Chapter 3

Third Order Half-linear Neutral Differential Equations with "Maxima"-I

3.Third Order Half-linear Neutral Differential Equations with "Maxima"-I

3.1 Introduction

In this chapter, we deal with the oscillatory behavior of third order half-linear neutral differential equations with "maxima" of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t) \max_{[\sigma(t),t]} x^{\alpha}(s) = 0, \quad t \geq t_0 \geq 0 \tag{3.1.1}$$

subject to the following conditions:

(C_1) $\tau(t) \leq t$ and $\sigma(t) \leq t$ are continuous functions in $[t_0, \infty)$ with $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C_2) α is a ratio of odd positive integers and $p(t) \in C^3([t_0,\infty),\ R)$ with $0 \leq p(t) \leq p < 1$, and $q(t) \in C([t_0,\infty),\ R_+)$ with $q(t)$ is not identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \geq t_0$;

(C_3) $a(t) \in C^1([t_0,\infty), R)$ and $a(t)$ is positive and nondecreasing for all $t \geq t_0$ and $\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(t)} dt = \infty$.

By a solution of equation (3.1.1) we mean a continuous function $x(t)$ defined on $[T_x, \infty)$, $T_x \geq t$, which has the property $((x(t) + p(t)x(\tau(t)))'')^{\alpha}$ are continuously differentiable and $x(t)$ satisfies the equation (3.1.1) on $[T_x, \infty)$. We consider only those solution $x(t)$ of equation (3.1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $t \geq T_x$. The great attention has been devoted to the oscillation of third order differential equations without "maxima", see, for example, [41, 44, 61, 62, 79] and the references cited therein. Compared to second order differential equations with "maxima", the third order differential equation with "maxima" received less attention. Motivated by these observations, in this chapter, we present some sufficient conditions for the oscillation of all solutions of equation (3.1.1). The results extend that of in [13] for equation (3.1.1) with $p(t) \equiv 0$ and without "maxima".

In Section 3.2, we obtain criteria for the oscillation of all solutions of equation (3.1.1) and in Section 3.3 we present some examples to illustrate the main results.

3.2 Oscillation Results

In this section, we obtain a oscillatory criteria for equation (3.1.1). For a solution $x(t)$ of (3.1.1) we define the corresponding function $z(t)$ by

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{3.2.1}$$

Throughout this chapter without loss of generality we can deal only with the positive solution of equation (3.1.1). Since proof for the negative case is similar. To obtain sufficient condition for the oscillation of solutions of equation (3.1.1), we need the following lemmas.

Lemma 3.2.1. *Let $x(t)$ be a positive solution of equation* (3.1.1)*, then there are only the following two cases for $z(t)$ defined in* (3.2.1) *hold:*

(I) $z(t) > 0,\ z'(t) > 0$ *and* $z''(t) > 0$;

(II) $z(t) > 0,\ z'(t) < 0$ *and* $z''(t) > 0$

for $t \geq t_0$, where t_0 is sufficiently large.

Proof. Assume that $x(t)$ is a positive solution of (3.1.1) on $[t_0, \infty)$. We see that $z(t) > x(t) > 0$ and

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' = -q(t) \max_{[\sigma(t),t]} x^{\alpha}(s) < 0. \tag{3.2.2}$$

Thus, $a(t)(z''(t))^\alpha$ is nonincreasing and of one sign eventually. Therefore $z''(t)$ is also of one sign eventually and so we have two possibilities

$$z''(t) < 0 \ \text{ or } \ z''(t) > 0 \ \text{ for } \ t \geq t_1.$$

If we admit that $z''(t) < 0$, then there exists a constant $M > 0$ such that

$$a(t)(z''(t))^\alpha \leq -M < 0.$$

Integrating the last inequality from t_1 to t we obtain

$$z'(t) \leq z'(t_1) - M^{1/\alpha} \int_{t_1}^{t} a^{-1/\alpha}(s)ds.$$

Letting $t \to \infty$ and using (C_2) we get $z'(t) \to -\infty$. Thus $z'(t) < 0$ eventually. But $z''(t) < 0$ and $z'(t) < 0$ eventually imply $z(t) < 0$ for $t \geq t_1$, a contradiction. This contradiction proves that $z''(t) > 0$ and we have only two cases (I) and (II) for $z(t)$. The proof is now complete. □

Lemma 3.2.2. *Let $x(t)$ be an eventually negative solution of equation* (3.1.1)*, then there are only the following two cases for $z(t)$ defined in* (3.2.1) *hold:*

(I) $z(t) < 0,\ z'(t) < 0$ *and* $(a(t)(z'(t))^\alpha)' \geq 0$;

(II) $z(t) < 0,\ z'(t) > 0$ *and* $(a(t)(z'(t))^\alpha)' \geq 0$.

The proof of Lemma 3.2.2 is analogous to that of Lemma 3.2.1.

Lemma 3.2.3. *Assume that $u(t) > 0,\ u'(t) \geq 0,\ u''(t) \leq 0$, on $[t_0, \infty)$. Then for each $\ell \in (0,1)$ there exists a $T_\ell \geq t_0$ such that*

$$\frac{u(\tau(t))}{u(t)} \geq \ell \frac{\tau(t)}{t} \quad \textit{for all } \ t \geq T_\ell.$$

Proof. It follows from the Mean Value Theorem and the monotone property of $u'(t)$ that

$$u(t) - u(\tau(t)) \leq u'(\tau(t))(t - \tau(t))$$

or

$$\frac{u(t)}{u(\tau(t)} \leq 1 + \frac{u'(\tau(t))}{u(\tau(t))}(t - \tau(t)). \tag{3.2.3}$$

Using the Mean Value Theorem once more, we see that

$$u(\tau(t)) \geq u(\tau(t)) - u(t_0) \geq u'(\tau(t))(\tau(t) - t_0).$$

So for each $\ell \in (0, 1)$ there is a $T_\ell \geq t_0$ such that

$$\frac{u(\tau(t))}{u'(\tau(t)} \geq l\tau(t), \quad t \geq T_\ell. \tag{3.2.4}$$

Combining (3.2.3) with (3.2.4), we get

$$\frac{u(t)}{u(\tau(t)} \leq 1 + \frac{1}{\ell\tau(t)}(t - \tau(t)) \leq \frac{t}{\ell\tau(t)}.$$

Now, the proof is complete. □

Lemma 3.2.4. *Assume that* $z(t) > 0,\ z'(t) > 0,\ z''(t) > 0,\ z'''(t) \leq 0,$ *on* $[T_\ell, \infty)$. *Then*

$$\frac{z(t)}{z'(t)} \geq \frac{t - T_\ell}{2} \quad \textit{for all} \ \ t \geq T_\ell.$$

Proof. Set $Z(t) = (t - T_\ell)z(t) - \frac{(t-T_\ell)^2}{2}z'(t)$. Then $Z(T_\ell) = 0$, and

$$Z'(t) = z(t) - \frac{(t - T_\ell)^2}{2}z''(t).$$

We shall prove that $Z(t) \geq 0$. By Taylor's Theorem, since $z''(t)$ is nonincreasing, we have

$$z(t) \geq z(T_\ell) + (t - T_\ell)z'(T_\ell) + \frac{(t - T_\ell)^2}{2}z''(t).$$

This implies

$$Z'(t) = z(t) - \frac{(t-T_\ell)^2}{2} z''(t) \geq z(T_\ell) + (t - T_\ell) z'(T_\ell) > 0.$$

Since $Z(t_\ell) = 0$, one gets $Z(t) \geq 0$ for $t \geq T_\ell$, which implies the desired inequality.

□

Lemma 3.2.5. *The function $x(t)$ is a negative solutions of equation* (3.1.1) *if and only if $-x(t)$ is a positive solution of the equation*

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t) \min_{[\sigma(t),t]} x^{\beta}(s) = 0. \tag{3.2.5}$$

Proof. The assertion of Lemma 3.2.5 can be verified easily. □

Lemma 3.2.6. *Let $x(t)$ be a positive solution of equation* (3.1.1) *and let the corresponding $z(t)$ satisfy Lemma 3.2.1 (II). If*

$$\int_{t_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{a(u)} \int_{u}^{\infty} q(s)ds\right)^{\frac{1}{\alpha}} dudv = \infty, \tag{3.2.6}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$

Proof. The proof is similar to that of in [13], and hence the details are omitted. □

Lemma 3.2.7. *Assume that $z'(t) > 0,\ z''(t) > 0,\ z'''(t) \leq 0$ on $[T_\ell, \infty)$. Then*

$$(t - T_\ell)\frac{z''(t)}{z'(t)} \leq 1 \quad \text{for} \quad t \geq T_\ell.$$

Proof. Since $z'''(t) \leq 0$ we have $z''(t)$ is decreasing and the result follows from the inequality

$$z'(t) \geq \int_{T_\ell}^{t} z''(s)ds \geq z''(t)(t - T_\ell).$$

□

For simplicity we introduce the following notations:

$$\begin{aligned} p_* &= \liminf_{t\to\infty} \frac{t^{\alpha}}{a(t)} \int_t^{\infty} P_\ell(s)ds, \\ q_* &= \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \frac{s^{\alpha+1}}{a(s)} P_\ell(s)ds, \end{aligned}$$

where

$$P_\ell(s) = \ell^\alpha \max_{[\sigma(t),t]}(1-p(s))^\alpha q(s)\left(\frac{\tau(s)}{s}\right)^\alpha \left(\frac{\tau(s)-T_\ell}{2}\right)^\alpha \tag{3.2.7}$$

with $\ell \in (0,1)$ arbitrarily chosen and T_ℓ large enough. Moreover for $z(t)$ satisfying case (I), we define

$$w(t) = a(t)\left(\frac{z''(t)}{z(t)}\right)^\alpha, \tag{3.2.8}$$

$$r = \liminf_{t\to\infty} \frac{t^\alpha}{a(t)},$$

and

$$R = \limsup_{t\to\infty} \frac{t^\alpha}{a(t)}. \tag{3.2.9}$$

Now, we present the main results.

Theorem 3.2.1. *Assume that condition* (3.2.6) *holds and* $a'(t) \geq 0$ *for all* $t \geq t_0$. *If*

$$p_* > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{3.2.10}$$

Then the solution $x(t)$ *of equation* (3.1.1) *is almost oscillatory.*

Proof. Assume that $x(t)$ is a positive solution of equation (3.1.1), and the corresponding function $z(t)$ satisfies case(I) of Lemma 3.2.1. First note that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq (1-p(t))z(t) \tag{3.2.11}$$

or

$$\max_{[\sigma(t),t]} x^\alpha(s) \geq z^\alpha \max_{[\sigma(t),t]}(1-p(s))^\alpha.$$

Using the above inequality in (3.1.1) we obtain

$$(a(t)(z''(t))^\alpha)' \leq 0. \tag{3.2.12}$$

The last inequality together with $a'(t) \geq 0$ gives that $z(t)$ satisfies $z(\tau(t)) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z'''(t) \leq 0$ for $t \in [T, \infty]$. From the definition of $w(t)$

we see that $w(t) > 0$, and from (3.1.1) we have

$$\begin{aligned} w'(t) &= \frac{(z'(t))^\alpha (a(t)(z''(t))^\alpha)' - (a(t)(z''(t))^\alpha)\alpha(z'(t))^{\alpha-1}z''(t)}{(z'(t))^{2\alpha}} \\ &= \frac{-q(t)z^\alpha(t)\max_{[\sigma(t),t]}(1-p(s))^\alpha}{(z'(t))^\alpha} - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t). \qquad (3.2.13) \end{aligned}$$

From Lemma 3.2.3 with $u(t) = z'(t)$, we have for ℓ the same in $P_\ell(t)$,

$$\frac{1}{z'(t)} \geq \ell \frac{\tau(t)}{t} \frac{1}{z'(\tau(t))}, \quad t \geq T_\ell$$

which with (3.2.13) gives

$$w'(t) \leq -q(t)\ell^\alpha \left(\frac{\tau(t)}{t}\right)^\alpha \frac{z^\alpha(t)}{(z'(\tau(t)))^\alpha} \max_{[\sigma(s),s]}(1-p(s))^\alpha - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t).$$

Using the fact from Lemma 3.2.4 that $z(t) \geq \frac{(t-T_\ell)}{2} z'(t)$, we have

$$w'(t) + P_\ell(t) + \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \leq 0. \qquad (3.2.14)$$

Since $P_\ell(t) > 0$ and $w(t) > 0$ for $t \geq T_\ell$, we have from (3.2.14) that $w'(t) \leq 0$ and

$$-\left(\frac{w'(t)}{\alpha w^{\frac{\alpha+1}{\alpha}}(t)}\right) > \frac{1}{a^{1/\alpha}(t)} \quad \text{for} \quad t \geq T_\ell. \qquad (3.2.15)$$

This implies that

$$\left(\frac{1}{w^{1/\alpha}(t)}\right)' > \frac{1}{a^{1/\alpha}(t)}. \qquad (3.2.16)$$

Integrating the last inequality from T_ℓ to t, we obtain

$$w(t) = \frac{1}{\left(\int_{T_\ell}^t \frac{ds}{a^{1/\alpha}(t)}\right)^\alpha} \qquad (3.2.17)$$

which inview of (C_3) implies that $\lim_{t\to\infty} w(t) = 0$. On the otherhand, from the definition of $w(t)$, and Lemma 3.2.4, we see that

$$0 \leq r \leq R \leq 1. \qquad (3.2.18)$$

Now, let $\varepsilon > 0$, then from the definitions of p_* and r we can pick $t_2 \in [T_\ell, \infty)$ sufficiently large that

$$\frac{t^\alpha}{a(t)} \int_t^\infty P_\ell(s)ds \geq p_* - \varepsilon \quad \text{for all} \quad t \geq t_2,$$

and

$$\frac{t^\alpha w(t)}{a(t)} \geq r - \varepsilon, \quad \text{for all} \quad t \in [t_2, \infty).$$

Integrating (3.2.14) from t to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w(t) \geq \int_t^\infty P_\ell(s)ds + \alpha \int_t^\infty \frac{w^{1+1/\alpha}(s)}{a^{1/\alpha}(s)}ds, \quad \text{for} \quad t \in [t_2, \infty). \tag{3.2.19}$$

Assume $p_* = \infty$, then from (3.2.19), we have

$$\frac{t^\alpha w(t)}{a(t)} \geq \frac{t^\alpha}{a(t)} \int_t^\infty P_\ell(s)ds.$$

Taking limit infimum on both sides as $t \to \infty$, we get in view of (3.2.18) that $1 \geq r \geq \infty$. This is a contradiction. Next assume that $p_* < \infty$. Now from (3.2.19) and the fact $a'(t) \geq 0$, we have

$$\begin{aligned}
\frac{t^\alpha}{a(t)} w(t) &\geq \frac{t^\alpha}{a(t)} \int_t^\infty P_\ell(s)ds + \alpha \frac{t^\alpha}{a(t)} \int_t^\infty \frac{s^{\alpha+1} a(s) w^{\frac{1}{\alpha}+1}(s)}{s^{\alpha+1} a^{\frac{1}{\alpha}+1}(s)} ds \\
&\geq (p_* - \varepsilon) + \frac{t^\alpha (r-\varepsilon)^{1+\frac{1}{\alpha}}}{a(t)} \int_t^\infty \frac{\alpha a(s)}{s^{\alpha+1}} ds \\
&\geq (p_* - \varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}} t^\alpha \int_t^\infty \frac{\alpha}{s^{\alpha+1}} ds
\end{aligned} \tag{3.2.20}$$

or

$$\frac{t^\alpha w(t)}{a(t)} \geq (p_* - \varepsilon) + (r - \varepsilon)^{1+\frac{1}{\alpha}}.$$

Taking limit infimum on both sides as $t \to \infty$, we get

$$r \geq p_* - \varepsilon + (r - \varepsilon)^{1+\frac{1}{\alpha}}.$$

Since $\varepsilon > 0$ is arbitrary, we get the desired result

$$p_* \leq r - r^{1+\frac{1}{\alpha}}.$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}$ with $A = B = 1$, we get $p_* \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}$, which contradicts (3.2.10). If $z(t)$ satisfies Case(II) of Lemma 3.2.1, then the proof follows from Lemma 3.2.6. This completes the proof. □

Corollary 3.2.1. *Assume that* (3.2.4) *holds and* $a'(t) \geq 0$, *for all* $t \geq t_0$. *If*

$$\liminf_{t\to\infty} \frac{t^\alpha}{a(t)} \int_t^\infty q(s) \max_{[\sigma(t),t]} (1-p(s))^\alpha \frac{\tau^{2\alpha}(s)}{s^\alpha} P_\ell(s) ds > \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{3.2.21}$$

then every solution of equation (3.1.1) *is almost oscillatory.*

Proof. We shall show that (3.2.21) implies (3.2.20). Then proof follows from Theorem 3.2.1. First note that for any $\ell \in (0,1)$ there exists a $t_1 \geq t_0$ such that $\tau(t) - T_\ell \geq \ell\tau(t),\ t \geq t_1$. Therefore

$$P_\ell \geq \frac{\ell^{2\alpha} \max_{[\sigma(t),t]} (1-p(t))^\alpha}{2^\alpha} \frac{q(t)\tau^{2\alpha}(t)}{t^\alpha}, \quad t \geq t_1. \tag{3.2.22}$$

On the otherhand, (3.2.21) implies that for some $\ell \in (0,1)$

$$\liminf_{t\to\infty} \frac{t^\alpha}{a(t)} \int_t^\infty q(s) \max_{[\sigma(t),t]} (1-p(s))^\alpha \frac{\tau^{2\alpha}(s)}{s^\alpha} > \frac{1}{\ell^{2\alpha}} \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{3.2.23}$$

Combining (3.2.22) with (3.2.23) we get (3.2.10). This completes the proof. □

Theorem 3.2.2. *Assume that the condition* (3.2.6) *holds and* $a'(t) \geq 0$ *for all* $t \geq t_0$. *If* $p_* + q_* > 1$, *then every solution of equation* (3.1.1) *is almost oscillatory.*

Proof. Assume that $x(t)$ is a positive solution of equation (3.1.1) and the corresponding function $z(t)$ satisfies case(I) of Lemma 3.2.1. Now multiply (3.2.14) by $\frac{t^{\alpha+1}}{a(t)}$, and integrating from $t_2 \geq t_0$ to t, we get

$$\int_{t_2}^t \frac{s^{\alpha+1}}{a(s)} w'(s) ds \leq \int_{t_2}^t \frac{s^{\alpha+1}}{a(s)} P_\ell(s) ds - \alpha \int_{t_2}^t \left(\frac{s^\alpha w(t)}{a(s)}\right)^{\frac{s+1}{s}} ds. \tag{3.2.24}$$

Using integration by parts, we obtain

$$\begin{aligned} \frac{t^{\alpha+1}}{a(t)} w(t) \quad &\leq \quad \frac{t_2^{\alpha+1} w(t_2)}{a(t_2)} - \int_{t_2}^t \frac{s^{\alpha+1}}{a(s)} P_\ell(s) ds \\ &\quad - \alpha \int_{t_2}^t \left(\frac{s^\alpha w(t)}{a(s)}\right)^{\frac{s+1}{s}} ds + \int_{t_2}^t \left(\frac{s^{\alpha+1}}{a(s)}\right)' w(s) ds. \end{aligned}$$

Since $a'(t) \geq 0$, we have

$$\left(\frac{s^{\alpha+1}}{a(s)}\right)' = \frac{a(s)(\alpha+1)s^\alpha - a'(s)s^\alpha}{(a(s))^2} \leq \frac{(\alpha+1)s^\alpha}{a(s)}.$$

Hence,

$$\frac{t^{\alpha+1}}{a(t)}w(t) \leq \frac{t_2^{\alpha+1}w(t_2)}{a(t_2)} - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)}P_\ell(s)ds + \int_{t_2}^{t} \left[\frac{(\alpha+1)s^\alpha w(s)}{a(s)} - \alpha\left(\frac{s^\alpha w(s)}{a(s)}\right)^{\alpha+1}\right] ds.$$

Using the inequality $Bu - Au^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha}$, with $u(s) = \frac{s^\alpha w(s)}{a(s)} > 0$, and positive constants $A = \alpha,\ B = \alpha+1$, we get

$$\frac{t^{\alpha+1}}{a(t)}w(t) \leq \frac{t_2^{\alpha+1}}{a(t_2)}w(t_2) - \int_{t_2}^{t} \frac{s^{\alpha+1}}{a(s)}P_\ell(s)ds + \frac{t-t_2}{t}. \tag{3.2.25}$$

Taking limit supremum on both sides as $t \to \infty$ we obtain $R \leq -q_* + 1$. Combining this with the inequality (3.2.21) we get

$$p_* + q_* \leq 1. \tag{3.2.26}$$

This is a contradiction. If $z(t)$ satisfies condition (3.2.6) then by Lemma 3.2.6, we have $\lim_{t\to\infty} x(t) = 0$.

If $z(t)$ satisfies case(II) of Lemma 3.2.1, then proof follows from Lemma 3.2.6. Now the proof is complete. □

Corollary 3.2.2. *Assume that* (3.2.4) *holds and* $a'(t) \geq 0$, *for all* $t \geq t_0$. *If*

$$q_* = \lim_{t\to\infty}\sup \frac{1}{t}\int_{t_0}^{t} \frac{s^{\alpha+1}}{a(s)}P_\ell(s)ds > 1, \tag{3.2.27}$$

then every solution of equation (3.1.1) *is almost oscillatory.*

Corollary 3.2.3. *Assume that* (3.2.6) *holds and* $a'(t) \geq 0$, *for all* $t \geq t_0$. *If*

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_{t_0}^{t} \frac{s\tau^{2\alpha}(s)q(s)\max_{[\sigma(t),t]}(1-p(s))^\alpha}{a(s)}ds > 2^\alpha,$$

then every solution of equation (3.1.1) *is almost oscillatory.*

3.3 Examples

In this section, we present some examples to illustrate the main results.

Example 3.3.1. *Consider the differential equation*

$$\left(t^3\left((x(t)+\frac{1}{3}x(t/2))''\right)^3\right)'+\frac{750}{27t^4}\max_{[t/2,t]}x^3(s)=0,\quad t\geq 0. \tag{3.3.1}$$

Here, $a(t)=t^3$, $p(t)=1/3$, $\tau(t)=\sigma(t)=t/2$, $q(t)=\frac{750}{27t^4}$, *and* $\alpha=3$. *One can easily verify that all conditions of Theorem 3.2.1 are satisfied and hence every solution of equation* (3.3.1) *is almost oscillatory. Infact* $x(t)=\frac{1}{t}$ *is one such almost oscillatory solution of equation* (3.3.1)*.*

Example 3.3.2. *Consider the differential equation*

$$\left(t^{1/3}\left((x(t)+\frac{1}{2}x(t/2))''\right)^{1/3}\right)'+\frac{1}{3}\left(\frac{2}{t}\right)^{4/3}\max_{[t/2,t]}x^{1/3}(s)=0,\quad t\geq 1. \tag{3.3.2}$$

Here, $a(t)=t^{1/3}$, $p(t)=1/2$, $\tau(t)=\sigma(t)=t/2$, $q(t)=\frac{1}{3}(\frac{2}{t})^{4/3}$, *and* $\alpha=1/3$. *One can easily verify that all conditions of Theorem 3.2.2 are satisfied and hence every solution of equation* (3.3.2) *is almost oscillatory. Infact* $x(t)=\frac{1}{t}$ *is one such almost oscillatory solution of equation* (3.3.2)*.*

We conclude this chapter with the following remark.

Remark 3.3.1. *It would be interesting to obtain results similar to that of in this chapter to the following equation*

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)'-q(t)\max_{[\sigma(t),t]}x^{\alpha}(s)=0,\quad t\geq t_0\geq 0.$$