Chapter 4

On Contra $\alpha$-$\mathcal{I}$-Continuous Functions

4.1 Introduction

An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ and if $\mathcal{P}(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^* : \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [36] of $\mathcal{I}$ with respect to $\tau$ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X|U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau|x \in U\}$.

We will make use of the basis facts about the local function [27], Theorem 2.3 without mentioning it explicitly. A Kuratowski closure operator $d^*(\cdot)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the *-topology finer than $\tau$ is defined by $d^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [27]. When there is no chance for confusion, we will simply write $A^*$ for $A^*(\mathcal{I}, \tau)$ and $\tau^*$ or $\tau^*(\mathcal{I})$ for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on $X$, then $(X, \tau, \mathcal{I})$ is called
an ideal topological space or simply an *ideal space*. By a space, we always mean a
topological space \((X, \tau)\) with no separation properties assumed. If \(A \subset X\), \(cl(A)\)
and \(int(A)\) will, respectively, denote the closure and interior of \(A\) in \((X, \tau)\) and
\(int^*(A)\) will denote the interior of \(A\) in \((X, \tau^*)\).

One of the important and basic topic in the theory of classical point set topology
and several branches of mathematics, which have been researched by many authors,
is continuity of functions. This concept has been extended to the setting of \(I\)-
continuity of functions. Jankovic and Hamlett \[27, 28\] introduced the notion of \(I\)-
open sets in topological spaces. Abd El-Monsef et al. \[2\] further investigated \(I\)-
open sets and \(I\)-continuous functions. The notion of semi-\(I\)-open sets to obtain
decomposition of continuity was introduced by Hatir and Noiri \[23, 24\]. In addition
to this, T. Noiri and S. Jafari \[23\], \[26\] have introduced the notions of \(\alpha\)-\(I\)-open
sets, \(\alpha\)-\(I\)-continuous functions and contra-\(\alpha\)-continuous functions.

The purpose of this chapter is to give a new class of functions called contra-\(\alpha\)-\(I\)-
continuous function in an ideal topological space. Some characterizations and several
basic properties of this class of functions are obtained.

A subset \(S\) of an ideal topological space \((X, \tau, \mathcal{I})\) is \(\alpha\)-\(I\)-open \[23\] (resp. \(\beta\)-
\(I\)-open \[23\]) if \(S \subset int(cl^*(int(S)))\) (resp. \(S \subset cl(int(cl^*(S)))\)). The complement
of a \(\alpha\)-\(I\)-open set is called \(\alpha\)-\(I\)-closed \[23\]. The intersection of all \(\alpha\)-\(I\)-closed
sets containing \(S\) is called the \(\alpha\)-\(I\)-closure of \(S\) and is denoted by \(\alpha_{I}cl(S)\). The
\(\alpha\)-\(I\)-interior of \(S\) is defined by the union of all \(\alpha\)-\(I\)-open sets contained in \(S\) and
is denoted by \(\alpha_{I}Int(S)\). The family of all \(\alpha\)-\(I\)-open (resp. \(\alpha\)-\(I\)-closed) sets of
\((X, \tau, \mathcal{I})\) is denoted by \(\alpha I O(X)\) (resp. \(\alpha I C(X)\)). The family of all \(\alpha\)-\(I\)-open
(resp. \(\alpha\)-\(I\)-closed) sets of \((X, \tau, \mathcal{I})\) containing a point \(x \in X\) is denoted by
4.2 Contra-$\alpha$-$\mathcal{I}$-Continuous Functions

We have introduced the following definition

**Definition 4.2.1** A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is called *contra-$\alpha$-$\mathcal{I}$-continuous* if $f^{-1}(V)$ is $\alpha$-$\mathcal{I}$-open in $(X, \tau, \mathcal{I})$ for every closed set $V$ of $Y$.

**Proposition 4.2.2** (i) Every contra-$\alpha$-$\mathcal{I}$-continuous function is contra $\alpha$-continuous.

(ii) Every contra-continuous function is contra-$\alpha$-$\mathcal{I}$-continuous.

**Proof.** Follows from Remark 3.4 in [45].

The converse of the Proposition are need not be true as shown in the following examples.

**Example 4.2.3** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Define $f : (X, \tau, \mathcal{I}) \rightarrow (X, \tau_2)$ by $f(a) = d$, $f(b) = a$, $f(c) = c$ and $f(d) = b$ is contra $\alpha$-continuous but not contra-$\alpha$-$\mathcal{I}$-continuous.

**Example 4.2.4** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{b\}, \{b, d\}, \{b, c, d\}\}$, $\tau_2 = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}, \{d\}\}$. Clearly, the identity function $f : (X, \tau_1, \mathcal{I}) \rightarrow (X, \tau_2)$ is contra $\alpha$-$\mathcal{I}$-continuous but not contra-continuous.
**Lemma 4.2.5** [25] The following properties hold for subsets $A, B$ of a space $X$:

1. $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for any closed set $F$ of $X$ containing $x$;
2. $A \subset \ker(A)$ and $A = \ker(A)$ if $A$ is open in $X$;
3. If $A \subset B$ then $\ker(A) \subset \ker(B)$.

**Theorem 4.2.6** The following are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:

1. $f$ is contra-$\alpha$-$\mathcal{I}$-continuous;
2. for every closed subset $F$ of $Y$, $f^{-1}(F) \in \alpha \mathcal{I}O(X)$;
3. for each $x \in X$ and each closed set $F$ of $Y$ containing $f(x)$, there exists $U \in \alpha \mathcal{I}O(X)$ such that $f(U) \subset F$;
4. $f(\alpha \mathcal{I}cl(A)) \subset \ker(f(A))$ for every subset $A$ of $X$;
5. $\alpha \mathcal{I}cl(f^{-1}(B)) \subset f^{-1}(\ker(B))$ for every subset $B$ of $Y$.

**Proof.** The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2): Let $F$ be any closed set of $Y$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \alpha \mathcal{I}O(X, x)$ such that $f(U_x) \subset F$. Therefore, we obtain $f^{-1}(F) = \bigcup \{U_x | x \in f^{-1}(F)\}$. Therefore $f^{-1}(F) \in \alpha \mathcal{I}O(X)$.

(2) $\Rightarrow$ (4): Let $A$ be any subset of $X$. Suppose that $y \notin \ker(f(A))$. Then by Lemma 4.2.5 there exists a closed set $F$ of $Y$ containing $y$ such that $f(A) \cap F = \emptyset$.

Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $\alpha \mathcal{I}cl(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(\alpha \mathcal{I}cl(A)) \cap F = \emptyset$ and $y \notin f(\alpha \mathcal{I}cl(A))$. This implies that $f(\alpha \mathcal{I}cl(A)) \subset \ker(f(A))$.

(4) $\Rightarrow$ (5): Let $B$ be any subset of $Y$. By (4) and Lemma 4.2.5, we have $f(\alpha \mathcal{I}cl(f^{-1}(B))) \subset \ker(f(f^{-1}(B))) \subset \ker(B)$ and $\alpha \mathcal{I}cl(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

(5) $\Rightarrow$ (1): Let $V$ be any open set of $Y$. Then by Lemma 4.2.5 we have
shows that 

\[ \alpha \text{cl}(f^{-1}(V)) \subset f^{-1}(\text{ker}(V)) = f^{-1}(V) \text{ and } \alpha \text{cl}(f^{-1}(V)) = f^{-1}(V). \]

This shows that \( f^{-1}(V) \) is \( \alpha \)-\( \mathcal{I} \)-closed in \( (X, \tau, \mathcal{I}) \).

\[ \boxed{\text{Theorem 4.2.7} \text{ If a function } f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \text{ is contra-} \alpha \text{-} \mathcal{I} \text{-continuous and } Y \text{ is regular, then } f \text{ is } \alpha \text{-} \mathcal{I} \text{-continuous.} } \]

**Proof.** Let \( x \) be an arbitrary point of \( X \) and \( V \) an open set of \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exists an open set \( W \) in \( Y \) containing \( f(x) \) such that \( cl(W) \subset V \). Since \( f \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous, so by Theorem 4.2.6 there exists \( U \in \alpha \mathcal{I}O(X, \tau) \) such that \( f(U) \subset cl(W) \). Then \( f(U) \subset cl(W) \subset V \).

Hence, \( f \) is \( \alpha \)-\( \mathcal{I} \)-continuous.

\[ \boxed{\text{Definition 4.2.8} \text{ A function } f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \text{ is said to satisfy the } \alpha \text{-} \mathcal{I} \text{-interiority condition if } \alpha \mathcal{I} \text{Int}(f^{-1}(cl(V))) \subset f^{-1}(V) \text{ for each open set } V \text{ of } (Y, \sigma).} \]

**Theorem 4.2.9**  If \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) is a contra-\( \alpha \)-\( \mathcal{I} \)-continuous function and satisfies the \( \mathcal{I} \)-interiority condition, then \( f \) is \( \alpha \)-\( \mathcal{I} \)-continuous.

**Proof.** Let \( V \) be any open set of \( Y \). Since \( f \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous and \( cl(V) \) is closed, by Theorem 4.2.6 \( f^{-1}(cl(V)) \) is \( \alpha \)-\( \mathcal{I} \)-open in \( X \). By hypothesis of \( f \),

\[ f^{-1}(V) \subset f^{-1}(cl(V)) \subset \alpha \mathcal{I} \text{Int}(f^{-1}(cl(V))) \subset \alpha \mathcal{I} \text{Int}(f^{-1}(V)) \subset f^{-1}(V). \]

Therefore, we obtain \( f^{-1}(V) = \alpha \mathcal{I} \text{Int}(f^{-1}(V)) \) and consequently \( f^{-1}(V) \in \beta \mathcal{I}O(X) \). This shows that \( f \) is a \( \alpha \mathcal{I} \)-continuous function.

**Theorem 4.2.10**  Let \( (X, \tau, \mathcal{I}) \) be any ideal topological space and let \( f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \) be a function and \( g : X \to X \times Y \) be the graph function, given by \( g(x) = (x, f(x)) \) for every \( x \in X \). Then \( f \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous if and only if \( g \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous.
**Proof.** Let $x \in X$ and let $W$ be a closed subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is closed in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to $Y$. Hence $\{y \in Y|(x, y) \in W\}$ is a closed subset of $Y$. Since $f$ is contra-$\alpha$-$\mathcal{I}$-continuous, $\bigcup\{f^{-1}(y) \in Y|(x, y) \in W\}$ is a $\alpha$-$\mathcal{I}$-open subset of $(X, \tau, \mathcal{I})$. Further, $x \in \bigcup\{f^{-1}(y)|(x, y) \in W\} \subset g^{-1}(W)$. Hence $g^{-1}(W)$ is $\alpha$-$\mathcal{I}$-open. Then $g$ is contra-$\alpha$-$\mathcal{I}$-continuous. Conversely, let $F$ be a closed subset of $X \times Y$. Since $g$ is contra-$\alpha$-$\mathcal{I}$-continuous, $g^{-1}(X \times F)$ is a $\alpha$-$\mathcal{I}$-open subset of $X$. Also, $g^{-1}(X \times F) = f^{-1}(F)$. Hence $f$ is contra-$\alpha$-$\mathcal{I}$-continuous.

**Definition 4.2.11** An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\alpha$-$\mathcal{I}$-$T_2$ if for each distinct point $x, y \in X$, there exist $U, V \in \alpha\mathcal{IO}(X)$ containing $x$ and $y$, respectively, such that $U \cap V = \phi$.

**Theorem 4.2.12** If $(X, \tau, \mathcal{I})$ is an ideal topological space and for each pair of distinct points $x_1$ and $x_2$ in $X$ there exists a function $f$ into a Urysohn space $(Y, \sigma)$ such that $f(x_1) \neq f(x_2)$ and $f$ is contra-$\alpha$-$\mathcal{I}$-continuous at $x_1$ and $x_2$, then the space $(X, \tau, \mathcal{I})$ is $\alpha$-$\mathcal{I}$-$T_2$.

**Proof.** Let $x_1$ and $x_2$ be any distinct points in $X$. Then by hypothesis there is a Urysohn space $(Y, \sigma)$ and a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$, which satisfies the conditions of this theorem. Let $y_i = f(x_i)$ for $i = 1, 2$. Then $y_1 \neq y_2$. Since $(Y, \sigma)$ is Urysohn, there exists open neighbourhoods $U_{y_1}$ and $U_{y_2}$ of $y_1$ and $y_2$ respectively in $Y$ such that $cl(U_{y_1}) \cap cl(U_{y_2}) = \phi$. Since $f$ is contra-$\alpha$-$\mathcal{I}$-continuous at $x_i$, there exists a $\alpha$-$\mathcal{I}$-open neighbourhoods $W_{x_i}$ of $x_i$ in $X$ such
that \( f(W_{x_1}) \subset cl(U_{y_1}) \) for \( i = 1, 2 \). Hence we get \( W_{x_1} \cap W_{x_2} = \phi \) because 
\( cl(U_{y_1}) \cap cl(U_{y_2}) = \phi \). Then \((X, \tau, \mathcal{I})\) is a \( \alpha-\mathcal{I}-T_2 \) space.

\[ \text{Corollary 4.2.13} \quad \text{If } f \text{ is contra-} \alpha-\mathcal{I}-\text{continuous injective function of an ideal topological space } (X, \tau, \mathcal{I}) \text{ into a Urysohn space } (Y, \sigma), \text{ then } (X, \tau, \mathcal{I}) \text{ is a } \alpha-\mathcal{I}-T_2 \text{ space.} \]

**Proof.** For each pair of distinct points \( x_1 \) and \( x_2 \) in \( X \), \( f \) is contra-\( \alpha-\mathcal{I} \)-continuous function of \( X \) into a Urysohn space \((Y, \sigma)\) such that \( f(x_1) \neq f(x_2) \) because \( f \) is injective. Hence by Theorem 4.2.12, the space \((X, \tau, \mathcal{I})\) is a \( \alpha-\mathcal{I}-T_2 \).

\[ \text{Theorem 4.2.14} \quad \text{If } f \text{ is a contra-} \alpha-\mathcal{I}-\text{continuous injective function of an ideal topological space } (X, \tau, \mathcal{I}) \text{ into ultra Hausdorff space } (Y, \sigma), \text{ then } (X, \tau, \mathcal{I}) \text{ is an } \alpha-\mathcal{I}-T_2 \text{ space.} \]

**Proof.** Let \( x_1 \) and \( x_2 \) be any distinct points in \( X \). Then since \( f \) is injective and \( Y \) is Ultra Hausdorff \( f(x_1) \neq f(x_2) \) and there exists clopen sets \( V_1, V_2 \) such that \( f(x_1) \in V_1, f(x_2) \in V_2 \) and \( V_1 \cap V_2 = \phi \). Then \( x_i \in f^{-1}(V_i) \in \alpha\mathcal{IO}(X) \) for \( i = 1, 2 \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \phi \). Thus, \((X, \tau, \mathcal{I})\) is an \( \alpha-\mathcal{I}-T_2 \) space.

\[ \text{Definition 4.2.15} \quad \text{The graph } G(f) \text{ of a function } f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \text{ is said to be contra-} \alpha-\mathcal{I}-\text{closed in } X \times Y \text{ if for each } (x, y) \in (X \times Y) \setminus G(f), \text{ there exists } U \in \alpha\mathcal{IO}(X) \text{ and a closed set } V \text{ of } Y \text{ containing } y \text{ such that } (U \times V) \cap G(f) = \phi. \]

\[ \text{Lemma 4.2.16} \quad \text{The graph } f : (X, \tau, \mathcal{I}) \to (Y, \sigma) \text{ is contra-} \alpha-\mathcal{I}-\text{closed in } X \times Y \text{ if and only if for each } (x, y) \in (X \times Y) \setminus G(f), \text{ there exists } U \in \alpha\mathcal{IO}(X, x) \text{ such that } f(U) \cap cl(W) = \phi. \]
Theorem 4.2.17 If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-$\alpha$-$\mathcal{I}$-continuous function and $Y$ is a Urysohn space, then $G(f)$ is contra-$\alpha$-$\mathcal{I}$-closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and there exist open set $V$, $W$ of $Y$ such that $f(x) \in V$, $y \in W$ and $cl(U) \cap cl(W) = \phi$. Since $f$ is contra-$\alpha$-$\mathcal{I}$-continuous, there exists $U \in \alpha IO(X, x)$ such that $f(U) \subset cl(V)$. Therefore we obtain $f(U) \cap cl(W) = \phi$. This shows that $G(f)$ is contra-$\alpha$-$\mathcal{I}$-closed.

Theorem 4.2.18 If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-$\alpha$-$\mathcal{I}$-continuous function and $(Y, \sigma)$ is $T_2$, then $G(f)$ is contra-$\alpha$-$\mathcal{I}$-closed.

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and there exist an open set $V$ in $Y$ such that $f(x) \in V$ and $y \notin V$. Since $f$ is $\alpha$-$\mathcal{I}$-continuous, there exists $U \in \alpha IO(X, \tau)$ such that $f(U) \subset cl(V)$. Therefore, $f(U) \cap (Y - V) = \phi$ and $Y - V$ is a closed set of $Y$ containing $y$. This shows that $G(f)$ is contra-$\alpha$-$\mathcal{I}$-closed.

Definition 4.2.19 An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\alpha$-$\mathcal{I}$-connected if $X$ cannot be expressed as the union of two nonempty disjoint $\alpha$-$\mathcal{I}$-open sets.

Theorem 4.2.20 A contra-$\alpha$-$\mathcal{I}$-continuous image of an $\alpha$-$\mathcal{I}$-connected space is connected.

Proof. The proof is clear.

Theorem 4.2.21 Let $(X, \tau, \mathcal{I})$ be a $\alpha$-$\mathcal{I}$-connected space and $Y$ be a $T_1$-space. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-$\alpha$-$\mathcal{I}$-continuous function then it is a constant function.
Proof. Since $Y$ is a $T_1$ space, $\Lambda = \{ f^{-1}(\{y\}) : y \in Y \}$ is a disjoint $\alpha-\mathcal{I}$-open partition of $X$. If $|\Lambda| \geq 2$, then $X$ is the union of at least two non-empty $\alpha-\mathcal{I}$-open sets. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-connected, $|\Lambda| = 1$. Hence $f$ is constant. 

Definition 4.2.22 An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\alpha-\mathcal{I}$-normal if each pair of nonempty disjoint closed sets can be separated by disjoint $\alpha-\mathcal{I}$-open sets.

Theorem 4.2.23 If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-$\alpha-\mathcal{I}$-continuous closed injective function and $Y$ is a ultra-normal space, then $(X, \tau, \mathcal{I})$ is an $\alpha-\mathcal{I}$-normal space.

Proof. Let $F_1$ and $F_2$ be a disjoint closed subsets of $X$. Since $f$ is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since $Y$ is ultra-normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets $V_1$ and $V_2$, of $Y$ respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in \alpha\mathcal{I}O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus, $(X, \tau, \mathcal{I})$ is a $\alpha-\mathcal{I}$-normal.

Definition 4.2.24 A collection $\{G_\alpha : \alpha \in \wedge\}$ is called a $\beta-\mathcal{I}$-closed cover of a subset $\wedge$ of an ideal space $(X, \tau, \mathcal{I})$ if $A \subset \bigcup\{G_\alpha : X \setminus G_\alpha \in \alpha\mathcal{I}O(X), \alpha \in \wedge\}$.

Definition 4.2.25 An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\alpha-\mathcal{I}$-closed compact if for every $\alpha-\mathcal{I}$-closed cover $\{W_i : i \in \Delta\}$, there exists a finite subset $\Delta_o$ of $\Delta$ such that $X - \bigcup\{U_i : i \in \Delta_o\} \in \mathcal{I}$.

Lemma 4.2.26 [53] For any function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$. $f(\mathcal{I})$ is an ideal on $Y$.  

54
Theorem 4.2.27  If a function \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma) \) is a contra-\( \alpha \)-\( \mathcal{I} \)-continuous and the set \( A \) is \( \alpha \)-\( \mathcal{I} \)-closed compact relative to \( X \), then \( f(A) \) is \( f(\mathcal{I}) \)-compact in \( Y \).

Proof. Let \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma) \) be a contra-\( \alpha \)-\( \mathcal{I} \)-continuous surjection and \( \{V_i : i \in \Delta\} \) be an open cover of \( Y \). Then \( \{f^{-1}(V_i) : i \in \Delta\} \) is a \( \alpha \)-\( \mathcal{I} \)-closed cover of \( X \). From the assumption, there exists a finite subset \( \Delta_o \) of \( \Delta \) such that \( X \setminus \bigcup \{f^{-1}(V_i) : i \in \Delta_o\} \in \mathcal{I} \). Therefore, \( Y \setminus \bigcup \{V_i : i \in \Delta_o\} \in f(\mathcal{I}) \) which shows that \( Y \) is \( f(\mathcal{I}) \)-compact.

Theorem 4.2.28  Let \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma) \) be a function and let \( \{U_\alpha : \alpha \in \Delta\} \) be an \( \alpha \)-\( \mathcal{I} \)-open cover of \( X \). If the restriction function \( f|U_\alpha : (U_\alpha, \tau|U_\alpha, \mathcal{I}|U_\alpha) \rightarrow (Y, \sigma) \) is \( \alpha \)-\( \mathcal{I} \)-continuous for each \( \alpha \in \Delta \), then \( f \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous.

Proof. Obvious.

Definition 4.2.29  A function \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J}) \) is said to be \( \alpha \)-\( \mathcal{I} \)-irresolute if \( f^{-1}(V) \) is \( \alpha \)-\( \mathcal{I} \)-open in \( X \) for every \( \alpha \)-\( \mathcal{I} \)-open set \( V \) of \( Y \).

Theorem 4.2.30  For the functions \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J}) \) and \( g : (Y, \sigma, \mathcal{J}) \rightarrow (Z, \eta) \) the following hold:

(i) \( g \circ f \) is \( \alpha \)-\( \mathcal{I} \)-continuous, if \( f \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous and \( g \) is contra-continuous.

(ii) \( g \circ f \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous, if \( f \) is contra-\( \alpha \)-\( \mathcal{I} \)-continuous and \( g \) is continuous.

(iii) \( g \circ f \) is \( \alpha \)-\( \mathcal{I} \)-continuous, if \( f \) is \( \alpha \)-\( \mathcal{I} \)-irresolute and \( g \) is contra \( \alpha \)-\( \mathcal{J} \)-continuous.
Remark 4.2.31  The following examples shows that composition of any contra-$\alpha$-$\mathcal{I}$-continuous functions need not be contra-$\alpha$-$\mathcal{I}$-continuous function in general.

Example 4.2.32  Let $X = \{a, b, c, d\}$ with topology $\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\tau_2 = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\}$, $\tau_3 = \{X, \phi, \{a\}, \{a, b, c\}\}$ and ideal $\mathcal{I}_{\infty} = \{\phi, \{a\}\}$, $\mathcal{I}_{\in} = \{\phi, \{b\}\}$. Define $f : (X, \tau_1, \mathcal{I}_{\infty}) \to (X, \tau_2, \mathcal{I}_{\in})$ by $f(a) = f(b) = d$, $f(c) = b$, $f(d) = c$ and $g : (X, \tau_2, \mathcal{I}_{\in}) \to (X, \tau_3)$ by $g(a) = d$, $g(b) = c$, $g(c) = b$ and $g(d) = a$. Then $f$ and $g$ are contra $\alpha$-$\mathcal{I}$-continuous, because $F = \{b, c, d\}$ is closed in $(X, \tau_3)$ but $(g \circ f)^{-1}(F) = \{b, d\}$ which is not $\alpha$-$\mathcal{I}$-open in $(X, \tau_1)$. 