Chapter 1

Introduction

1.Introduction

1.1 Differential Equations

Differential equations with "maxima" are a special type of differential equations that contain the maximum of the unknown function over previous interval(s). Such equations adequately model real world process whose present state significantly depends on the maximum value of the state on the past time interval. Since the maximum function has very specific properties, it makes these equations strongly nonlinear. As a result, these equations gain an important place in the theory of differential equations and are called differential equations with "maxima". Differential equation with maximum arise naturally when solving practical and phenomenon problems, for example, in the theory of automatic control in various technical systems often the law of regulation depends on the maximum values of some regulated state parameters over certain time intervals, see for example E. P. Popov [90] considered the system for regulating the voltage of a generator of constant current. the object of the experiment was a generator of constant current with parallel simulation and the regulated quantity was the voltage at the source electric current. The equation describing the work of the regulator involves the maximum of the unknown function and it has the form

$$T_0 u'(t) + u(t) + q \max_{[t-h,t]} u(s) = f(t),$$

where T_0 and q are constants characterizing the object, $u(t)$ is the regulated voltage and $f(t)$ is the perturbed effect. Note that some modifications of the above differential equation are used to model the vision process in the compound eye, see for example Hadeler [54] and Hale [55].

This requires the use of differential equations with "maxima" in the control theory. Recently, the interest in differential equations with "maxima" has increased, see for examples, [17, 20, 60, 72, 88]. The differential equations with "maxima" have very different properties than the differential equations with delay. For example, let us consider the following two scalar equations:

$$x' = (x(t - \tau(t))^4 , \tag{1.1.1}$$

where the function $\tau \in C(R, R)$, $\tau(t) \neq 0$; and

$$x' = \left(\max_{s\in[t-h,t]} x(s) \right)^4 , \tag{1.1.2}$$

where $h \geq 0$.

Note that equation (1.1.1) seems to be very similar to (1.1.2), especially in the case of $\tau(t) \equiv h$.

In both equations the right part is nonnegative and the solution $x(t)$ is nondecreasing. Hence, the equation (1.1.2) is reduced to ordinary differential equation $x' = x^4$ and the initial condition is required at one single point. On the otherhand, the equation (1.1.1) could not be reduced to an ordinary differential equation for any nonzero value of $\tau(t)$. Next, consider the scalar differential equation with "maxima"

$$x'(t) = \max_{s\in[t-\frac{\pi}{2},t]} x(s) \quad \text{for} \quad t \geq 0, \tag{1.1.3}$$

with initial condition

$$x(t) = -asint \quad \text{for} \quad t \in [-\frac{\pi}{2}, 0], \tag{1.1.4}$$

where $a > 0$ is a constant. The solution of the initial value problem (1.1.3) is given by

$$x(t) = \begin{cases} -asint & \text{for} \quad t \in \left[-\frac{\pi}{2}, 0 \right] \\ asint & \text{for} \quad t \in \left[0, \frac{\pi}{4} \right] \\ a\frac{1}{\sqrt{2}}e^{t-\frac{\pi}{4}} & \text{for} \quad t \geq \frac{\pi}{4}. \end{cases} \tag{1.1.5}$$

Therefore the solution of equation(1.1.3) with initial condition (1.1.4) is unbounded. Now let us consider the differential equation with "maxima" (1.1.3) with initial condition.

$$x(t) = asint \quad \text{for} \quad t \in [-\frac{\pi}{2}, 0], \tag{1.1.6}$$

where $a > 0$ is a constant. The solution of the initial value problem (1.1.3) is given by

$$x(t) = \begin{cases} -asint & \text{for} \quad t \in \left[-\frac{\pi}{2}, 0 \right] \\ 0 & \text{for} \quad t \geq 0. \end{cases} \tag{1.1.7}$$

Now let us consider the differential equation with delay

$$x'(t) = x\left(t - \frac{\pi}{2}\right) \quad \text{for} \quad t \geq 0, \tag{1.1.8}$$

with initial condition (1.1.4). The solution of equation (1.1.8) with initial condition (1.1.4) is

$$x(t) = asint \quad \text{for} \quad t \geq -\frac{\pi}{2} \quad \text{and it is bounded.}$$

The above examples illustrate the difference between the behavior of the solutions of the differential equations with "maxima" and the differential equations with delay. Therefore it is necessary to study the properties of solutions of differential equations with "maxima" as a separate subject.

1.2 Motivation

In the past few decades the mathematical importance of various types of differential equations with "maxima" has grown exponentially, see [73–75, 110], and the references cited therein. However the qualitative theory of these equations is little developed, see for example, [111]. The existence of periodic solutions of the equations with "maxima" is considered in [9], and the asymptotic stability of solutions is investigated in [24, 113]. The oscillatory and asymptotic behavior of solutions of differential equations with "maxima" is studied in [23, 30].

In the qualitative theory of differential equations, oscillatory and asymptotic behavior of solutions $x(t)$ plays an important role. A solution of differential equation is said to be oscillatory if it has infinitely many zeros in $[t_0, \infty)$, $t_0 \geq 0$. Otherwise it is called nonoscillatory. The solution $x(t)$ is called almost oscillatory if it is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

The oscillatory and asymptotic behavior of solutions of functional differential equations have been studied extensively in the literature, see for example, [3, 4, 6, 7, 10–13, 23, 27, 28, 31, 33, 38, 55, 83–86, 99] and the references cited therein. However from the review of literature, it is known that only few results are available on the oscillatory behavior of second order differential equations with "maxima", see example [18, 19, 22, 66, 101, 102].

Keeping in view of the importance of the subject and in the light of the above trend, the author has obtained some significant results on the following topics:

1. **Second order quasilinear neutral delay differential equations with "maxima".**

2. **Third order half-linear neutral differential equations with "maxima"-I.**

3. **Third order half-linear neutral differential equations with "maxima"-II.**

4. **Third order nonlinear neutral delay differential equations with "maxima".**

5. **Fourth order nonlinear neutral differential equations with "maxima".**

1.3 Summary of the Thesis

This thesis consists of six chapters including this introductory chapter.

In Chapter 2, we study the oscillation results for second order quasilinear neutral delay differential equations with "maxima" of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))'\right)^{\alpha}\right)'+q(t)\max_{[t-\sigma,t]}x^{\beta}(s)=0,\quad t\geq t_0\geq 0, \tag{1.3.1}$$

subject to the following conditions:

(C_1) $\tau(t)$ is a continuous function in $[t_0,\ \infty)$ with $\tau(t)\leq t$, σ is a nonnegative integer and $\alpha,\ \beta$ are the ratio of odd positive integers;

(C_2) $p(t)\in C^2([t_0,\infty),R)$ with $0\leq p(t)<p<1$, and $q(t)\in C([t_0,\infty),R)$;

(C_3) $a(t)\in C([t_0,\infty),(0,\infty))$ and $\int_{t_0}^{\infty}\frac{ds}{a^{\frac{1}{\alpha}}(t)}<\infty$.

Section 2.1 presents necessary introduction. Section 2.2 deals with the sufficient conditions for the almost oscillation of equation (1.3.1). In Section 2.3, we provide sufficient conditions for the existence of nonoscillatory solutions of equation (1.3.1). Examples are presented to illustrate the main results in Section 2.4. The results presented in this chapter extend and generalize some of the known results in [1, 18, 26, 43, 45, 47, 48, 65, 66, 70, 71, 78, 97, 100].

Chapter 3, deals with the third order half-linear neutral differential equations of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)'+q(t)\max_{[\sigma(t),t]}x^{\alpha}(s)=0,\quad t\geq t_0\geq 0, \tag{1.3.2}$$

subject to the following conditions:

(C_1) $\tau(t)\leq t$ and $\sigma(t)\leq t$ are continuous functions in $[t_0,\infty)$ with $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

(C_2) α is a ratio of odd positive integers and $p(t) \in C^3([t_0, \infty),\ R)$ with $0 \le p(t) \le p < 1$, and $q(t) \in C([t_0, \infty),\ (0, \infty))$ with $q(t)$ is not identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \ge t_0$;

(C_3) $a(t) \in C^1([t_0, \infty)$, is positive and nondecreasing for all $t \ge t_0$, and

$$\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(t)} dt = \infty.$$

Section 3.1 provides necessary introduction, and in Section 3.2, we present some oscillatory criteria for equation (1.3.2). In Section 3.3, we provide some examples to illustrate the main results. The results obtained in this chapter generalize and complement to those given in [13, 14, 16].

In Chapter 4, we continue the study of oscillatory behavior of third order half-linear neutral differential equation of the form

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t) \max_{[\sigma(t), t]} x^{\alpha}(s) = 0, \quad t \ge t_0 \ge 0, \tag{1.3.3}$$

subject to the following conditions:

(C_1) $\tau(t) \le t$ and $\sigma(t) \le t$ are continuous functions in $[t_0, \infty)$ with $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C_2) α is a ratio of odd positive integers and $p(t) \in C^3([t_0, \infty),\ R)$ with $0 \le p(t) \le p < 1$, and $q(t) \in C([t_0, \infty),\ R_+)$ with $q(t)$ is not identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \ge t_0$;

(C_3) $a(t) \in C^1([t_0, \infty)$, is positive and nonincreasing for all $t \ge t_0$, and

$$\int_{t_0}^{t} a^{-\frac{1}{\alpha}}(s) ds \to \infty \quad \text{ as } \ t \to \infty.$$

In Section 4.1, we present necessary introduction and motivation, and in Section 4.2, we obtain conditions for all solutions of equation (1.3.2) to be almost oscillatory. In

Section 4.3, we present some examples to illustrate the main results. The results presented in this chapter generalize and improve those obtained in [34, 39, 41].

In Chapter 5 we are concern with the oscillation of third order nonlinear neutral differential equations with "maxima" of the form

$$\Big(a(t)\left(b(t)\left(x(t)+p(t)x(\tau(t))\right)'\right)'\Big)'+q(t)\max_{[\sigma(t),t]}x^{\gamma}(s)=0,\quad t\geq t_0\geq 0, \tag{1.3.4}$$

where γ is the ratio of odd positive integers. Throughout this chapter, it is always assumed that

(C_1) γ is a ratio of odd positive integers;

(C_2) $a(t)\in C^3([t_0,\infty),R)$, $b(t)\in C^2([t_0,\infty),R)$, $p(t)\in C^1([t_0,\infty),R)$, $q(t)\in C([t_0,\infty),R)$, $a(t)>0$, $b(t)>0$, $q(t)>0$ for all $t\geq t_0$;

(C_3) $\sigma(t)\in C([t_0,\infty),R)$, $\sigma(t)<t$, $\sigma(t)$ is nondecreasing and $\lim\limits_{t\to\infty}\sigma(t)=\infty$.

Section 5.1 provides necessary introduction and motivation. In Section 5.2, we obtain some oscillation criteria for equation (1.3.4). In Section 5.3, we present some examples to support our main results. The results obtained in this chapter generalize and complement to the results established in [15, 25, 35–37, 59, 76, 77, 82, 91–95].

Finally in Chapter 6 we consider fourth order nonlinear neutral differential equations with "maxima" of the form

$$(r(t)(x(t)+p(t)x(h(t)))'')''+q(t)\max_{[\sigma(t),t]}f(x(s))=0,\quad t\geq t_0\geq 0, \tag{1.3.5}$$

subject to the conditions:

(C_1) $r(t)$ is positive and continuous for $t\geq t_0$ and $\int_{t_0}^{\infty}\frac{1}{r(t)}dt=\infty$;

(C_2) $p(t)$ is continuous for $t\geq t_0$, and $0\leq p(t)\leq p<1$;

(C_3) $h(t) < t$ and $\sigma(t) < t$ are continuous functions for $t \geq t_0$ and $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C_4) $q(t)$ is continuous and nonnegative and non identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \geq t_0$;

(C_5) $f : R \to R$ is continuous and $f(x)$ is nondecreasing with $xf(x) > 0$, $x \neq 0$.

Section 6.1, presents necessary introduction and motivation. In Section 6.2, we obtain some criteria for the existence of nonoscillatory solutions of equation (1.3.5) some specific asymptotic behavior. Section 6.3, deals with the conditions for the oscillation of all solutions of equation(1.3.5). In Section 6.4, examples are included to illustrate the main results. The results obtained in this chapter generalize and complement to that of in [40, 63, 64, 67–69, 104, 112].

Thus, we obtained some new results, improve and extended some of the existing results on the oscillatory and asymptotic behavior of second, third and fourth order neutral differential equations using comparison theorems, integral averaging method and inequalities. Further, examples are presented in the text to illustrate the importance of the results.