Chapter 2

Second Order Quasilinear Neutral Delay Differential Equations with "Maxima"

2.Second Order Quasilinear Neutral Delay Differential Equations with "Maxima"

2.1 Introduction

In this chapter, we concerned with the oscillation problem of the second order quasilinear neutral delay differential equation with "maxima" of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))'\right)^{\alpha}\right)' + q(t)\max_{[t-\sigma,t]} x^{\beta}(s) = 0, \quad t \geq t_0 \geq 0. \tag{2.1.1}$$

Throughout this chapter, we will assume that the following conditions hold:

(C_1) $\tau(t)$ is a continuous function in $[t_0, \ \infty)$ with $\tau(t) \leq t$, σ is a nonnegative integer and $\alpha, \ \beta$ are the ratios of odd positive integers;

(C_2) $p(t) \in C^2([t_0,\infty), R)$ with $0 \leq p(t) < p < 1$, and $q(t) \in C([t_0,\infty), R)$;

(C_3) $a(t) \in C([t_0,\infty),(0,\infty))$ and $\int_{t_0}^{\infty} \frac{ds}{a^{\frac{1}{\alpha}}(t)} < \infty$.

By a solution of equation (2.1.1), we mean a continuous function $x(t)$ defined on the interval $[t_0,\infty)$ such that $x(t) + p(t)x(\tau(t))$ is twice differentiable and $((x(t)+p(t)x(\tau(t)))')^{\alpha}$ is twice differentiable and $x(t)$ satisfies the equation (2.1.1) for $t \geq t_0$. In [19], the authors have established some oscillation criteria for equation (2.1.1) when $\alpha = 1$ and $a(t) \equiv 1$. Also in [100], the authors discussed oscillatory behavior of equation (2.1.1) when $\alpha = 1$ and $\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty$. Motivated by these observations, in this chapter, we analyse the oscillatory and asymptotic behavior of solutions of equation (2.1.1) under the condition $\int_{t_0}^{\infty} \frac{dt}{a^{\frac{1}{\alpha}}(t)} < \infty$.

In Section 2.1, we establish sufficient conditions for the almost oscillation of all solutions of equation (2.1.1). In Section 2.3, we present some sufficient conditions for the existence of nonoscillatory solutions of the equation (2.1.1) using contraction mapping principle. In Section 2.4, we present some examples to illustrate the main results.

2.2 Oscillation Results

In this section, we derive some new sufficient conditions for the almost oscillation of all solutions of equation (2.1.1). Throughout this chapter we use the following notation without further mention:

$$z(t) = x(t) + p(t)x(\tau(t)),$$

and

$$A(t) = \int_t^\infty \frac{ds}{a^{\frac{1}{\alpha}}(s)}.$$

Lemma 2.2.1. *Let $x(t)$ be an eventually positive solution of equation* (2.1.1). *Then one of the following holds for $z(t)$:*

(I) $z(t) > 0,\ z'(t) > 0$ *and* $(a(t)(z'(t))^\alpha)' \leq 0$*;*

(II) $z(t) > 0,\ z'(t) < 0$ *and* $(a(t)(z'(t))^\alpha)' \leq 0$*.*

Proof. Let $x(t)$ be an eventually positive solution of equation (2.1.1). Then we may assume that $x(t) > 0,\ x(\tau(t)) > 0$ for all $t \geq T \geq 0$. Then inview of (C_2), we have $z(t) > 0$ for all $t \geq T \geq 0$. From the equation (2.1.1) we obtain

$$\left(a(t)\left(z'(t)\right)^\alpha\right)' = -q(t) \max_{[t-\sigma,t]} x^\beta(s) \leq 0.$$

Hence, $a(t)\left(z'(t)\right)^\alpha$ is of one sign eventually, and hence $z(t)$ is of one eventually since $a(t) > 0$. This completes the proof. □

Lemma 2.2.2. *Let $x(t)$ be an eventually negative solution of equation* (2.1.1). *Then one of the following holds:*

(I) $z(t) < 0,\ z'(t) < 0$ *and* $(a(t)(z'(t))^\alpha)' \geq 0$;

(II) $z(t) < 0,\ z'(t) > 0$ *and* $(a(t)(z'(t))^\alpha)' \geq 0$.

Proof. Let $x(t)$ be a negative solution of equation (2.1.1). Then the exists $t_0 \geq 0$ such that $x(t) \leq 0$, and $x(\tau(t)) \leq 0$ for all $t \geq t_0$. Then by definition of $z(t) \leq 0$ for all $t \geq t_0$. For equation (2.1.1) $(a(t)(z'(t))^\alpha)' = -q(t) \max_{[t-\sigma,t]} x^\beta(s) \geq 0$. Therefore $a(t)(z'(t))^\alpha$ is of one sign eventually and hence $z'(t)$ is of one sign eventually since $a(t) > 0$. Now the proof is complete. □

Lemma 2.2.3. *The function $x(t)$ is a negative solutions of equation* (2.1.1) *if and only if $-x(t)$ is a positive solution of the equation*

$$\left(a(t)\left(\left(x(t) + p(t)x(\tau(t))\right)'\right)^\alpha\right)' + q(t) \min_{[t-\sigma,t]} x^\beta(s) = 0.$$

Proof. Let $x(t)$ be a negative solution of equation (2.1.1). Let $x(t) = -y(t)$ where $y(t) > 0$. For equation (2.1.1)

$$\left(a(t)\left(\left(-y(t) - p(t)y(\tau(t))\right)'\right)^\alpha\right)' - q(t) \min_{[t-\sigma,t]} y^\beta(s) = 0.$$

Hence $y(t)$ is a positive solution of equation (2.1.1). Thus $-x(t) = y(t)$ is a positive solution of equation (2.1.1). By retracing the steps the converse can be proof. □

Lemma 2.2.4. *Let $x(t)$ be a positive solution of equation* (2.1.1) *and let the corresponding $z(t)$ satisfies Lemma 2.2.1 (II). If*

$$\int_T^\infty \left(\frac{1}{a(s)} \int_s^\infty q(u)du\right)^{\frac{1}{\alpha}} ds = \infty, \tag{2.2.1}$$

then

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$$

Proof. Let $x(t)$ be a positive solution of equation (2.1.1). Then by Lemma 2.2.1(II), we have $z(t) > 0$ and $z'(t) < 0$ for all $t \geq T \geq t_0$. Therefore $z(t) \to L \geq 0$ as $t \to \infty$, we prove that $L = 0$. If $L > 0$, then for $\epsilon = \frac{L(1-p)}{2p} > 0$, there exists $T \geq t_0$ such that $L < z(t) < L + \epsilon$ for $t \geq T$. Then for $t \geq T$, we have

$$x(t) = z(t) - p(t)x(\tau(t)) > L - p(t)z(t) > L - p(L+\epsilon) = L_1.$$

From equation (2.1.1), we have

$$\left(a(t)\left(z(t))'\right)^{\alpha}\right)' \leq -q(t) \max_{[t-\sigma,t]} x^{\beta}(s).$$

Integrating from t to ∞ and using the fact that $a(t)\left(z'(t)\right)^{\alpha}$ is positive and decreasing we obtain

$$a(t)\left(z'(t)\right)^{\alpha} \geq \int_t^{\infty} q(s) \max_{[s-\sigma,s]} x^{\beta}(s)ds \geq L_1^{\beta} \int_t^{\infty} q(s)ds.$$

Dividing the last inequality by $a(t)$ and then integrating the resulting inequality from T to ∞ we obtain

$$z(t) - z(T) \geq L_1^{\beta/\alpha} \int_T^t \left(\frac{1}{a(s)}\left(\int_s^{\infty} q(u)du\right)\right)^{1/\alpha} ds$$

or

$$\infty > z(t) \geq L_1^{\beta/\alpha} \int_T^t \left(\frac{1}{a(s)}\left(\int_s^{\infty} q(u)du\right)\right)^{1/\alpha} ds.$$

Letting $t \to \infty$ we see that

$$L_1^{\beta/\alpha} \int_T^t \left(\frac{1}{a(s)}\left(\int_s^{\infty} q(u)du\right)\right)^{1/\alpha} ds < \infty$$

which is a contradiction to (2.2.1). Hence $L = 0$, that is, $z(t) \to 0$. Since, $z(t) > x(t) > 0$ we have, $x(t) \to 0$ as $t \to \infty$. □

Theorem 2.2.1. *Let $\alpha = \beta = 1$ in the equation* (2.1.1). *If* (2.2.1) *and*

$$\lim_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m \left[q(s) \max_{[s-\sigma,s]} (1-p(u)) - \frac{m^2 a(s)}{4(t-s)^2}\right] ds = \infty \qquad (2.2.2)$$

hold, where $m \geq 1$ is an integer, then every solution of equation (2.1.1) *is almost oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (2.1.1). Then either $x(t) > 0$ eventually or $x(t) < 0$ eventually. We shall consider the case when $x(t) > 0$, since the other case can be proved analogously.

Let $x(t-\sigma) > 0,\ x(\tau(t)) > 0$ for all $t \geq T$, where T is chosen so large enough that the conclusions of Lemma 2.2.1 hold for all $t \geq T$.

First we assume Lemma 2.2.1(I) holds. Then $z(t) = x(t) + p(t)x(\tau(t))$ implies that

$$x(t) \geq (1 - p(t))z(t) \tag{2.2.3}$$

and

$$\max_{[t-\sigma,t]} x(s) \geq \max_{[t-\sigma,t]}(1 - p(s))z(s) = z(t) \max_{[t-\sigma,t]}(1 - p(t)). \tag{2.2.4}$$

From the equation (2.1.1) and (2.2.4), we have

$$(a(t)(z'(t))^{\alpha})' + q(t)z(t) \max_{[t-\sigma,t]}(1 - p(s)) \leq 0, \quad t \geq T. \tag{2.2.5}$$

Define

$$w(t) = \frac{a(t)z'(t)}{z(t)}, \quad t \geq T. \tag{2.2.6}$$

Then from (2.2.5) and (2.2.6), we have

$$w'(t) + \frac{w^2(t)}{a(t)} + q(t) \max_{[t-\sigma,t]}(1 - p(s)) \leq 0.$$

Multiplying the last inequality by $(t-s)^m$ and then integrating from T to t, we obtain

$$\begin{aligned} -w(T)(t-T)^m + \int_T^t m(t-s)^{m-1}w(s)ds + \int_T^t (t-s)^m \frac{w^2(s)}{a(s)}dt \\ + \int_T^t (t-s)^m q(s) \max_{[s-\sigma,s]}(1 - p(u))ds \leq 0 \end{aligned}$$

or

$$\int_T^t (t-s)^m q(s) \max_{[s-\sigma,s]}(1-p(u))ds$$
$$\leq w(T)(t-T)^m - \int_T^t \left[(t-s)^m \frac{w^2(s)}{a(s)} + m(t-s)^{m-1}w(s)\right] ds$$
$$\leq w(T)(t-T)^m - \int_T^t \frac{(t-s)^m}{a(s)} \left[w(s) + \frac{1}{2}\frac{ma(s)}{(t-s)}\right]^2 ds$$
$$+ \frac{1}{4}\int_T^t m^2 a(s)(t-s)^{m-2}ds$$
$$\leq w(T)(t-T)^m + \int_T^t \frac{m^2}{4} a(s)(t-T)^{m-2}ds$$

or

$$\int_T^t (t-s)^m \left[q(s) \max_{[s-\sigma,s]}(1-p(u)) - \frac{m^2 a(s)}{4(t-s)^2}\right] ds \leq w(T)(t-T)^m.$$

Dividing the last inequality by t^m and then taking limit supremum, we obtain

$$\lim_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m \left[q(s) \max_{[s-\sigma,s]}(1-p(u)) - \frac{m^2 a(s)}{4(t-s)^2}\right] ds \leq w(T) < \infty$$

which is a contradiction to (2.2.2). Hence every solution of equation (2.1.1) is almost oscillatory.

Next assume that Lemma 2.2.1(II) holds. Then by Lemma 2.2.4 we obtain that $\lim_{t\to\infty} x(t) = 0$, then every solution of equation (2.1.1) is almost oscillatory. This completes the proof. □

Theorem 2.2.2. *Assume that* $\beta > 1$. *If* (2.2.1) *and*

$$\int_{t_0}^{\infty} q(t)A^{\beta}(t-\sigma) \max_{[t-\sigma,t]}(1-p(s))^{\beta}dt = \infty \tag{2.2.7}$$

hold, then every solution of equation (2.1.1) *is almost oscillatory.*

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of equation (2.1.1) such that $x(t) > 0,\ x(\tau(t)) > 0$ for all $t \leq T$ where T is chosen large enough that the conclusion of Lemma 2.2.1 hold for all $t \geq T$. First assume that case(I) of lemma 2.2.1 holds. Integrating (2.1.1) from T to t yields,

$$a(t)(z'(t))^{\alpha} - a(T)(z'(T))^{\alpha} + \int_T^t q(s) \max_{[s-\sigma,s]} x^{\beta}(u)ds = 0. \tag{2.2.8}$$

Letting $t \to \infty$, we have

$$\int_T^{\infty} q(s) \max_{[s-\sigma,s]} x^{\beta}(u)ds < \infty. \tag{2.2.9}$$

In this case $z(t)$ is increasing, so there exists a positive number C such that $z(t) > C$ for $t \geq T$. This, together with (2.2.3) yields.

$$x^{\beta}(t) \geq C^{\beta}(1 - p(t))^{\beta} \quad \text{for all} \quad t \geq T.$$

Now $A^{\beta}(t) \to 0$ as $t \to \infty$. So there exists $T_1 \geq T$ such that

$$x^{\beta}(t) \geq C^{\beta}(1 - p(t))^{\beta} A^{\beta}(t) \quad \text{for all} \quad t \geq T_1.$$

Then

$$\begin{aligned} \max_{[t-\sigma,t]} x^{\beta}(s) &\geq C^{\beta} \max_{[t-\sigma,t]}(1 - p(s))^{\beta} \max_{[t-\sigma,t]} A^{\beta}(s) \\ &\geq C^{\beta} \max_{[t-\sigma,t]}(1 - p(s))^{\beta} A^{\beta}(t - \sigma). \end{aligned} \tag{2.2.10}$$

Combining (2.2.9) and (2.2.10) we have

$$C^{\beta} \int_T^{\infty} q(s)A^{\beta}(s - \sigma) \max_{[s-\sigma,s]}(1 - p(u))^{\beta}ds < \infty \tag{2.2.11}$$

which contradicts (2.2.7). Hence every solution of equation (2.1.1) is almost oscillatory.

Next assume that Lemma 2.2.1(II) holds. Then by Lemma 2.2.4, we obtain that $\lim_{t\to\infty} x(t) = 0$. Then every solution of equation (2.1.1) is almost oscillatory. This completes the proof. $\square$

Theorem 2.2.3. *Assume that* $0 < \beta < 1$. *If* (2.2.1) *and*

$$\int_{t_0}^{\infty} q(t)A(\sigma(t))dt = \infty \tag{2.2.12}$$

hold, then every solution of equation (2.1.1) *is almost oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.2.2, we have that Lemma 2.2.1 holds. For Case(I), we have (2.2.9) and (2.2.10). For large t, we have $A(t) \leq 1$ and $A^{\beta}(t) \geq A(t)$. So (2.2.11) implies

$$\int_{T}^{\infty} q(s) \max_{[t-\sigma,t]} (1 - p(s))^{\beta} A(s)ds < \infty.$$

This contradicts (2.2.12). Next assume that Lemma 2.2.1(II) holds. Then by Lemma 2.2.4, we obtain that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

2.3 Existence of Nonoscillatory Solutions

In this section, we provide some sufficient conditions for the existence of nonoscillatory solutions of equation (2.1.1) in case $\alpha > \beta > 1$ or $\beta < \alpha < 1$.

Theorem 2.3.1. *Assume that* $\alpha > \beta > 1$. *If*

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)} \int_{t}^{\infty} q(s)ds \right)^{1/\alpha} dt < \infty, \tag{2.3.1}$$

and

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_{t}^{\infty} q(s)ds \right) dt < \infty, \tag{2.3.2}$$

then the equation (2.1.1) *has a bounded nonoscillatory solution.*

Proof. From (2.3.1) and (2.3.2), we can choose $T \geq t_0$ sufficiently large so that

$$\int_{T}^{\infty} \left(\frac{1}{a(t)} \int_{t}^{\infty} q(s)ds \right)^{1/\alpha} dt \leq \frac{1-p}{8},$$

and

$$\int_{T}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_{t}^{\infty} q(s)ds \right) dt \leq \frac{(1-p)^2}{16}.$$

Let ψ be the set of all bounded continuous functions on $[t_0, \infty)$ with norm

$$||x|| = \sup_{t \geq t_0} \{x(t)\},$$

and let

$$S = \{x \in \psi : \frac{1-p}{4} \leq x(t) \leq 1,\ t \geq t_0\}.$$

Define an operator $T : S \to \psi$ by

$$(Tx)(t) = \begin{cases} \frac{5p+3}{8} - p(t)x(t-\tau) \\ \quad + \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v) du \right)^{1/\alpha} ds, & t \geq T \\ \\ (Tx)(t), & t_0 \leq t < T. \end{cases}$$

Clearly, T is continuous. Now for every $x \in S$ and $t \geq T$, we have

$$\begin{aligned} (Tx)(t) &\geq \frac{5p+3}{8} - p(t)x(t-\tau) \\ &\geq \frac{5p+3}{8} - p \\ &\geq \frac{3(1-p)}{8} > \frac{1-p}{4}. \end{aligned}$$

Also,

$$\begin{aligned} (Tx)(t) &\leq \frac{5p+3}{8} + \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v) du \right)^{1/\alpha} ds \\ &\leq \frac{5p+3}{8} + \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) du \right)^{1/\alpha} ds \\ &\leq \frac{5p+3}{8} + \frac{1-p}{8} \\ &\leq \frac{p+1}{2} < 1. \end{aligned}$$

Thus we proved that $TS \subset S$. Since S is bounded, closed and convex subset of ψ, we only need to show that T is a contraction mapping on S, in order to apply contraction principle. For $x, y \in S$ and $t \geq T$, we have

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &\leq p(t) \max_{[t-\sigma,t]} \mid x(s-\tau) - y(s-\tau) \mid \\
&\quad + \left| \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v) du \right)^{1/\alpha} ds \right. \\
&\quad \left. - \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} y^\beta(v) du \right)^{1/\alpha} ds \right| \\
&\leq p||x-y|| + \int_T^\infty \frac{1}{a^{1/\alpha}}(t) \left| \left(\int_t^\infty q(s) \max_{[s-\sigma,s]} x^\beta(s) ds \right)^{1/\alpha} \right. \\
&\quad \left. - \left(\int_t^\infty q(t) \max_{[t-\sigma,t]} y^\beta(t) dt \right)^{1/\alpha} \right| dt.
\end{aligned}$$

By applying the Mean Value Theorem for derivatives to the function $V(u) = u^\alpha$ or $u^\beta,\ \alpha > \beta > 1$, we see that for any $x, y \in S$, we have

$$\left| x^{1/\alpha} - y^{1/\alpha} \right| \leq \frac{4}{\alpha(1-p)} |x-y|$$

and

$$\left| x^\beta - y^\beta \right| \leq 2\beta |x-y|.$$

Thus,

$$\begin{aligned}
||Tx - Ty|| &\leq p||x-y|| + \frac{4}{\alpha(1-p)} \int_T^\infty \frac{1}{a^{1/\alpha}(t)} \\
&\quad \left(\int_t^\infty q(s) \max_{[s-\sigma,s]} \left| x^\beta(v) - y^\beta(v) \right| ds \right) dt \\
&\leq p||x-y|| + \frac{4}{\alpha(1-p)} 2\beta \\
&\quad \int_T^\infty \left(\frac{1}{a^{1/\alpha}(t)} \int_t^\infty q(s) ds \right) dt ||x-y|| \\
&\leq \left(p + \frac{8\beta}{\alpha(1-p)} \frac{(1-p)^2}{16} \right) ||x-y|| \\
&\leq \left(\frac{p+1}{2} \right) ||x-y|| < ||x-y||.
\end{aligned}$$

Therefore, T is a contraction on S. Hence, by contraction principle T has a unique fixed point, which is clearly a positive solution of equation (2.1.1). This completes the proof. □

Theorem 2.3.2. *Assume that $\beta < \alpha < 1$. If* (2.3.1) *and* (2.3.2) *hold, then equation* (2.1.1) *has a bounded nonoscillatory.*

Proof. By (2.3.1) and (2.3.1), we can choose $T \geq t_0$ sufficiently large so that,

$$\int_T^\infty \left(\frac{1}{a(t)}\int_t^\infty q(s)ds\right)^{1/\alpha} dt < \frac{1-p}{9},$$

and

$$\int_T^\infty \left(\frac{1}{a^{1/\alpha}(t)}\int_t^\infty q(s)ds\right) dt \leq \frac{5(1-p)^2}{27}.$$

Let ψ be the set of all bounded continuous function on $[t_0, \infty)$ with norm

$$||x|| = \sup_{t \geq t_0}\{x(t)\},$$

and let

$$S = \{x \in \psi : \frac{5(1-p)}{9} \leq x(t) \leq 1,\ t \geq t_0\}.$$

Define an operator $T : S \to \psi$ by

$$(Tx)(t) = \begin{cases} \frac{2p+7}{9} - p(t)x(t-\tau) \\ \quad + \int_T^t \left(\frac{1}{a(s)}\int_s^\infty q(u)\max_{[u-\sigma,u]} x^\beta(v)du\right)^{1/\alpha} ds, & t \geq T \\ (Tx)(t), & t_0 \leq t < T. \end{cases}$$

Clearly, T is continuous. Now for every $x \in S$ and $t \geq T$.

$$Tx(t) \geq \frac{2p+7}{9} - p(t)x(t-\tau) \geq \frac{2p+7}{9} - p > \frac{5(1-p)}{9}.$$

Also,

$$\begin{aligned} Tx(t) &\leq \frac{2p+7}{9} + \int_T^t \left(\frac{1}{a(s)}\int_s^\infty q(u)\max_{[u-\sigma,u]} x^\beta(v)du\right)^{1/\alpha} ds \\ &\leq \frac{2p+7}{9} + \int_T^t \left(\frac{1}{a(s)}\int_s^\infty q(u)du\right)^{1/\alpha} ds \\ &\leq \frac{2p+7}{9} + \frac{1-p}{9} \leq \frac{p+8}{9} \leq 1. \end{aligned}$$

Thus, we proved that $TS \subset S$. To apply contraction principle we have to prove T is a contraction on S. Since S is bounded closed and convex subset of ψ. For $x, y \in S$ and $t \geq T$, we have

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| &\leq p(t) \max_{[t-\sigma,t]} |x(s-\tau) - y(s-\tau)| \\
&\quad + |\int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v)du \right)^{1/\alpha} ds \\
&\quad - \int_T^t \left(\frac{1}{a(s)} \int_s^\infty q(u) \max_{[u-\sigma,u]} y^\beta(v)du \right)^{1/\alpha} ds| \\
&\leq p||x-y|| + \int_T^\infty \frac{1}{a^{1/\alpha}(t)} | \left(\int_s^\infty q(u) \max_{[u-\sigma,u]} x^\beta(v)du \right)^{1/\alpha} \\
&\quad - \left(\int_s^\infty q(u) \max_{[u-\sigma,u]} y^\beta(v)du \right)^{1/\alpha} |dt.
\end{aligned}$$

By applying the Mean Value Theorem for derivatives to the function $V(u) = u^\alpha$ or $u^\beta,\ \beta < \alpha < 1$, we see that for any $x, y \in S$, we have

$$\left| x^{1/\alpha} - y^{1/\alpha} \right| \leq \frac{2}{\alpha} |x - y|$$

and

$$\left| x^\beta - y^\beta \right| \leq \frac{9\beta}{5(1-p)} |x - y|.$$

Hence

$$\begin{aligned}
||Tx - Ty|| &\leq p||x-y|| + \frac{9\beta}{5(1-p)} \int_T^\infty \frac{1}{a^{1/\alpha}(t)} \\
&\quad \left(\int_t^\infty q(s) \max_{[s-\sigma,s]} \left| x^\beta(v) - y^\beta(v) \right| ds \right) dt \\
&\leq p||x-y|| + \frac{9\beta}{5(1-p)} \frac{2}{\alpha} \\
&\quad \int_T^\infty \left(\frac{1}{a^{1/\alpha}(t)} \int_t^\infty q(s)ds \right) dt ||x-y|| \\
&\leq p \left(1 + \frac{18}{5(1-p)} \frac{5(1-p)^2}{27} \right) ||x-y|| \\
&\leq \left(\frac{p+2}{3} \right) ||x-y|| < ||x-y||.
\end{aligned}$$

Thus, T is a Contraction mapping. So by contraction principle, T has a unique fixed point x, that is, x is clearly a positive solution of equation (2.1.1). This completes the proof. $\square$

2.4 Examples

In this section, we present some examples to illustrate the main results.

Example 2.4.1. *Consider the following second order differential equation*

$$\left(e^{2t}(x(t)+\frac{1}{2}x(t-1))'\right)' + \frac{e^{2t}(2-e)}{2e}\max_{[t-1,t]} x(s) = 0, \qquad t \geq 1. \tag{2.4.1}$$

Here, $a(t) = e^{2t}$, $p(t) = 1/2$, $q(t) = \frac{e^{2t}(2-e)}{2e}$, $\tau(t) = t-1$, $\sigma = 1$ *and* $\alpha = \beta = 1$. *One can easily verify that all conditions of Theorem 2.2.1 are satisfied and hence every solution of equation* (2.4.1) *is almost oscillatory. Infact* $x(t) = e^{-t}$ *is one such almost oscillatory solution of equation* (2.4.1).

Example 2.4.2. *Consider the differential equation*

$$\left(e^{2t}\left((x(t)+\frac{1}{2}x(t-1))'\right)^3\right)' + e^{4t-3}\left(1+\frac{e}{2}\right)^3 \max_{[t-1,t]} x^3(s) = 0. \tag{2.4.2}$$

Here, $a(t) = e^{2t}$, $p(t) = 1/2$, $q(t) = e^{4t-3}(1+\frac{e}{2})^3$, $\tau(t) = t-1$, $\sigma = 1$ *and* $\alpha = \beta = 3$. *It is easy to see that the condition*

$$\lim_{t\to\infty}\int_T^t \left(\frac{1}{e^{2s}}\int_s^\infty \left(1+\frac{e}{2}\right)^3 du\right) ds = \infty$$

holds. Further other conditions of Theorem 2.2.2 are also satisfied, and hence every solution of equation (2.4.2) *is almost oscillatory.*

Example 2.4.3. *Consider the differential equation*

$$\left(e^{5t}\left((x(t)+\frac{1}{3}x(t-1))'\right)^3\right)' + 2e^{\frac{7t+1}{3}}\left(1+\frac{e}{3}\right)^3 \max_{[t-1,t]} x^{1/3}(s) = 0. \tag{2.4.3}$$

Here, $a(t) = e^{5t},\ p(t) = \frac{1}{3},\ q(t) = 2e^{\frac{7t+1}{3}}(1+\frac{e}{3})^3,\ \tau(t) = t-1,\ \sigma = 1\ \alpha = 3,$ *and* $\beta = 1/3$. *Then,*

$$\int_T^\infty \left(\frac{1}{a(s)}\int_s^\infty q(u)du\right)^{\frac{1}{\alpha}} ds = \int_T^\infty \left(\frac{1}{e^{5s}}\int_s^\infty 2e^{\frac{7u+1}{3}}(1+\frac{e}{3})^3 du\right)^{\frac{1}{3}} ds = \infty.$$

Hence by Theorem 2.2.3, every solution of equation (2.4.3) *is almost oscillatory.*

Example 2.4.4. *Consider the differential equation*

$$\left(e^{7t}\left((x(t)+\frac{1}{2}x(t-2))'\right)^5\right)' + 2e^{5t-6}\left(1+\frac{e^2}{3}\right)^5 \max_{[t-2,t]} x^3(s) = 0. \tag{2.4.4}$$

Hence $a(t) = e^{7t},\ p(t) = 1/2,\ q(t) = 2e^{5t-6}(1+\frac{e^2}{3})^5,\ \tau(t) = t-2,\ \sigma = 2\ \alpha = 5,$ *and* $\beta = 3$. *Then*

$$\begin{aligned}\int_{t_0}^\infty \left(\frac{1}{a(t)}\int_t^\infty q(s)ds\right)^\alpha dt &= \int_{t_0}^\infty \left(\frac{1}{e^{7t}}\int_t^\infty 2e^{5s-6}(1+\frac{e^2}{3})^5 ds\right)^{1/5} dt \\ &= \frac{2(3+e^2)}{3e^6}\left(\int_{t_0}^\infty \frac{1}{e^{7t}}\int_t^\infty e^s ds\right)^{1/5} dt < \infty.\end{aligned}$$

Hence by Theorem 2.3.1, every nonoscillatory solution of equation (2.4.4) *tends to zero as* $t \to \infty$. *Infact* $x(t) = e^{-t}$ *is one such solution of equation* (2.4.4), *which is almost oscillatory.*

Example 2.4.5. *Consider the differential equation*

$$\left(e^t\left((x(t)+x(t-1))'\right)^{1/5}\right)' + 4e^{\frac{29t}{30}}(1+e)^{1/5}\max_{[t-1,t]} x^{1/6}(s) = 0,\ t \geq 1. \tag{2.4.5}$$

Here, $a(t) = e^t,\ p(t) = 1,\ q(t) = 4e^{\frac{29t}{30}}(1+e)^{1/5},\ \tau(t) = t-1,\ \sigma = 1\ \alpha = 1/5,$ *and* $\beta = 1/6$. *It is easy to see that all conditions of Theorem 2.3.2 are satisfies and hence every bounded nonoscillatory solution of equation* (2.4.5) *tends to zero as* $t \to \infty$. *Infact* $x(t) = e^{-t}$ *is one such solution of equation* (2.4.5), *which bounded nonoscillatory.*

We conclude this chapter with the following remark.

Remark 2.4.1. *It would be interesting to obtain results similar to that of in this chapter to the following equation*

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))'\right)^{\alpha}\right)' - q(t)\max_{[t-\sigma,t]} x^{\beta}(s) = 0, \quad t \geq t_0 \geq 0.$$