true as seen from the following Example.

\textbf{Example 7.5.6} Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\} \), and \( Y = \{p, q\}, \sigma_1 = \{Y, \phi, \{p\}\} \) and \( \sigma_2 = \{Y, \phi, \{q\}\} \). Define a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(d) = q \), and \( f(b) = f(c) = p \). Then \( f \) is \( D_B(1, 2) \)-\( \sigma_1 \)-continuous but it is not \( C(1, 2) \)-\( \sigma_1 \)-continuous, since for the \( \sigma_1 \)-closed set \( \{q\} \), \( f^{-1}(\{q\}) = \{a, d\} \) which is not \( (1, 2) \)-\( w \)-closed set.

\textbf{Theorem 7.5.7} If a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous, then it is \( D_r(i, j) \)-\( \sigma_k \)-continuous.

\textbf{Proof.} Let \( V \) be a \( \sigma_k \)-closed set. Since \( f \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous, \( f^{-1}(V) \) is \( (i, j) \)-\( Bg \)-closed. By Theorem 7.2.8, \( f^{-1}(V) \) is \( (i, j) \)-\( rg \)-closed in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( D_r(i, j) \)-\( \sigma_k \)-continuous.

The converse of this Theorem 7.5.7 need not be true as seen from the following Example.

\textbf{Example 7.5.8} Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\} \), and \( Y = \{p, q\}, \sigma_1 = \{Y, \phi, \{p\}\} \) and \( \sigma_2 = \{Y, \phi, \{q\}\} \). Define a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(b) = q \), and \( f(c) = f(d) = p \). Then \( f \) is \( D_r(1, 2) \)-\( \sigma_1 \)-continuous but it is not \( D_B(1, 2) \)-\( \sigma_1 \)-continuous, since for the \( \sigma_1 \)-closed set \( \{q\} \), \( f^{-1}(\{q\}) = \{a, b\} \) which is not \( (1, 2) \)-\( Bg \)-closed set.

\textbf{Theorem 7.5.9} If a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous, then it is \( \wp(i, j) \)-\( \sigma_k \)-continuous.

\textbf{Proof.} Let \( V \) be a \( \sigma_k \)-closed set. Since \( f \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous, \( f^{-1}(V) \) is \( (i, j) \)-\( B \)-closed. By Theorem 7.2.14, \( f^{-1}(V) \) is \( (i, j) \)-\( gpr \)-closed in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( \wp(i, j) \)-\( \sigma_k \)-continuous.

The converse of this Theorem need not be true as seen from the following Example.
Example 7.5.10 Let \( X = \{a, b, c, d\}, \ \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b, c\}\} \), and \( Y = \{p, q\}, \ \sigma_1 = \{Y, \phi, \{p\}\} \) and \( \sigma_2 = \{Y, \phi\} \). Define a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(b) = q \), and \( f(c) = f(d) = p \). Then this function \( f \) is \( \sigma(1, 2) \)-continuous but it is not \( D_B(1, 2)\)-continuous, since for the \( \sigma_2 \)-closed set \( \{q\} \), \( f^{-1}(\{q\}) = \{a, b\} \) which is not \( (1, 2)\)-closed set in \( (X, \tau_1, \tau_2) \).

Theorem 7.5.11 If a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( D(i, j)\)-\( \sigma_k \)-continuous, then it is \( D_B(i, j)\)-\( \sigma_k \)-continuous.

Proof. Let \( V \) be a \( \sigma_k \)-closed set. Since \( f \) is \( D(i, j)\)-\( \sigma_k \)-continuous, \( f^{-1}(V) \) is \( (i, j)\)-\( g \)-closed. By Theorem 7.2.12, \( f^{-1}(V) \) is \( (i, j)\)-\( Bg \)-closed in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( D_B(i, j)\)-\( \sigma_k \)-continuous.

The converse of this Theorem need not be true as seen from the following Example.

Example 7.5.12 Let \( X = \{a, b, c\}, \ \tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \), and \( Y = \{p, q\}, \ \sigma_1 = \{Y, \phi, \{p\}\} \) and \( \sigma_2 = \{Y, \phi, \{q\}\} \). Define a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(c) = p \), and \( f(b) = q \). Then this function \( f \) is \( D_B(1, 2)\)-\( \sigma_1 \)-continuous but it is not \( D(1, 2)\)-\( \sigma_1 \)-continuous, since for the \( \sigma_2 \)-closed set \( \{q\} \), \( f^{-1}(\{q\}) = \{q\} \) which is not \( (1, 2)\)-\( g \)-closed set in \( (X, \tau_1, \tau_2) \).

Theorem 7.5.13 If a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( W(i, j)\)-\( \sigma_k \)-continuous, then it is \( D_B(i, j)\)-\( \sigma_k \)-continuous.

Proof. Let \( V \) be a \( \sigma_k \)-closed set. Since \( f \) is \( W(i, j)\)-\( \sigma_k \)-continuous, \( f^{-1}(V) \) is \( (i, j)\)-\( Wg \)-closed. By Theorem 7.2.16, \( f^{-1}(V) \) is \( (i, j)\)-\( Bg \)-closed in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( D_B(i, j)\)-\( \sigma_k \)-continuous.

Example 7.5.14 Let \( X = \{a, b, c\}, \ \tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \), and \( Y = \{p, q\}, \ \sigma_1 = \{Y, \phi, \{p\}\} \) and \( \sigma_2 = \{Y, \phi, \{q\}\} \). Define a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = q \), and
Let $f(b) = f(c) = p$. Then this function $f$ is $D_B(1, 2)-\sigma_1$-continuous but it is not $W(1, 2)-\sigma_1$-continuous, since for the $\sigma_1$-closed set $\{q\}$, $f^{-1}(\{q\}) = \{a\}$ which is not $(1, 2)$-$wg$-closed set in $(X, \tau_1, \tau_2)$.

**Remark 7.5.15** $D_B(i, j)-\sigma_k$-continuous maps and $D_{rw}(i, j)-\sigma_k$-continuous maps are independent.

**Example 7.5.16** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{q\}\}$ and $\sigma_2 = \{Y, \phi, \{p\}\}$. Define a map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = f(b) = p$, and $f(c) = f(d) = q$. Then this function $f$ is $D_{rw}(1, 2)-\sigma_1$-continuous but it is not $D_B(1, 2)-\sigma_1$-continuous, since for the closed set $\{p\}$, $f^{-1}(\{p\}) = \{a, b\}$ which is not $(1, 2)$-$Bg$-closed set in $(X, \tau_1, \tau_2)$.

**Example 7.5.17** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{q\}\}$ and $\sigma_2 = \{Y, \phi, \{p\}\}$. Define a map $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ by $f(a) = f(c) = q$, and $f(b) = f(d) = p$. Then this function $f$ is $D_B(1, 2)-\sigma_1$-continuous but it is not $D_{rw}(1, 2)-\sigma_1$-continuous, since for the closed set $\{p\}$, $f^{-1}(\{p\}) = \{b, d\}$ which is not $(1, 2)$-$rw$-closed set in $(X, \tau_1, \tau_2)$.

**Remark 7.5.18** From the above discussions and known results we have the following implications. Here

$A \rightarrow B$ we mean $A$ implies $B$ but not conversely and

$A \leftrightarrow B$ means $A$ and $B$ are independent of each other.
**Theorem 7.5.19** The following statements are equivalent:

(i) A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous.

(ii) The inverse image of every \( \sigma_k \)-open set in \( Y \) is \((i, j)\)-\( B_g \)-open in \( X \).

**Proof.**  
(i) \( \Rightarrow \) (ii) Let \( G \) be a \( \sigma_k \)-open set in \( Y \). Then \( G^c \) is \( \sigma_k \)-closed set in \( Y \). Since \( f \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous, \( f^{-1}(G^c) \) is \((i, j)\)-\( B_g \)-closed in \( X \), that is \( f^{-1}(G^c) = (f^{-1}(G))^c \) and so \( f^{-1}(G) \) is \((i, j)\)-\( B_g \)-open in \((X, \tau_1, \tau_2)\).

(ii) \( \Rightarrow \) (i) Let \( F \) be a \( \sigma_k \)-closed set in \( Y \). Then \( F^c \) is \( \sigma_k \)-open set in \( Y \). By hypothesis, \( f^{-1}(F^c) \) is \((i, j)\)-\( B_g \)-open in \( X \). That is \( f^{-1}(F^c) = (f^{-1}(F))^c \) and so \( f^{-1}(F) \) is \((i, j)\)-\( B_g \)-closed in \((X, \tau_1, \tau_2)\). Therefore \( f \) is is \( D_B(i, j) \)-\( \sigma_k \)-continuous.

**Theorem 7.5.20** If a map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous, then \( f((i, j) \cdot B_g \cdot cl(A)) \subset \sigma_k \cdot cl(f(A)) \) holds for every subset \( A \) of \( X \).

**Proof.** Let \( A \) be any subset of \( X \). Then \( f(A) \subset \sigma_k \cdot cl(f(A)) \) and \( \sigma_k \cdot cl(f(A)) \) is \( \sigma_k \)-closed set in \( Y \). Also \( f^{-1}(f(A)) \subset f^{-1}(\sigma_k \cdot cl(f(A))) \). That is \( A \subset f^{-1}(\sigma_k \cdot cl(f(A))) \). Since \( f \) is \( D_B(i, j) \)-\( \sigma_k \)-continuous, \( f^{-1}(\sigma_k \cdot cl(f(A))) \) is \((i, j)\)-\( B_g \)-closed in \((X, \tau_1, \tau_2)\). By Theorem 7.4.2 (iii), \((i, j)\)-\( B_g \)-\( cl(A) \subset f^{-1}(\sigma_k \cdot cl(f(A))) \). Therefore \( f((i, j) \cdot B_g \cdot cl(A)) \subset f(f^{-1}(\sigma_k \cdot cl(f(A)))) \subset f \cdot cl(f(A)) \). Hence \( f((i, j) \cdot B_g \cdot cl(A)) \subset \sigma_k \cdot cl(f(A)) \) for every subset \( A \) of \( X \).
Converse of the above Theorem 7.5.20 is not true as seen from the following Example.

**Example 7.5.21** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{q\}\}$ and $\sigma_2 = \{Y, \phi\}$. Then $D_B(2, 1) = \{X, \phi, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}$. Define a map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a) = f(b) = p$, and $f(c) = f(d) = q$. Then $f((1, 2) - Bg-cl(A)) \subset \sigma_1 - cl(f(A))$ for every subset $A$ of $X$. But $f$ is not $(1, 2) - \sigma_1$-continuous, since for the closed set $\{p\}$, $f^{-1}(\{p\}) = \{a, b\}$ which is not $(1, 2) - \sigma_1-Bg$-closed set in $(X, \tau_1, \tau_2)$.

**Theorem 7.5.22** If a map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $D_B(i, j)$-$\sigma_k$-continuous and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ is $\sigma_k$-$\eta_n$-continuous, then $g \circ f$ is $D_B(i, j)$-$\eta_n$-continuous.

**Proof.** Let $F$ be $\eta_n$-closed set in $(Z, \eta_1, \eta_2)$. Since $g$ is $\sigma_k$-$\eta_n$-continuous, $g^{-1}(F)$ is $\sigma_k$-closed set in $(Y, \sigma_1, \sigma_2)$. Since $f$ is $D_B(i, j)$-$\sigma_k$-continuous, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $(i, j)$-$Bg$-closed set in $(X, \tau_1, \tau_2)$ and hence $g \circ f$ is $D_B(i, j)$-$\eta_n$-continuous.

**Definition 7.5.23** (i) A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $gB$-bi-continuous if $f$ is $D_B(1, 2)$-$\sigma_2$-continuous and is $D_B(2, 1)$-$\sigma_1$-continuous.

(ii) A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $gB$-strongly-bi-continuous (briefly $gB$-s-bi-continuous) if $f$ is $gB$-bi-continuous, $D_B(2, 1)$-$\sigma_2$-continuous and $D_B(1, 2)$-$\sigma_1$-continuous.

**Theorem 7.5.24** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a map.

(i) If $f$ is bi-continuous then $f$ is $gB$-bi-continuous.

(ii) If $f$ is s-bi-continuous then $f$ is $gB$-s-bi-continuous.

**Proof.** (i) Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a bi-continuous map. Then $f$ is $\tau_1$-$\sigma_1$-continuous and $\tau_2$-$\sigma_2$-continuous and so by Theorem 7.5.7, $f$ is $D_B(1, 2)$-$\sigma_2$-continuous and $D_B(2, 1)$-$\sigma_1$-continuous. Thus $f$ is $gB$-bi-continuous.
(ii) Similar to (i), using Theorem 7.5.3.

The converse of this Theorem 7.5.24 need not be true in general as seen from the following Example.

**Example 7.5.25** Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}$, and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{p\}$ and $\sigma_2 = \{Y, \phi, \{q\}$. Define a map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a) = f(c) = q$, and $f(b) = p$. Then $f$ is $gB$-s-bi-continuous but not s-bi-continuous. This map is also $gB$-bi-continuous but not bi-continuous.

**Theorem 7.5.26** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a map.

(i) If $f$ is $w$-bi-continuous then $f$ is $gB$-bi-continuous.

(ii) If $f$ is $w$-s-bi-continuous then $f$ is $gB$-s-bi-continuous.

**Proof.** (i) Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $w$-bi-continuous map. Then $f$ is $C(2, 1)$-$\sigma_1$-continuous and $C(1, 2)$-$\sigma_2$-continuous and so by Theorem 7.5.5, $f$ is $D_B(1, 2)$-$\sigma_2$-continuous and $D_B(2, 1)$-$\sigma_1$-continuous. Thus $f$ is $gB$-bi-continuous.

(ii) Similar to (i), using Theorem 7.5.5.

The converse of this Theorem 7.5.26 need not be true in general as seen from the following Example.

**Example 7.5.27** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}$, and $Y = \{p, q\}$, $\sigma_1 = \{Y, \phi, \{p\}$ and $\sigma_2 = \{Y, \phi, \{q\}$. Define a map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a) = f(c) = q$, and $f(a) = f(d) = q$. Then this function $f$ is $gB$-bi-continuous but not $w$-bi-continuous, since $f$ is not $C(1, 2)$-$\sigma_2$-continuous. This map is also $gB$-s-bi-continuous but not $w$-s-bi-continuous.

**Theorem 7.5.28** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a map.

(i) If $f$ is $gB$-bi-continuous then $f$ is $rg$-bi-continuous.

(ii) If $f$ is $gB$-s-bi-continuous then $f$ is $rg$-s-bi-continuous.
Proof. (i) Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be \( gB \)-bi-continuous map. Then \( f \) is \( D_B(2, 1) - \sigma_1 \)-continuous and \( D_B(1, 2) - \sigma_2 \)-continuous and so by Theorem 7.5.7, \( f \) is \( D_i(1, 2) - \sigma_2 \)-continuous and \( D_e(2, 1) - \sigma_1 \)-continuous. Therefore \( f \) is \( rg \)-bi-continuous.

(ii) Similar to (i), using Theorem 7.5.7.

The converse of this Theorem 7.5.28 need not be true in general as seen from the following Example.

Example 7.5.29 Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\} \), and \( Y = \{p, q\} \), \( \sigma_1 = \{Y, \phi, \{q\}\} \) and \( \sigma_2 = \{Y, \phi, \{p\}\} \). Define a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(b) = p \), and \( f(c) = f(d) = q \). Then \( f \) is \( rg \)-bi-continuous but it is not \( gB \)-bi-continuous, since \( f \) is not \( D_B(1, 2) - \sigma_1 \)-continuous. This map is also \( rg \)-s-bi-continuous but not \( gB \)-s-bi-continuous.

Remark 7.5.30 The following diagram summarizes the above discussions.

\[
\begin{array}{cccccc}
\text{bi-continuity} & \text{w-bi-continuity} & \text{gB-bi-continuity} & \text{rg-bi-continuity} \\
\text{s-bi-continuity} & \text{ws-bi-continuity} & \text{gB-s-bi-continuity} & \text{rgs-bi-continuity}
\end{array}
\]

Definition 7.5.31 A map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called pairwise \( gB \)-irresolute if \( f^{-1}(A) \in D_B(i, j) \) in \( (X, \tau_1, \tau_2) \) for every \( A \in D_B(k, e) \) in \( (Y, \sigma_1, \sigma_2) \).

Remark 7.5.32 If \( \tau_1 = \tau_2 \) and \( \sigma_1 = \sigma_2 \) simultaneously, then \( f \) becomes a \( gB \)-irresolute map.

Theorem 7.5.33 If a map \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise \( gB \)-irresolute, then \( f \) is \( D_B(i, j) - \sigma_e \)-continuous.
Theorem 7.2.10. By hypothesis, therefore a \text{Theorem 7.5.36}

Proof. Let \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) be pairwise \( gB \)-irresolute and \( F \) be a \( \sigma_e \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( F \) is \((k, e)\)-\( gB \)-closed in \( (Y, \sigma_1, \sigma_2) \) by Theorem 7.2.10. By hypothesis, \( f^{-1}(F) \) is \((i, j)\)-\( gB \)-closed set in \( (X, \tau_1, \tau_2) \). Therefore \( f \) is \( D_B(i, j)\)-\( \sigma_e \)-continuous.

The converse of this Theorem 7.5.33 is not true in general as seen from the following Example.

Example 7.5.34 Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \), and \( Y = \{p, q\}, \sigma_1 = \{Y, \emptyset\} \) and \( \sigma_2 = \{Y, \emptyset, \{p\}\} \). Define a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(b) = p \), and \( f(c) = f(d) = q \). Then \( f \) is \((1, 2)\)-\( gB \)-\( \sigma_2 \)-continuous but it is not pairwise \( gB \)-irresolute, since for the \((1, 2)\)-\( gB \)-closed set \( \{p\} \) in \( (Y, \sigma_1, \sigma_2) \), \( f^{-1}(\{p\}) = \{a, b\} \) which is not \((1, 2)\)-\( Bg \)-closed set in \( (X, \tau_1, \tau_2) \).

Theorem 7.5.35 The following statements are equivalent
(i) A map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise \( gB \)-irresolute
(ii) The inverse image of every \((k, e)\)-\( Bg \)-open set in \((Y, \sigma_1, \sigma_2)\) is \((i, j)\)-\( Bg \)-open set in \((X, \tau_1, \tau_2)\).

Proof. Proof is similar to that of Theorem 7.5.19.

Theorem 7.5.36 If \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) are two pairwise \( gB \)-irresolute maps, then their composition \( g \circ f \) is also pairwise \( gB \)-irresolute.

Proof. Let \( A \in D_B(m, n) \) in \((Z, \eta_1, \eta_2)\). Since \( g \) is pairwise \( gB \)-irresolute, \( g^{-1}(A) \in D_B(k, e) \) in \((Y, \sigma_1, \sigma_2)\). Since \( f \) is pairwise \( gB \)-irresolute, \( f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A) \in D_B(i, j) \). Hence \( g \circ f \) is pairwise \( gB \)-irresolute.

Theorem 7.5.37 If a map \( f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is pairwise \( gB \)-irresolute and \( g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2) \) is \( D_B(k, e)\)-\( \eta_m \)-continuous, then \( g \circ f : (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2) \) is \( D_B(i, j)\)-\( \eta_m \)-continuous.
Proof. Let \( F \) be a \( \eta_n \)-closed set in \((Z, \eta_1, \eta_2)\). Since \( g \) is \( D_B(k, e) - \eta_n \)-continuous, \( g^{-1}(F) \in D_B(k, e) \) in \((Y, \sigma_1, \sigma_2)\). Since \( f \) is pairwise \( gB \)-irresolute, \( f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A) \in D_B(i, j) \) in \((X, \tau_1, \tau_2)\) and hence \( g \circ f \) is \( D_B(i, j) - \eta_n \)-continuous. \( \blacksquare \)