Chapter 6

Fourth Order Nonlinear Neutral Differential Equations with "Maxima"

6. Fourth order Nonlinear Neutral Differential Equations with "Maxima"

6.1 Introduction

In this final chapter, we study the oscillation and asymptotic behavior of solutions of fourth order nonlinear neutral differential equations with "maxima" of the form

$$(r(t)(x(t)+p(t)x(h(t)))'')'' + q(t)\max_{[\sigma(t),t]} f(x(s)) = 0, \quad t \geq t_0 \geq 0, \tag{6.1.1}$$

subject to the following conditions:

(C_1) $r(t)$ is positive and continuous for $t \geq t_0$ and $\int_{t_0}^{\infty} \frac{1}{r(t)} dt = \infty$;

(C_2) $p(t)$ is continuous for $t \geq t_0$, and $0 \leq p(t) \leq p < 1$;

(C_3) $h(t) < t$ and $\sigma(t) < t$ are continuous functions for $t \geq t_0$ and $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C_4) $q(t)$ is continuous and nonnegative and non identically zero on any ray of the form $[t_*, \infty)$ for any $t_* \geq t_0$;

(C_5) $f : R \to R$ is continuous and $f(x)$ is nondecreasing with $xf(x) > 0,\ x \neq 0$.

By a solution of equation (6.1.1) we mean a continuous function $x(t) \in C^2([T_x, \infty))$, $T_x \geq t_0$, which has the property $(x(t) + p(t)x(h(t)))''$ are continuously differentiable and $x(t)$ satisfies the equation (6.1.1) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of equation (6.1.1) which satisfy $\sup\{|x(t)| : t \geq T_x\} > 0$ for all $t \geq T_x$.

There are many results available in the literature on the oscillatory and asymptotic behavior of fourth order differential equation without"maxima", see for example,[6, 40, 63, 64, 67–69, 112], and the references cited therein. As far as the authors know that there is no result available in the literature on the oscillation of fourth order differential equations with "maxima". Motivated by these observation, in this chapter, we present some necessary and sufficient conditions for all solutions of equation (6.1.1) to be oscillatory. The result extends that of in [104] for the equation (6.1.1) without "maxima".

In Section 6.2, we establish necessary and sufficient conditions for equation (6.1.1) to have nonoscillatory solutions with specified asymptotic behavior and in Section 6.3, we obtain some criteria for the oscillation of all solutions of equation (6.1.1) and in Section 6.4, we present some examples to illustrate the main results.

6.2 Existence of Nonoscillatory Solutions

In this section, we give some criteria for the existence of nonoscillatory solutions of equation (6.1.1) with some specific asymptotic behavior. We state and prove some lemmas, which are useful in establishing our main results. We define the corresponding function $z(t)$ by $z(t) = x(t) + p(t)x(h(t))$,

$$R(t) = \int_{t_0}^{t} \int_{t_0}^{s} \left(\frac{\tau}{r(\tau)} \right) d\tau ds,$$

and

$$R(t,T) = \int_{T}^{t} \int_{T}^{s} \left(\frac{\tau - T}{r(\tau)} \right) d\tau ds.$$

Lemma 6.2.1. *Let $x(t)$ be an eventually positive solution of equation* (6.1.1), *then there are only the following two cases for $z(t)$, for large t hold:*

(I) $z(t) > 0,\ z'(t) > 0,\ r(t)z''(t) > 0$ *and* $(r(t)z''(t))' \geq 0$*;*

(II) $z(t) > 0,\ z'(t) > 0,\ r(t)z''(t) < 0$ *and* $(r(t)z''(t))' \geq 0$ *for all* $t \geq t_1 \geq t_0$*;*

The proof of Lemma 6.2.1 can be modelled as that of in [104].

Lemma 6.2.2. *Let $x(t)$ be an eventually negative solution of equation* (6.1.1)*. Then only the following two cases for $z(t)$, for large t hold:*

(I) $z(t) < 0,\ z'(t) < 0,\ r(t)z''(t) < 0$ *and* $(r(t)z''(t))' \geq 0$*;*

(II) $z(t) < 0,\ z'(t) < 0,\ r(t)z''(t) > 0$ *and* $(r(t)z''(t))' \geq 0$ *for* $t \geq t_1 \geq t_0$.

The proof of Lemma 6.2.2 is analogous to that of Lemma 6.2.1.

Lemma 6.2.3. *If $T \geq t_0$, then*

$$\lim_{t\to\infty} \frac{R(t,T)}{R(t)} = 1.$$

Lemma 6.2.4. *Let $x(t)$ be an eventually positive solution of equation* (6.1.1)*, then there exists $T \geq t_0$ and $k > 0$ such that*

$$\frac{1}{2}\left(r(t)z''(t)\right)' R(t) \leq z(t) \leq kR(t). \tag{6.2.1}$$

for $T \geq t_0$.

Lemma 6.2.5. *Let $x(t)$ be an eventually positive solution of equation* (6.1.1)*, then there exists $t_1 \geq t_0$ such that for any $T \geq t_1$ we have*

$$z(t) \geq \int_T^t R(s,T)q(s) \max_{[\sigma(t),t]} f(x(s)) \quad for \quad t \geq T. \tag{6.2.2}$$

The proofs of Lemmas 6.2.3, 6.2.4 and 6.2.5 can be modelled as that of in [71, 113], and hence the details are omitted.

Lemma 6.2.6. *Let $x(t)$ be an eventually positive solution of equation* (6.1.1)*, then there exists $T \geq t_0$ such that*

$$(1 - p(t))z(t) \leq x(t) \leq z(t) \ for \ t \geq T. \tag{6.2.3}$$

Proof. Let $x(t)$ be an eventually positive solution of equation (6.1.1) for $t \geq T$. Then from the definition of $z(t)$, we have $z(t) \geq x(t)$ for $t \geq T$. From Lemma 6.2.1, we have $z(t) > 0$ and $z'(t) > 0$ for $t \geq T$. Hence

$$z(t) - p(t)z(h(t)) = x(t) - p(t)p(h(t))x(h(t)) \leq x(t)$$

or

$$(1 - p(t))z(t) \leq z(t) - p(t)z(h(t)) \leq x(t) \ for\ t \geq T.$$

This completes the proof of the lemma. □

Lemma 6.2.7. *Let $x(t)$ be an eventually positive solution of equation (6.1.1), then there exists $T \geq t_0$ such that*

$$z'(t) \geq \frac{1}{2}(r(t)z''(t))'R'(t) for\ t \geq T.$$

Also if $\sigma(t) \leq t$, then

$$z'(\sigma(t)) \geq \frac{1}{2}(r(t)z''(t))'R'(\sigma(t)) for\ t \geq T. \tag{6.2.4}$$

Proof. From Lemma 6.2.1, we have for $t \geq t_1 \geq t_0$,

$$z(t) > 0,\ z'(t) > 0,\ (r(t)z''(t))' > 0 \text{ and } (r(t)z''(t))'' < 0.$$

Hence,

$$\begin{aligned} z'(t) &\geq \int_{t_1}^{t} z''(s)ds = \int_{t_1}^{t} \frac{1}{r(s)} r(s)z''(s)ds \\ &\geq \int_{t_1}^{t} \frac{1}{r(s)} \left(\int_{t_1}^{s} (r(\tau)z''(\tau))'d\tau \right) ds \\ &\geq (r(t)z''(t))' \int_{t_1}^{t} \frac{s - t_1}{r(s)} ds = (r(t)z''(t))'R'(t, t_1). \end{aligned}$$

From Lemma 6.2.2, we conclude that there exists $T \geq t_1$ such that

$$R'(t, t_1) \geq \frac{1}{2}R'(t) \ for\ t \geq T,$$

and hence

$$z'(t) \geq \frac{1}{2}(r(t)z''(t))'R'(t) \ for \ t \geq T.$$

Since, $(r(t)z''(t))'' < 0$ and $\sigma(t) \leq t$,we have

$$z'(\sigma(t)) \geq \frac{1}{2}(r(\sigma(t)z''(\sigma(t)))' \geq \frac{1}{2}(r(t)z''(t))'R'(\sigma(t)) \ for \ t \geq T.$$

This completes the proof. □

Theorem 6.2.1. *A necessary and sufficient condition for the equation* (6.1.1) *to have a nonoscillatory solution* $x(t)$ *such that* $\lim_{t\to\infty} \frac{z(t)}{R(t)} = \alpha \neq 0$ *is*

$$\int_{t_0}^{\infty} q(t)|\max_{[\sigma(t),t]} f(c(1-p(s))R(s))|dt < \infty, \tag{6.2.5}$$

for some $c \neq 0$.

Proof. Necessity: Let $x(t)$ be a nonoscillatory solution of equation (6.1.1) such that $\lim_{t\to\infty} \frac{z(t)}{R(t)} = \alpha \neq 0$. We may assume that $x(t)$ is eventually positive. Then there exist positive constants α_1 and α_2 such that

$$\alpha_1 R(\sigma(t)) \leq z(\sigma(t)) \leq \alpha_2 R(\sigma(t)) \text{ for } \ t \geq T \geq t_0. \tag{6.2.6}$$

In view of (6.2.3) and (6.2.6), we obtain

$$x(\sigma(t)) \geq (1 - p(\sigma(t))\alpha_1 R(\sigma(t)) \text{ for } \ t \geq T. \tag{6.2.7}$$

Since, $(r(t)z''(t))' > 0$ by Lemma 6.2.1 on integrating (6.1.1) from T to ∞, we have

$$\int_{T}^{\infty} q(t)\max_{[\sigma(t),t]} f(x(s))dt < \infty. \tag{6.2.8}$$

From (6.2.7) and (6.2.8) we conclude that,

$$\int_{T}^{\infty} q(t)\max_{[\sigma(t),t]} f\left(1 - p(\sigma(s))\alpha_1 R(\sigma(s))\right) dt < \infty.$$

Sufficiency: Suppose that (6.2.5) holds for some $c \neq 0$. We may assume that $c > 0$ since a similar argument holds if $c < 0$. Let $d > 0$ be such that $\frac{4d}{1-p} < c$ and choose $T \geq t_0$ so large that

$$\int_T^\infty q(t) \max_{[\sigma(t),t]} f(c(1-p(\sigma(s))R(\sigma(s)))dt < \frac{(1-p)d}{8},$$

and

$$T_0 = \min\{T, \inf_{t\geq T} h(t), \inf_{t\geq T} \sigma(t)\} \geq t_0.$$

Let $C[T_0,\infty)$ be locally convex space of all continuous function on $[T_0,\infty)$ with the topology of uniform convergence on any compact subintervals of $[T_0,\infty)$. Define a closed convex subset X of $[T_0,\infty)$ by $X = \{x \in C[T_0,\infty) : 2(1-p)dR(t) \leq x(t) \leq 4dR(t)$ on $[T,\infty)$ and $x(t) = 0$ on $[T_0,T]\}$.

Now we define an operator $F : X \to C[T_0,\infty)$ by

$$(Fx)(t) = \begin{cases} (3+p)dR(t) - p(t)x(h(t)) \\ \quad + \int_T^t \int_T^{s_3} \frac{1}{r(s_2)} \int_T^{s_2} \int_{s_1}^\infty q(t) \max_{[\sigma(t),t]} f(x(s))ds ds_1 ds_2 ds_3, & t \geq T \\ 0, & T_0 \leq t \leq T. \end{cases}$$

It is a matter of routine calculation to verify that F is continuous mapping which sends X into relatively compact subset of X. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of an element $x \in X$ such that $Fx = x$. It is easy to see that $x(t)$ is a solution of (6.1.1) for $t \geq T_0$. Since, by L'Hospitals rule

$$\begin{aligned} \lim_{t\to\infty} \frac{z(t)}{R(t)} &= \lim_{t\to\infty} \frac{z'(t)}{R'(t)} \\ &= \lim_{t\to\infty} \frac{r(t)z''(t)}{m} r(t)R''(t) \\ &= \lim_{t\to\infty} \frac{r(t)z''(t)}{t} \lim_{t\to\infty} (r(t)z''(t)) \\ &= (3+p)d, \end{aligned}$$

it turns out that $x(t)$ is a nonoscillatory solution of (6.1.1) with the desired asymptotic property. The proof is complete. □

Theorem 6.2.2. *A necessary and sufficient condition for the equation* (6.1.1) *to have a nonoscillatory solution* $x(t)$ *such that*

$\lim_{t\to\infty} z(t) = \beta \neq 0$ *is that*

$$\int_{t_0}^{\infty} R(t)q(t)|\max_{[\sigma(t),t]} f(c(1-p(s)))|dt < \infty \tag{6.2.9}$$

for some $c \neq 0$.

Proof. Necessity: Let $x(t)$ be a nonoscillatory solution of equation (6.1.1) such that $\lim_{t\to\infty} z(t) = \beta \neq 0$. We may assume that $x(t)$ is eventually positive. Then there exist constants $\beta_1 > 0, \beta_2 > 0$ for which $\beta_1 \leq z(\sigma(t)) \leq \beta_2$ for $t \geq T \geq t_0$. Hence from (6.2.3) we have

$$x(\sigma(t)) \geq \beta_1(1 - p(\sigma(t)) \text{ for } t \geq T. \tag{6.2.10}$$

Multiplying equation (6.1.1) by $R(t)$ and integrating it from T to t, we obtain

$$\int_T^t R(s)q(s) \max_{[\sigma(s),s]} f(x(u))ds = -\int_T^t R(s)(r(s)z''(s))''ds$$
$$- R(t)(r(t)z''(t))' + R'(t)r(t)z''(t) = tz''(t) + z(t) + K, \tag{6.2.11}$$

where K is a constant. Observe that $z(t)$ is subject to the Case(II) of Lemma 6.2.1, we deduce from (6.2.11) that

$$\int_T^{\infty} R(t)q(t) \max_{[\sigma(t),t]} f(x(s))dt < \infty. \tag{6.2.12}$$

From (6.2.10) and (6.2.12) we have,

$$\int_T^{\infty} R(t)q(t) \max_{[\sigma(t),t]} f(\beta_1(1-p(s)))dt < \infty.$$

Sufficiency: Suppose that (6.2.9) holds for some constant $c \neq 0$. We may assume that $c > 0$ since a similar argument holds if $c < 0$. Let $d > 0$ be such that $d \leq \frac{c(1-p)}{2}$ and take $T \geq t_0$ so large that

$$\int_T^{\infty} R(t)q(t) \max_{[\sigma(t),t]} f(1-p(s))dt < \frac{(1-p)d}{4}.$$

We denote by X the set of function $x \in [T_0, \infty)$ satisfying

$$(1-p(t))d \le x(t) \le 2d \text{ on } [T.\infty)$$

and $x(t) = x(T)$ on $[T_0, T]$. and define the operator $F : X \to C[T_0, \infty]$ by

$$(Fx)(t) = \begin{cases} (1+p)d - p(t)x(h(t)) \\ \quad + \int_T^t \int_{s_3}^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty q(t) \max_{[\sigma(t),t]} f(x(s)) ds ds_1 ds_2 ds_3, \ t \ge T \\ (Fx)(T), \qquad T_0 \le t \le T. \end{cases}$$

By Schauder-Tychonoff fixed point theorem F has a fixed point $x \in X$ is a nonoscillatory solution of (6.1.1) for $t \ge T_0$. Since,

$$z'(t) = \int_t^\infty \frac{1}{r(s_2)} \int_{s_2}^\infty \int_{s_1}^\infty q(s) \max_{[\sigma(t),t]} f(x(s)) ds ds_1 ds_2 > 0,$$

it follows that

$$\lim_{t \to \infty} z(t) = \beta \in [(1-p)d, 2d].$$

The proof is complete. □

6.3 Oscillation Results

In this section, we establish conditions for the oscillation of all solutions of equation (6.1.1). We begin with the following definition:

Definition 6.3.1. *Equation* (6.1.1) *is called strongly sublinear if there exists a number* $\alpha < 1$ *such that*

$$\frac{|f(x_1)|}{|x_1|^\alpha} \le \frac{|f(x_2)|}{|x_2|^\alpha}$$

for $|x_1| > |x_2|$, $x_1 x_2 > 0$.

Theorem 6.3.1. *A necessary and sufficient condition for all solutions of equation* (6.1.1) *are oscillatory is that*

$$\int_{t_0}^\infty q(t) | \max_{[\sigma(t),t]} f(c(1-p(s))R(s))| dt = \infty \qquad (6.3.1)$$

for all $c \ne 0$.

Proof. The necessity part follows from the sufficiency part of Theorem 6.2.1. Now we prove the sufficiency of condition (6.3.1).Assume that there exist a nonoscillatory solution $x(t)$ of equation (6.1.1). Without loss of generality we may assume that $x(t)$ is eventually positive. From Lemmas 6.2.1, 6.2.4 and 6.2.6, there exist $T \geq t_0$ and $K > 0$ such that $z(t) > 0,\ z'(t) > 0$ and $(r(t)z''(t))' > 0$ for $t \geq T$,

$$x(\sigma(t)) \geq (1 - p(\sigma(t))z(\sigma(t)) \text{ for } t \geq T. \tag{6.3.2}$$

and

$$\frac{1}{2}(r(t)z''(t))'R(t) \leq z(t) \leq KR(t),\ t \geq T. \tag{6.3.3}$$

Since $\sigma(t) \leq t$ and $(r(t)z''(t))' < 0$, we have from Lemma 6.2.4,

$$z(\sigma(t)) \geq \frac{1}{2}(r(t)z''(t))'R(\sigma(t)),\ t \geq T. \tag{6.3.4}$$

From (6.3.2)-(6.3.4) and the strongly sublinearity of f, we have

$$\begin{aligned}\left(-((r(t)z''(t))')^{1-\alpha}\right)' &= (1-\alpha)((r(t)z''(t))')^{-\alpha}q(t)\max_{[\sigma(t),t]} f(x(s)) \\ &\geq \frac{1-\alpha}{(2K)^\alpha}q(t)\max_{[\sigma(t),t]} f((1-p(s))R(\sigma(s))).\end{aligned}$$

Integrating the last inequality from T to t, we obtain

$$\frac{1-\alpha}{(2K)^\alpha}\int_T^t q(s)\max_{[\sigma(t),t]} f((1-p(s))R(s))ds \leq \left((r(T)z''(t))'\right)^{1-\alpha}$$

which leads to

$$\int_T^\infty q(s)\max_{[\sigma(t),t]} f((1-p(s))R(s))ds < \infty,$$

which contradicts (6.3.1). This completes the proof. □

Theorem 6.3.2. *Assume that there exist a constant M such that*

$$\frac{f(u)}{u} \geq M > 0$$

for all $u \neq 0$. *If there exist a positive differential function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)\left[q(s)\max_{[\sigma(s),s]}(1-p(u)) - \frac{(\rho'(s))^2}{2cMR'(s)\rho^2(s)}\right] ds = \infty, \tag{6.3.5}$$

then all solutions of equation (6.1.1) *are oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (6.1.1) and assume without loss of generality that $x(t)$ is eventually positive. From Lemmas 6.2.1 and 6.2.6, we have $z(t) > 0,\ z(\sigma(t)) > 0,\ z'(t) > 0$ and $(r(t)z''(t))' > 0$ for $t \geq T$

and

$$x(\sigma(t)) \geq ((1 - p(\sigma(t)))z(\sigma(t)).$$

Define

$$w(t) = \frac{\rho(t)(r(t)z''(t))'}{z(\sigma(t))},\ t \geq T.$$

Then in view of (6.2.4) and Definition 6.3.1, we have

$$\begin{aligned} w'(t) &\leq -M\rho(t)q(t)\max_{[\sigma(t),t]}(1-p(\sigma(s)) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{cR'(t)w^2(t)}{2\rho(t)} \\ &\leq -M\rho(t)q(t)\max_{[\sigma(t),t]}(1-p(\sigma(s)) + \frac{(\rho'(t))^2}{2cM\rho^2(s)R'(s)}. \end{aligned}$$

Integrating the last inequality from T to $t \geq T$, we obtain

$$\int_{T}^{t} \rho(s)\left[q(s)\max_{[\sigma(s),s]}(1-p(u)) - \frac{(\rho'(s))^2}{2cMR'(s)\rho^2(s)}\right] ds \leq \frac{w(t)}{M}$$

and this contradicts (6.3.5). Thus the proof is complete. □

Corollary 6.3.1. *Suppose that* $q(t) \geq 0$ *for all* $t \geq t_0$ *and there exists a positive differential function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \rho(s)\left[q(s)\max_{[\sigma(s),s]}(1-p(u)) - \left(\frac{\rho'(s)}{s\rho(s)}\right)^2\right] ds = \infty,$$

then all solutions of equation (6.1.1) *are oscillatory.*

6.4 Examples

In this section, we present some examples to illustrate the main results.

Example 6.4.1. *Consider the neutral differential equation of the form*

$$\left(t\left(x(t)+\frac{1}{2}x(t-2)\right)''\right)''+e^{-t}\max_{[t/2,t]}x^3(s)=0,\quad t\geq 1. \tag{6.4.1}$$

Here, $r(t)=t,\ p(t)=\frac{1}{2},\ q(t)=e^{-t},\ \tau(t)=t-2,\ \sigma(t)=t/2$ *and* $f(x)=x^3$. *It is easy to see that all conditions of Theorem 6.2.1 are satisfied and hence every solution* $x(t)$ *of equation* (6.4.1) *with* $\lim_{t\to\infty}\frac{z(t)}{R(t)}=\alpha\neq 0$, *is nonoscillatory.*

Example 6.4.2. *Consider the neutral differential equation of the form*

$$\left(t\left(x(t)+\frac{1}{2}x(t-2)\right)''\right)''+\frac{1}{t^4}\max_{[t/2,t]}x^3(s)=0,\quad t\geq 1. \tag{6.4.2}$$

Here, $r(t)=t,\ p(t)=\frac{1}{2},\ q(t)=\frac{1}{t^4},\ \tau(t)=t-2,\ \sigma(t)=t/2$ *and* $f(x)=x^3$. *It is easy to see that all conditions of Theorem 6.2.2 are satisfied and hence every solution* $x(t)$ *of equation* (6.4.2) *with* $\lim_{t\to\infty}z(t)=\beta\neq 0$, *is nonoscillatory.*

Example 6.4.3. *Consider the neutral differential equation of the form*

$$\left(t^2\left(x(t)+\frac{1}{\sqrt{t-1}}x(t-1)\right)''\right)''+t\max_{[t-1,t]}x^{1/3}(s)=0,\quad t\geq 2. \tag{6.4.3}$$

Here, $r(t)=t^2,\ p(t)=\frac{1}{\sqrt{(t-1)}},\ q(t)=t,\ \tau(t)=t-1,\ \sigma=1$ *and* $f(x)=x^{1/3}$. *It is easy to see that all conditions of Theorem 6.3.1 are satisfied and hence every solution of equation* (6.4.3) *is oscillatory.*

Example 6.4.4. *Consider the neutral differential equation of the form*

$$\left(\frac{1}{t+\pi}\left(x(t)+\frac{1}{t}x(t-\pi)\right)''\right)''+t^3\max_{[t-\pi,t]}(1+x^2(s))=0. \tag{6.4.4}$$

Here, $r(t)=\frac{1}{t+\pi},\ p(t)=\frac{1}{t},\ q(t)=t^3,\tau(t)=t-\pi,\sigma=\pi$ *and* $f(x)=1+x^2$. *By taking* $\rho(t)=1$, *it is easy to see that all conditions of Theorem 6.3.2 are satisfied and hence every solution of equation* (6.4.4) *is oscillatory.*

We conclude this chapter with the following remark.

Remark 6.4.1. *It is interesting to obtain oscillation criteria for the equation* (6.1.1) *when f is strongly superlinear and/or*

$$\int_{t_0}^{\infty} \frac{1}{r(t)} dt < \infty.$$