Chapter 6

New Approaches for Homeomorphism in Topological Spaces

6.1 Introduction

The study of generalized closed ($g$-closed) sets in a topological space was initiated by Levine 1963 and concept of $T_2$-spaces was introduced. Balachandran et al. 1991 [9], introduced the concept of generalized continuous maps, generalized homeomorphism in a topological space. In this chapter we shall introduce a new class of open sets namely $B^{**}$-open sets and investigate some of their properties. Further, we introduce the concept of $B^{**}$-continuous maps which includes the class of continuous maps in a new topological space. Also we introduce $B^{**}$-irresolute maps analogy to irresolute maps in a topological space and investigate some of their properties. Moreover, we introduce the concept of $B^{**}$-compactness and $B^{**}$-connectedness.
on a topological space. Finally, we introduce a new class of maps namely $B^{**}$-homeomorphism in a topological space and study some of their properties.

6.2 $B^{**}$-Open Set

**Definition 6.2.1** Let $(X, \tau)$ be a topological space and a subset $A$ of $X$ is said to be $B^{**}$-open if and only if there exist an open set $U$ of $X$ such that $U \subseteq A \subseteq Bcl(U)$.

**Example 6.2.2** If $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, a non open set $\{b\}$ then $B^{**}$-open sets of $(X, \tau)$ are $\{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$.

**Theorem 6.2.3** Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is $B^{**}$-open in $X$ if and only if $A \subseteq Bcl(int(A))$.

**Proof.** If $A$ is $B^{**}$-open set of $X$, then there exists an open set $U$ such that $U \subseteq A \subseteq Bcl(U)$, $U \subseteq A$, implies $U \subseteq int(A)$. Hence $Bcl(U) \subseteq Bcl(int(A))$. Therefore $A \subseteq Bcl(int(A))$.

Conversely, let $A \subseteq Bcl(int(A))$. To prove $A$ is a $B^{**}$-open set in $X$, let $U = int(A)$. Then $U \subseteq A \subseteq Bcl(U)$. Hence $A$ is $B^{**}$-open set in $X$.

**Remark 6.2.4** If $U$ is an open set in $(X, \tau)$, then $U$ is $B^{**}$-open set.

**Proof.** Let $U$ be an open set in $X$, it implies $U = int(U) \subseteq Bcl(int(U))$. Hence $U$ is a $B^{**}$-open set in $X$. 

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Remark 6.2.5 The following example shows that the converse of the above remark need not be true.

Example 6.2.6 Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \), then \( A = \{a, c\} \) is a \( B^{**} \)-open set in \( X \) but not open in \( X \).

Definition 6.2.7 A topological space \((X, \tau)\) is said to be \( B^{**}-T_{\frac{1}{2}} \) space if every \( B^{**} \)-open set of \( X \) is open in \( X \).

Theorem 6.2.8 If \( A \) and \( B \) are \( B^{**} \)-open sets of a topological space \( X \), then \( A \cup B \) is also \( B^{**} \)-open set in \( X \).

Proof. Given \( A \) and \( B \) are \( B^{**} \)-open set of \( X \), then there exists an open set \( U \) and \( V \) respectively such that \( U \subseteq A \subseteq Bcl(U) \) and \( V \subseteq B \subseteq Bcl(V) \) and also \( Bcl(U \cup V) = [Bcl(U)] \cup [Bcl(V)] \). Hence \( U \cup V \subseteq A \cup B \subseteq Bcl(U \cup V) \). Hence \( A \cup B \) is also \( B^{**} \)-open set in \( X \).

Remark 6.2.9 The following example shows that if \( A \) and \( B \) are \( B^{**} \)-open in \( X \), then \( A \cap B \) need not be \( B^{**} \)-open set in \( X \).

Example 6.2.10 Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \), then \( A = \{a, d\} \) and \( B = \{b, d\} \) are \( B^{**} \)-open set in \( X \) and \( A \cap B = \{d\} \) is not a \( B^{**} \)-open set in \( X \) with a non open set \( \{a, c\} \).

Theorem 6.2.11 Let \( A \) be a \( B^{**} \)-open set in \( X \) and \( B \) be any set such that \( A \subseteq B \subseteq Bcl(int(A)) \) then \( B \) is also a \( B^{**} \)-open set in \( X \).
Proof. Given $A$ is $B^{**}$-open set in $X$, therefore by Theorem 6.2.3 $A \subseteq Bcl(int(A))$. $A \subseteq B$ implies $int(A) \subseteq int(B)$, hence, $Bcl(int(A)) \subseteq Bcl(int(B))$. Therefore, $B \subseteq Bcl(int(A)) \subseteq Bcl(int(B))$. Hence $B$ is a $B^{**}$-open set in $X$.

Theorem 6.2.12 Let $X$ be a topological space and if $A$ is $B^{**}$-open set in $X$, then $A$ is semi open in $X$.

Proof. Given $A$ is $B^{**}$-open set in $X$, therefore there exists an open set $U$ such that $U \subseteq A \subseteq Bcl(U)$. Since $Bcl(U) \subseteq cl(U)$. Hence $U \subseteq A \subseteq cl(U)$, implies $A$ is semi-open.

Remark 6.2.13 The converse of the above Theorem 6.2.12 need not be true for from the following example.

Example 6.2.14 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, then $A = \{b, c, d\}$ is a semi open set but not a $B^{**}$-open set with a non open set $B = \{a, c\}$.

Remark 6.2.15 In $T_2$ space every semi open set is $B^{**}$-open set.

Theorem 6.2.16 Let $f : X \rightarrow Y$ be $B$-continuous open map. If $A$ is a $B^{**}$-open set in $X$, then $f(A)$ is a semi open set in $Y$.

Proof. Given $A$ is $B^{**}$-open set in $X$, therefore there exists an open set $U$ such that $U \subseteq A \subseteq Bcl(U)$. Also we have $f(Bcl(A)) \subseteq cl(f(A))$. Hence $f(U) \subseteq f(A) \subseteq f(Bcl(U)) \subseteq cl(f(U))$. Since $f$ is an open map, $f(U)$ is open in $Y$. This implies $f(A)$ is a semi open set in $Y$. 

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Theorem 6.2.17 Let $f : X \rightarrow Y$ be a homeomorphism from a topological space $X$ into a topological space $Y$. If $A$ is a $B^{**}$-open set in $X$, then $f(A)$ is $B^{**}$-open set in $Y$.

Proof. Given $A$ is $B^{**}$-open set in $X$, therefore there exists an open set $U$ such that $U \subseteq A \subseteq Bcl(U)$, implies $f(U) \subseteq f(A) \subseteq f(Bcl(U))$. Since $f$ is a homeomorphism and also we have $f(Bcl(U)) \subseteq Bcl(f(U))$. Therefore $f(U) \subseteq f(A) \subseteq Bcl(f(U))$, and hence $f(A)$ is $B^{**}$-open set in $Y$. ■

Theorem 6.2.18 Let $f : X \rightarrow Y$ be a homeomorphism from a topological space $X$ into a topological space $Y$. If $A$ is a $B^{**}$-open set in $Y$, then $f^{-1}(A)$ is $B^{**}$-open in $X$.

Proof. Given $A$ is $B^{**}$-open set in $Y$, therefore there exist an open set $U$ in $Y$ such that $U \subseteq A \subseteq Bcl(U)$ implies $f^{-1}(U) \subseteq f^{-1}(A) \subseteq f^{-1}(Bcl(U))$. Since $f$ is a homeomorphism and also we have $f^{-1}(Bcl(U)) \subseteq Bcl(f^{-1}(U))$. Therefore $f^{-1}(U) \subseteq f^{-1}(A) \subseteq Bcl(f^{-1}(U))$, hence $f^{-1}(A)$ is $B^{**}$-open in $X$. ■

Definition 6.2.19 A subset $A$ of $X$ is said to be $B^{**}$-closed iff its complement is $B^{**}$-open set.

Definition 6.2.20 Let $(X, \tau)$ be a topological space. Let $A$ be a subset of $X$. Then the $B^{**}$-closure of $A$ is defined as intersection of all $B^{**}$-closed sets containing $A$ and it is denoted by $B^{**}cl(A)$. That is $B^{**}cl(A) = \cap\{F : F$ is $B^{**}$-closed and $A \subseteq F\}$.

Remark 6.2.21 By the above Definition $B^{**}cl(A)$ is the smallest $B^{**}cl(A)$-closed set containing $A$. 73
Definition 6.2.22 Let \((x, \tau)\) be a topological space and let \(A\) be a subset of \(X\). Let \(x \in X\) is said to be \(B^{**}\)-limit point of \(A\) if and only if every \(B^{**}\)-open set containing \(x\) contains at least one point other than \(x\).

Definition 6.2.23 Let \(A\) be a subset of a topological space \((X, \tau)\). Then the set of all \(B^{**}\)-limit points of \(A\) is said to be \(B^{**}\)-derived set of \(A\) it is denoted by \(DB^{**}(A)\).

Theorem 6.2.24 Let \(A\) be a subset of a topological space \((X, \tau)\), then \(x \in B^{**}cl(A)\) iff every \(B^{**}\)-open set \(U\) contains \(x\) intersect with \(A\).

Proof. We prove this Theorem in contra positive way. If \(x \notin B^{**}cl(A)\), then \(x \in X - B^{**}cl(A)\). Let \(U = X - B^{**}cl(A)\) then by remark 6.2.21 \(U\) is \(B^{**}\)-open set which does not intersect \(A\). This implies \(x \notin B^{**}cl(A)\). Conversely, if \(U\) is \(B^{**}\)-open set of \(x\) which does not intersect with \(A\) then \(X - U\) is a \(B^{**}\)-closed set containing \(A\). This implies \(x \notin B^{**}cl(A)\).

Theorem 6.2.25 Let \(A\) be a subset of a topological space \((X, \tau)\), let \(DB^{**}(A)\) be set of all \(B^{**}\)-limit points of \(A\). Then \(B^{**}cl(A) = A \cup DB^{**}(A)\).

Proof. Let \(x \in A \cup DB^{**}(A)\), this implies either \(x \in A\) or \(x \in DB^{**}(A)\). If \(x \in A\), then \(x \in B^{**}cl(A)\). If \(x \in DB^{**}(A)\), then every \(B^{**}\)-open set contains \(x\) will intersect with \(A\). Therefore \(x \in B^{**}cl(A)\). This implies \(A \cup DB^{**}(A) \subseteq B^{**}cl(A)\).

If \(x \in B^{**}(A)\), then we have to prove \(x \in A \cup DB^{**}(A)\). If \(x \in A\) then \(x \in A \cup DB^{**}(A)\). If \(x \notin A\), since \(x \in B^{**}cl(A)\) implies every \(B^{**}\)-open set of \(x\) intersects with \(A\). Hence \(x \in DB^{**}(A)\). Therefore \(B^{**}cl(A) = A \cup DB^{**}(A)\).
6.3 \( B^{**} \)-Continuous Map

**Definition 6.3.1** A function \( f : X \rightarrow Y \) is said to be \( B^{**} \)-continuous map if the inverse image of every open set in \( Y \) is \( B^{**} \)-open in \( X \).

**Theorem 6.3.2** Let \( f : X \rightarrow Y \) be a continuous map from \( X \) into \( Y \) then it is \( B^{**} \)-continuous.

**Proof.** Let \( U \) be an open set in \( Y \). Since \( f \) is continuous, \( f^{-1}(U) \) is open in \( X \). By the remark 6.2.4 \( f^{-1}(U) \) is \( B^{**} \)-open in \( X \). Hence \( f \) is \( B^{**} \)-continuous. \( \blacksquare \)

**Remark 6.3.3** The following example shows that the converse of the above Theorem 6.3.2 need not be true.

**Example 6.3.4** Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \), \( Y = \{p, q\} \) and \( \sigma = \{Y, \phi, \{p\}\} \). Let \( f : X \rightarrow Y \) be a map defined by \( f(a) = f(c) = p \), \( f(b) = q \), then \( f \) is \( B^{**} \)-continuous but it is not a continuous map.

**Theorem 6.3.5** Let \( f : X \rightarrow Y \) be a mapping from a topological space \( X \) into a topological space \( Y \), then the following statements are equivalent

(i) \( f \) is \( B^{**} \)-continuous,

(ii) the inverse image of each closed set in \( Y \) is \( B^{**} \)-closed in \( X \).

**Proof.** \((i) \rightarrow (ii)\) Let \( C \) be any closed set in \( Y \), then \( Y - C \) is open in \( Y \). Since \( f \) is \( B^{**} \)-continuous, \( f^{-1}(Y - C) \) is \( B^{**} \)-open in \( X \). Therefore \( X - f^{-1}(C) \) is \( B^{**} \)-open in \( X \) which implies \( f^{-1}(C) \) is \( B^{**} \)-closed set in \( X \).
(ii) → (i) Let $U$ be an open set in $Y$, then $Y - U$ is closed in $Y$. This implies $f^{-1}(Y - U)$ is $B^{**}$-closed in $X$, which implies $X - f^{-1}(U)$ is $B^{**}$-closed in $X$. Therefore $f^{-1}(U)$ is $B^{**}$-open in $X$. Hence $f$ is $B^{**}$-continuous. \[\square\]

**Theorem 6.3.6** If $f : X \to Y$ is a $B^{**}$-continuous map from $X$ into $Y$ then $f(B^{**}cl(A)) \subseteq cl(f(A))$.

**Proof.** Since $f(A) \subseteq cl(f(A))$, implies $A \subseteq f^{-1}(cl(f(A))$. But $cl(f(A))$ is a closed set in $Y$ and $f$ is $B^{**}$-continuous map. Therefore $f^{-1}(cl(f(A))$ is $B^{**}$-closed in $X$. Hence $B^{**}cl(A) \subseteq f^{-1}(cl(f(A)))$. Therefore $f(B^{**}cl(A)) \subseteq cl(f(A))$.

\[\square\]

**Theorem 6.3.7** If $f : X \to Y$ be a mapping from a topological space $X$ into a topological space $Y$, then the following statements are equivalent

(i) For each $x \in X$ and each open set $V$ containing $f(x)$, there exists a $B^{**}$-open set $U$ containing $x$ such that $f(U) \subseteq V$

(ii) $f(B^{**}cl(A)) \subseteq cl(f(A))$ for all subset $A$ of $X$.

**Proof.** (ii) → (i) Let $x \in X$ and $V$ be an open set containing $f(x)$, then $f^{-1}(V)$ is $B^{**}$-open in $X$. Let $A = X - f^{-1}(V)$ then $A$ is $B^{**}$-closed in $X$. Since $f(B^{**}cl(A)) \subseteq cl(f(A))$ implies $f(B^{**}cl(A)) \subseteq cl(f(X - f^{-1}(V))) \subseteq cl(Y - V) = V'$. Since $x \in V$, $x \notin V'$ hence $x \notin f(B^{**}cl(A))$. Therefore there exist a $B^{**}$-open set $U$ of $x$ such that $U \cap A' = \emptyset$ which implies $U \subseteq A'$. Hence $f(U) \subseteq f(A') \subseteq V$.

(i) ⇒ (ii) Let $y \in f(B^{**}cl(A))$, therefore, there exist $x \in B^{**}cl(A)$ such that $f(x) = y$. Let $V$ be a any open set containing $f(x)$, then by hypothesis there exist a $B^{**}$-open set $U$ containing $x$ such that $f(U) \subseteq V$ and $U \cap A \neq \emptyset$ which
implies $f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A) \neq \phi$. Therefore $x \in cl(f(A))$. Hence
\[ f(B^{**}cl(A)) \subseteq clf(A). \]

6.4 Relation Between $B^{**}$-Continuous Maps and $B^{**}$-Irresolute Maps

Definition 6.4.1 A map $f : X \to Y$ is called $B^{**}$-irresolute if the inverse image of every $B^{**}$-open set of $Y$ is $B^{**}$-open in $X$.

Remark 6.4.2 A map $f : X \to Y$ is $B^{**}$-irresolute map if and only if the inverse image of every $B^{**}$-closed set in $Y$ is $B^{**}$-closed in $X$.

Theorem 6.4.3 If $f : X \to Y$ is a $B^{**}$-irresolute map, then $f$ is $B^{**}$-continuous.

Proof. Let $F$ be an open set in $X$. Since $f$ is $B^{**}$-irresolute map, implies $f^{-1}(F)$ is $B^{**}$-open in $X$. Hence $f$ is $B^{**}$-continuous.

Remark 6.4.4 The following example shows that the converse of the above Theorem 6.4.3 need not be true.

Example 6.4.5 Let $X = Y = \{a, b, c\}$, and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Let $f : X \to Y$ be a map defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$ then $f$ is $B^{**}$-continuous but $B^{**}$-irresolute with nonopen set $\{c\}$.

Theorem 6.4.6 Let $f : X \to Y$ be a $B^{**}$-continuous map from $X$ into $Y$ and $Y$ is $B^{**}$-$T_{\frac{1}{2}}$-space then $f$ is $B^{**}$-irresolute.
Proof. Let $A$ be $B^\ast\ast$-open set in $Y$. Since $Y$ is $B^\ast\ast$-$T_2$ implies $A$ is an open set in $Y$. Since $f$ is $B^\ast\ast$-continuous, implies $f^{-1}(A)$ is $B^\ast\ast$-open in $X$. Therefore, $f$ is $B^\ast\ast$-irresolute map. \hfill

**Theorem 6.4.7** Let $X$, $Y$, $Z$ be any topological spaces. For any $B^\ast\ast$-irresolute map $f : X \to Y$ and any $B^\ast\ast$-irresolute map $g : Y \to Z$ then the composition $g \circ f : X \to Z$ is $B^\ast\ast$-irresolute.

Proof. Let $U$ be a $B^\ast\ast$-open set in $Z$, then $g^{-1}(U)$ is $B^\ast\ast$-open in $Y$ which implies $f^{-1}(g^{-1}(U))$ is $B^\ast\ast$-open in $X$. Therefore, $(g \circ f)^{-1}(U)$ is $B^\ast\ast$-open in $X$. Hence $g \circ f$ is $B^\ast\ast$-irresolute. \hfill

6.5 $B^\ast\ast$-Compact Sets

**Definition 6.5.1** A collection $\{A_\alpha : \alpha \in J\}$ of $B^\ast\ast$-open sets is said to be $B^\ast\ast$-open cover for a subset $B$ of $X$ if $B \subseteq \bigcup \{A_\alpha : \alpha \in J\}$ holds.

**Definition 6.5.2** A topological space $X$ is said to be $B^\ast\ast$-compact if for every $B^\ast\ast$-open cover of $X$ has finite subcover.

**Definition 6.5.3** A subset $B$ of $X$ is said to be $B^\ast\ast$-compact relative to $X$, if for every collection $\{A_\alpha : \alpha \in J\}$ of $B^\ast\ast$-open subsets of $X$ such that $B \subseteq \bigcup \{A_\alpha : \alpha \in J\}$, then there exists a finite subcollection such that $B \subseteq A_1 \cup A_2 \cup \ldots \cup A_n$.

**Theorem 6.5.4** If $A$ is $B^\ast\ast$-closed subset of a $B^\ast\ast$-compact space $X$, then $A$ is $B^\ast\ast$-compact relative to $X$.
Proof. Let \( A \) be a \( B^{**} \)-closed subset of \( X \). Then \( A' \) is a \( B^{**} \)-open set in \( X \). Let \( \{ A_\alpha : \alpha \in J \} \) be a \( B^{**} \)-open cover for \( A \), then \( \{ A'_\alpha : A_\alpha \in \alpha \in J \} \) forms a \( B^{**} \)-open cover for \( X \). Since \( X \) is a \( B^{**} \)-compact, then \( B^{**} \)-open cover has a finite subcover \( \{ G_1, G_2, \ldots, G_n \} \). If this finite subcover contains \( A' \) discard it otherwise leave the subcover as it is. Thus we obtained a finite \( B^{**} \)-open subcover for \( A \). Therefore \( A \) is compact relative to \( X \).

Theorem 6.5.5 The \( B^{**} \)-continuous image of \( B^{**} \)-compact space is compact.

Proof. Let \( f : X \rightarrow Y \) be \( B^{**} \)-continuous map from \( X \) onto \( Y \). Let \( \{ A_\alpha : \alpha \in J \} \) be an open cover for \( Y \). Then \( \{ f^{-1}(A_\alpha) : \alpha \in J \} \) is a \( B^{**} \)-open cover for \( X \). Since \( X \) is \( B^{**} \)-compact. Therefore this \( B^{**} \)-open cover of \( X \) has a finite subcover \( \{ f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n) \} \). Since \( f \) is onto \( \{ A_1, A_2, \ldots, A_n \} \) be an open cover of \( Y \). Therefore \( Y \) is compact.

Theorem 6.5.6 If \( f : X \rightarrow Y \) is \( B^{**} \)-irresolute map and a subset \( B \) of \( X \) is \( B^{**} \)-compact relative to \( X \), then the image \( f(B) \) is compact relative to \( Y \).

Proof. Given \( f : X \rightarrow Y \) is \( B^{**} \)-irresolute map from \( X \) onto \( Y \). Let \( \{ A_\alpha : \alpha \in J \} \) be an \( B^{**} \)-open cover for \( f(B) \) relative to \( Y \). Then \( \{ f^{-1}(A_\alpha) : \alpha \in J \} \) is a \( B^{**} \)-open cover for \( B \) relative to \( X \). Since \( B \) is \( B^{**} \)-compact relative to \( X \), this \( B^{**} \)-open cover has a finite subcover \( \{ f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n) \} \). Since \( f \) is onto, therefore \( \{ A_1, A_2, \ldots, A_n \} \) is \( B^{**} \)-open cover for \( f(B) \). Hence \( f(B) \) is \( B^{**} \)-compact.
6.6 $B^{**}$-Connected Sets

**Definition 6.6.1** A topological space $X$ is said to be $B^{**}$-connected if $X$ cannot be written as disjoint union of two nonempty $B^{**}$-open sets. A subset of $X$ is $B^{**}$-connected if it is $B^{**}$-connected as a subspace.

**Theorem 6.6.2** For a topological space $X$, the following statements are equivalent.

(i) $X$ is $B^{**}$-connected.

(ii) The only subsets of $X$ which are both $B^{**}$-open and $B^{**}$-closed are empty set and $X$.

(iii) Every $B^{**}$-continuous map of $X$ into a discrete space $Y$ with at least two points is a constant map.

**Proof.** (i) $\Rightarrow$ (ii) Let $U$ be a $B^{**}$-open and $B^{**}$-closed subset of $X$, then $X - U$ is both $B^{**}$-open and $B^{**}$-closed. Since $X$ is the disjoint union of $B^{**}$-open set $U$ and $X - U$ implies one of these must be empty, that is $U = \emptyset$ or $X - U = \emptyset$.

(ii) $\Rightarrow$ (i) Suppose $X = A \cup B$ where $A$ and $B$ are disjoint non-empty $B^{**}$-open set of $X$, then $A = X - B$ is $B^{**}$-closed. Hence $A$ is both $B^{**}$-open and $B^{**}$-closed subset of $X$, by assumption $A = \emptyset$ or $A = X$. This implies $X$ is $B^{**}$-connected.

(ii) $\Rightarrow$ (iii) Let $f : X \rightarrow Y$ be a $B^{**}$-continuous, then $X$ is covered by $B^{**}$-open and $B^{**}$-closed covering $\{f^{-1}(y) : y \in Y\}$. By assumption $f^{-1}(y) = \emptyset$, then $f$ fails to be $B^{**}$-continuous. Therefore, $f^{-1}(y) = X$. This implies $f$ is a constant map.

(iii) $\Rightarrow$ (ii) Let $U$ be both $B^{**}$-open and $B^{**}$-closed in $X$. Suppose $U \neq \emptyset$. Let
$f : X \to Y$ be $B^{**}$-continuous map defined by $f(U) = \{y\}$ and $f(X - U) = \{w\}$ for some distinct points $y$ and $w$ in $Y$. By assumption $f$ is a constant map, therefore, we have $U = X$. 

**Theorem 6.6.3** (i) If $f : X \to Y$ is a $B^{**}$-continuous surjection map and $X$ is $B^{**}$-connected, then $Y$ is connected. (ii) If $f : X \to Y$ is a $B^{**}$-irresolute surjection map and $X$ is $B^{**}$-connected, then $Y$ is $B^{**}$-connected.

**Proof.** (i) Suppose that $Y$ is not connected, then $Y = A \cup B$, where $A$ and $B$ are disjoint nonempty open sets in $Y$. Since $f$ is $B^{**}$-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A) \cup f^{-1}(B)$ are disjoint nonempty $B^{**}$-open sets which is a contradiction to our assumption, that $X$ is $B^{**}$-connected. Hence $Y$ is connected.

(ii) It follows from the definition 6.6.1.

#### 6.7 $B^{**}$-Homeomorphism

**Definition 6.7.1** A map $f : X \to Y$ is said to be $B^{**}$-open map if $f(U)$ is $B^{**}$-open in $Y$ for every open set $U$ in $X$.

**Theorem 6.7.2** If $f : X \to Y$ is an open map, then it is $B^{**}$-open map.

**Proof.** Given $f : X \to Y$ is an open map. Let $G$ be any open set in $X$, then $f(G)$ is open in $Y$. By remark 6.2.4 $f(G)$ is $B^{**}$-open in $Y$. Hence $f$ is $B^{**}$-open map.
Remark 6.7.3 The following example shows that the converse of the above Theorem 6.7.2 need not be true.

Example 6.7.4 Let $X = Y = \{a, b, c\}$, and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\} \{a, b\}\}$, then define $f : X \to Y$ as $f(a) = a$, $f(b) = c$ and $f(c) = b$ then $f$ is $B^{**}$-open but not open from $X$ into $Y$.

Definition 6.7.5 A map $f : X \to Y$ is said to be $B^{**}$-closed map if $f(U)$ is $B^{**}$-closed in $Y$ for every closed set $U$ in $X$.

Remark 6.7.6 If $f : X \to Y$ is closed map then it is a $B^{**}$-closed map and converse need not be true.

Proof. Similar proof of the above Theorem 6.7.2.

Definition 6.7.7 A bijection map $f : (X, \tau) \to (Y, \sigma)$ is called $B^{**}$-homeomorphism if $f$ is both $B^{**}$-continuous and $B^{**}$-open.

Remark 6.7.8 Every homeomorphism is a $B^{**}$-homeomorphism but converse need not be true.

Theorem 6.7.9 For any bijection map $f : X \to Y$ the following statement are equivalent: (i) the inverse map $f^{-1} : Y \to X$ is $B^{**}$-continuous, (ii) $f$ is $B^{**}$-open map, (iii) $f$ is $B^{**}$-closed map.

Proof. $(i) \Rightarrow (ii)$ Let $G$ be any open set in $X$. Since $f^{-1}$ is $B^{**}$-continuous, the inverse image of $G$ under $f^{-1}$, namely $f(G)$ is $B^{**}$-open in $Y$. Hence $f$ is $B^{**}$-open map.
(ii) ⇒ (iii) Let $F$ be any closed set in $X$, then $F'$ is open in $X$. Since $f$ is $B^{**}$-open map, $f(F')$ is $B^{**}$-open in $Y$. But $f(F') = Y - f(F)$ implies $f(F)$ is $B^{**}$-closed set in $Y$. Therefore $f$ is $B^{**}$-closed map.

(iii) ⇒ (i) Let $F$ be any closed set in $X$. Then the inverse image of $F$ under $f^{-1}$, namely $f(F)$ is $B^{**}$-closed in $Y$. Since $f$ is a $B^{**}$-closed map. Therefore $f$ is $B^{**}$-continuous.

Theorem 6.7.10 Let $f : (X, \tau) \to (Y, \sigma)$ be a bijection and $B^{**}$-continuous map, then the following statements are equivalent

(i) $f$ is $B^{**}$-open map.

(ii) $f$ is $B^{**}$-homeomorphism. (iii) $f$ is $B^{**}$-closed map.

Proof. (i) ⇒ (ii) By assumption, $f$ is bijective, $B^{**}$-continuous and $B^{**}$-open. Then by definition, $f$ is a $B^{**}$-homeomorphism.

(ii) ⇒ (iii) By assumption, $f$ is $B^{**}$-open and bijective, By Theorem 6.7.9 $f$ is $B^{**}$-closed map.

(iii) ⇒ (i) By assumption, $f$ is $B^{**}$-closed and bijective, By Theorem 6.7.9 $f$ is a $B^{**}$-open map.