Chapter 5

Third Order Nonlinear Neutral Delay Differential Equations with "Maxima"

5. Third order Nonlinear Neutral Delay Differential Equations with "Maxima"

5.1 Introduction

In this chapter, we investigate the oscillatory behavior of third order nonlinear neutral delay differential equations with "maxima" of the form

$$\big(a(t)\left((b(t)(x(t)+p(t)x(\tau(t))')'\right)\big)' + q(t)\max_{[\sigma(t),t]} x^{\gamma}(s) = 0, \quad t \geq t_0 \geq 0 \tag{5.1.1}$$

subject to the following conditions:

(C_1) γ is the ratio of odd positive integers;

(C_2) $a(t) \in C^1([t_0,\infty),R)$, $b(t) \in C^2([t_0,\infty),R)$, $p(t) \in C^3([t_0,\infty),R)$, $q(t) \in C([t_0,\infty),R)$, $a(t) > 0$, $b(t) > 0$, $q(t) > 0$ for all $t \geq t_0$;

(C_3) $\tau(t)$, $\sigma(t) \in C([t_0,\infty),R)$, $\tau(t) \leq t, \sigma(t) < t$, $\sigma(t)$ is nondecreasing and $\lim_{t\to\infty} \sigma(t) = \infty$.

Further, we will consider the following two cases

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty, \quad \int_{t_0}^{\infty} \frac{dt}{b(t)} = \infty, \tag{5.1.2}$$

and

$$\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty, \quad \int_{t_0}^{\infty} \frac{dt}{b(t)} < \infty. \tag{5.1.3}$$

By a solution of equation (5.1.1), we mean a continuous function $x(t) \in C([T_x,\infty))$, $T_x \geq t_0$, such that $x(t)+p(t)x(\tau(t))$ is thrice differentiable and $x(t)$ satisfying equation (5.1.1) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (5.1.1) which satisfy $\sup\{|x(t)| : t \geq T_x\} > 0$ for all $t \geq T_x$.

In the previous chapter, we have discussed the oscillatory and asymptotic behavior of all solutions of equation (4.1.1) when $\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty$ and $\int_{t_0}^{\infty} \frac{dt}{b(t)} = \infty$. But in this chapter, we study the oscillatory behavior of all solutions of equation (5.1.1) when the conditions (5.1.2) or (5.1.3) holds.

In Section 5.2, we obtain some criteria for the almost oscillation of all solutions of equation (5.1.1) and in Section 5.3, we present some examples to illustrate the main results.

5.2 Oscillation Results

In the following, we will establish some oscillation criteria for equation (5.1.1). To simplify our notation , let us denote

$$\begin{aligned} z(t) &= x(t) + p(t)x(\tau(t)), \\ y(t) &= -w(t) = -a(t)(b(t)z'(t))', \end{aligned}$$

and

$$B(t) = \int_t^{\infty} \frac{ds}{b(s)}.$$

We begin with the following theorem.

Theorem 5.2.1. *Let condition* (5.1.2) *holds. Assume that there exist numbers* $\alpha \geq \gamma,\ \beta \geq \gamma$ *and a function* $\delta \in C([t_0, \infty), R)$ *such that* $\alpha,\ \beta$ *are ratio of odd positive integers,* $\delta(t)$ *is nondecreasing and* $\delta(t) > t$*. If the condition* (3.2.6) *and for all sufficiently large for all* $t_2 \geq t_1 \geq t_0 \geq 0$*, the first order delay differential equation*

$$w'(t) + c_1^{\gamma-\alpha} q(t) \left(\max_{[\sigma(t),t]} (1 - p(s))^{\gamma} \right) \left(\int_{t_2}^{\sigma(t)} \frac{\int_{t_1}^{s} \frac{du}{a(u)}}{b(s)} \right)^{\alpha} w^{\alpha}(\sigma(t)) = 0 \qquad (5.2.1)$$

is oscillatory for all positive constants c_1 *and the first order advanced differential* *equation*

$$y'(t) - c_2^{\gamma-\beta} q(t) \left(\max_{[\sigma(t),t]} (1-p(s))^{\gamma} \right) \left(\int_{\delta(t)}^{\infty} \frac{ds}{a(s)} \right)^{\beta} \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)} \right)^{\gamma} y^{\beta}(\delta(t)) = 0 \tag{5.2.2}$$

is oscillatory for all positive constants c_2*, then every solution of equation* (5.1.1) *is almost oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (5.1.1). Without loss of generality we may assume that $x(t)$ is positive , since the proof for the case negative is similar. Then the corresponding function $z(t)$ satisfies the following three possible cases:

Case(I) $z(t) > 0,\ z'(t) > 0,\ (b(t)z'(t))' > 0,\ (a(t)(b(t)z'(t))')' \leq 0;$

Case(II) $z(t) > 0,\ z'(t) > 0,\ (b(t)z'(t)) < 0',\ (a(t)((b(t)z'(t))')' \leq 0;$

and

Case(III) $z(t) > 0,\ z'(t) < 0,\ (b(t)z'(t))' > 0,\ (a(t)(b(t)z'(t))')' \leq 0$ for all $t \geq t_1 \geq t_0$.

Assume that Case(I) holds. From the definition of $z(t)$ we have

$$z(t) = x(t) + p(t)x(\tau(t))$$

or

$$x(t) = z(t) - p(t)x(\tau(t)) \geq (1-p(t))z(t). \tag{5.2.3}$$

From equation (5.1.1) and (5.2.3) we have

$$\left(a(t)\left((b(t)(z'(t))'\right)' = -q(t) \left(\max_{[\sigma(t),t]} ((1-p(s))^{\gamma} \right) z^{\gamma}(t) \leq 0. \tag{5.2.4}$$

Using $\left(a(t)\left((b(t)(z'(t))'\right)' < 0$, we obtain

$$\begin{aligned} b(t)(z'(t)) &\geq \int_{t_1}^{t} \frac{a(s)(b(s)z'(s))'}{a(s)} ds \\ &\geq a(t)(b(t)z'(t))' \int_{t_1}^{t} \frac{ds}{a(s)}. \end{aligned}$$

Dividing the last inequality by $b(t)$ and then integrating from t_2 to t, we get

$$z(t) \geq a(t)(b(t)z'(t))' \int_{t_2}^{t} \left(\frac{\int_{t_1}^{s} \frac{du}{a(u)}}{b(s)} \right) ds. \tag{5.2.5}$$

From (5.2.4) and the fact that $z'(t) > 0$, we see that there is a constant $c_1 > 0$ such that

$$\left(a(t)\left((b(t)(z'(t))'\right)' + c_1^{\gamma-\alpha} q(t) \left(\max_{[\sigma(t),t]} ((1-p(s))^{\gamma} \right) z^{\alpha}(\sigma(t)) \leq 0. \tag{5.2.6}$$

Using (5.2.5) in (5.2.6), we obtain

$$w'(t) + c_1^{\gamma-\alpha} q(t) \left(\max_{[\sigma(t),t]} (1-p(s))^{\gamma} \right) \left(\int_{t_2}^{\sigma(t)} \frac{\int_{t_1}^{\sigma(t)} \frac{du}{a(u)}}{b(s)} ds \right)^{\alpha} w^{\alpha}(\sigma(t)) \leq 0.$$

In view of Theorem 2.1 of [89], the associated delay differential equation (5.2.1) also has a positive solution, which is a contradiction.

Assume that Case (II) holds. Since $a(t)(b(t)z'(t))' \leq 0$, we see that $a(t)(b(t)z'(t))'$ is decreasing. Then we get

$$a(s)(b(s)z'(s))' \leq a(t)(b(t)z'(t))' \text{ for all} s \geq t \geq t_1.$$

Dividing the last inequality by $a(s)$ and then integrating the resulting inequality from t to ℓ we have

$$b(l)z'(l) \leq b(t)z'(t) + a(t)(b(t)z'(t))' \int_{t}^{\ell} \frac{ds}{a(s)}.$$

Letting $\ell \to \infty$, we have

$$b(t)z'(t) \leq -a(t)(b(t)z'(t))' \int_{t}^{\ell} \frac{ds}{a(s)}. \tag{5.2.7}$$

Using conditions $z(t) > 0$ and $(b(t)z'(t))' \leq 0$, we have

$$z(t) \geq b(t)z'(t) \int_{t_1}^{t} \frac{ds}{b(s)}. \tag{5.2.8}$$

Thus,

$$\left(\frac{z(t)}{\int_{t_1}^{t}\frac{ds}{b(s)}}\right)' \le 0. \tag{5.2.9}$$

Combining (5.2.7) and (5.2.8), we have

$$z(t) \ge -a(t)(b(t)z'(t))' \int_t^{\infty} \frac{ds}{a(s)} \int_{t_1}^{t} \frac{ds}{b(s)}. \tag{5.2.10}$$

On the other hand, we have by (5.2.9) and $\delta(t) \ge \sigma(t)$ that

$$\begin{aligned} z^{\gamma}(\sigma(t)) &\ge \left(\frac{\int_{t_1}^{\sigma(t)}\frac{ds}{b(s)}}{\int_{t_1}^{\delta(t)}\frac{ds}{b(s)}}\right)^{\gamma} z^{\gamma}(\delta(t)) \\ &= \left(\frac{\int_{t_1}^{\sigma(t)}\frac{ds}{b(s)}}{\int_{t_1}^{\delta(t)}\frac{ds}{b(s)}}\right)^{\gamma} z^{\gamma}(\delta(t)) z^{\gamma-\beta}(\delta(t)). \end{aligned} \tag{5.2.11}$$

By (5.2.8), there exists a constant c_2 such that $z(t) \ge c_2 \int_{t_1}^{t} \frac{ds}{b(s)}$. By (5.2.11), we get

$$\begin{aligned} z^{\gamma}(\sigma(t)) &\ge c_2^{\gamma-\beta} \left(\frac{\int_{t_1}^{\sigma(t)}\frac{ds}{b(s)}}{\int_{t_1}^{\sigma(t)}\frac{ds}{b(s)}}\right)^{\gamma} z^{\beta}(\delta(t)) \left(\int_{t_1}^{\delta(t)} \frac{ds}{b(s)}\right)^{\gamma-\beta} \\ &= c_2^{\gamma-\beta} \left(\int_{t_1}^{\delta(t)} \frac{ds}{b(s)}\right)^{-\beta} \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)}\right)^{\gamma} z^{\beta}(\delta(t)). \end{aligned} \tag{5.2.12}$$

Combining (5.2.10) and (5.2.12), we obtain

$$z^{\gamma}(\sigma(t)) \ge c_2^{\gamma-\beta} \left(\int_{\delta(t)}^{\infty} \frac{ds}{a(s)}\right)^{\beta} \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)}\right)^{\gamma} (-w(\delta(t)))^{\beta}. \tag{5.2.13}$$

Using (5.2.13) in (5.2.4), we have

$$w'(t) + c_2^{\gamma-\beta} q(t) \left(\max_{[\sigma(t),t]} (1-p(s))^{\gamma}\right) \left(\int_{\delta(t)}^{\infty} \frac{ds}{a(s)}\right)^{\beta} \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)}\right)^{\gamma} \left(-w(\delta(t))^{\beta}\right) \le 0,$$

or

$$y'(t) - c_2^{\gamma-\beta} q(t) \left(\max_{[\sigma(t),t]} (1-p(s))^{\gamma}\right) \left(\int_{\delta(t)}^{\infty} \frac{ds}{a(s)}\right)^{\beta} \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)}\right)^{\gamma} y^{\beta}(\delta(t)) \ge 0. \tag{5.2.14}$$

We deduce from Lemma 2.3 of [10], the associated advanced differential equation (5.2.2) also has a positive solution, which is a contradiction.

Next assume that Case (III) holds. Then by Lemma 3.2.6, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. $\square$

Corollary 5.2.1. *Let condition* (5.1.2) *holds and let* $\gamma = 1$. *Assume that there exist a function* $\delta \in C([t_0, \infty), R)$ *such that,* $\delta(t)$ *is nondecreasing and* $\delta(t) > t$. *If* (3.2.6) *and for all sufficiently large* $t_2 > t_1 \geq t_0$,

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s) \max_{[\sigma(s),s]} (1 - p(s)) \int_{t_2}^{\sigma(s)} \left(\frac{\int_{t_1}^{v} \frac{du}{a(u)}}{b(v)} \right) dvds > \frac{1}{e}, \tag{5.2.15}$$

and

$$\liminf_{t\to\infty} \int_{t}^{\delta(t)} q(s) \max_{[\sigma(s),s]} (1 - p(s)) \int_{\delta(s)}^{\infty} \frac{du}{a(u)} \int_{t_1}^{\sigma(s)} \frac{du}{b(u)} > \frac{1}{e}, \tag{5.2.16}$$

then every solution of equation (5.1.1) *is almost oscillatory.*

Proof. By taking $\alpha = \beta = 1$ and proceeding as in Theorem 5.2.1, we get

$$w'(t) + q(t) \left(\max_{[\sigma(t),t]} (1 - p(s)) \right) \left(\int_{t_2}^{\sigma(t)} \frac{\int_{t_1}^{s} \frac{du}{a(u)}}{b(s)} \right) w(\sigma(t)) \leq 0$$

and

$$y'(t) - q(t) \left(\max_{[\sigma(t),t]} (1 - p(s)) \right) \left(\int_{\delta(t)}^{\infty} \frac{ds}{a(s)} \right) \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)} \right) y(\delta(t)) \geq 0,$$

respectively. Applications of Theorem 5.2.1 with Lemma 2.1 of [38], we obtain the desired result. This completes the proof. $\square$

Corollary 5.2.2. *Let condition* (5.1.2) *holds and let* $\gamma < 1$. *Assume that there exist a number* $\beta > 1$ *and a function* $\delta \in C([t_0, \infty), R)$ *such that,* β *is the ratio of odd positive integers,* $\delta(t)$ *is nondecreasing and* $\delta(t) > t$. *If* (3.2.6) *and for all sufficiently large* $t_3 > t_2 > t_1 \geq 0$,

$$\int_{t_3}^{\infty} q(t) \left(\max_{[\sigma(t),t]} (1 - p(s))^{\gamma} \right) \left(\int_{t_2}^{\sigma(t)} \frac{\int_{t_1}^{s} \frac{du}{a(u)}}{b(s)} ds \right)^{\gamma} dt = \infty, \tag{5.2.17}$$

and

$$\int_{t_2}^{\infty} q(t) \left(\max_{[\sigma(t),t]} (1 - p(s))^{\gamma} \right) \left(\int_{\delta(t)}^{\infty} \frac{ds}{a(s)} \right)^{\beta} \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)} \right)^{\gamma} dt = \infty,$$

then every solution of equation (5.1.1) *is almost oscillatory.*

Proof. Let $\alpha = \gamma$. Applications of Theorem 5.2.1 with Lemma 2.1 of [65], we obtain the desired result. This completes proof. $\square$

Corollary 5.2.3. *Let condition* (5.1.2) *holds and let* $\gamma > 1$, *and* $\sigma \in C'([t_0, \infty), R)$. *Assume that there exist functions* $\xi, \delta \in C([t_0, \infty), R)$ *such that* $\xi(t) > t$, $\xi(t)$ *is nondecreasing,* $\sigma(\xi(\xi(t))) < t$, $\delta(t)$ *is nondecreasing and* $\delta(t) > t$. *Suppose also that there exists a function* $\phi \in C'([t_0, \infty), R)$ *such that* $\phi'(t) > 0$, $\lim_{t\to\infty} \phi(t) = \infty$. *If* (3.2.6) *and*

$$\limsup_{t\to\infty} \frac{\phi'(\sigma(t))(\sigma'(t))}{\phi'(t)} \leq 1,$$

$$\liminf_{t\to\infty} \frac{\left(\frac{1}{b(t)} \int_t^{\xi(t)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1) ds_1 ds_2\right) e^{-\phi(t)}}{\phi'(t)} > 0.$$

If for all sufficiently large $t_1 \geq t_0$ *and* $t_3 > t_2 > t_1$,

$$\int_{t_2}^{\infty} q(t) \left(\max_{[\sigma(t),t]} (1 - p(s))^{\gamma}\right) \left(\int_{\delta(t)}^{\infty} \frac{ds}{a(s)}\right)^{\beta} \left(\int_{t_1}^{\sigma(t)} \frac{ds}{b(s)}\right)^{\gamma} dt = \infty,$$

then every solution of equation (5.1.1) *is almost oscillatory.*

Proof. Let $\beta = \gamma$. Applications of Theorem 5.2.1 with Lemma 2.1 of [65], we obtain the desired result. This completes proof. $\square$

Theorem 5.2.2. *Let all conditions of Theorem 5.2.1 hold with* (5.1.2) *replaced by* (5.1.3). *If*

$$\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{t_0}^{v} \frac{1}{a(u)} \int_{t_0}^{u} q(s) \left(\max_{[\sigma(t),t]} (1 - p(s))\right)^{\gamma} B^{\gamma}(\sigma(s)) ds du dv = \infty, \tag{5.2.18}$$

then every solution of equation (5.1.1) *is almost oscillatory.*

Proof. Suppose $x(t)$ be a nonoscillatory solution of equation (5.1.1). Without loss of generality that $x(t)$ is positive. Then there exists four possible Cases (I), (II), (III)(as those of Theorem 5.2.1), and

Case(IV) $z(t) > 0$, $z'(t) < 0$, $(b(t)z'(t))' < 0$, $(a(t)(b(t)z'(t))')' < 0$ for $t \geq t_1$, where $t_1 \geq t_0$ is large enough. From the proof of Theorem 2.1, we can eliminate Cases (I),(II) and (III).

Consider now the Case (IV). Since $(b(t)z'(t))' \leq 0$, we get

$$z'(s) \geq \frac{b(t)z'(t)}{b(s)} \text{ for } s \geq t.$$

Integrating the last inequality from t to ℓ and letting $\ell \to \infty$ implies that

$$z(t) \geq -B(t)b(t)z'(t) \geq LB(t)$$

for some constants $L > 0$. From equation(5.2.2) we have,

$$\left(a(t)\left(b(t)(z'(t))\right)'\right)' + L^\gamma q(t) \max_{[\sigma(t),t]} \left((1-p(s))^\gamma\right) B^\gamma(\sigma(s))dsdu \quad \leq \quad 0.$$

Integrating again, we have

$$z'(t) \geq L^\gamma \int_{t_1}^{t} \frac{1}{b(v)} \int_{t_1}^{v} \frac{1}{a(v)} \int_{t_1}^{u} q(s) \left(\max_{[\sigma(t),t]} \left((1-p(s))^\gamma\right) \right) B^\gamma(\sigma(s))dsdudv + z(t),$$

which contradicts (5.2.18). This completes the proof. $\square$

Based on Theorem 5.2.2, similar to Corollaries 5.2.1, 5.2.2 and 5.2.3, we have obtain the following corollaries.

Corollary 5.2.4. *Let all conditions of Corollary 5.2.1 are hold with* (5.1.2) *replaced by* (5.1.3). *If*

$$\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{t_0}^{v} \frac{1}{a(u)} \int_{t_0}^{u} q(s) \left(\max_{[\sigma(t),t]} (1-p(s)) \right) B(\sigma(s))dsdudv = \infty,$$

then every solution of equation (5.1.1) *is almost oscillatory.*

Corollary 5.2.5. *Let condition* (5.1.3) *holds and let* $\gamma < 1$. *If*

$$\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{t_0}^{v} \frac{1}{a(u)} \int_{t_0}^{u} q(s) \left(\max_{[\sigma(t),t]} (1-p(s)) \right)^\gamma B^\gamma(\sigma(s))dsdudv = \infty,$$

then every solution of equation (5.1.1) *is almost oscillatory.*

Corollary 5.2.6. *Let condition* (5.1.3) *holds and let* $\gamma > 1$. *If*

$$\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{t_0}^{v} \frac{1}{a(u)} \int_{t_0}^{u} q(s) \left(\max_{[\sigma(t),t]} (1-p(s)) \right)^{\gamma} B^{\gamma}(\sigma(s)) ds du dv = \infty,$$

then every solution of equation (5.1.1) *is almost oscillatory.*

5.3 Examples

In this section, we present some examples to illustrate the main results.

Example 5.3.1. *Consider the neutral differential equation*

$$\left(e^t \left((x(t) + \frac{1}{2} x(t-1) \right)'' \right)' + \sqrt{2} e^t \max_{[t-\frac{15}{2},t]} x(s) = 0, \quad t \geq 1. \tag{5.3.1}$$

Here, $a(t) = e^t$, $p(t) = 1/2$, $\gamma = 1$, $q(t) = \sqrt{(}2)e^t$, $\tau(t) = t-1$ *and* $\sigma = 15/2$. *By taking* $\delta(t) = t+1$, *one can easily verify that all conditions of Corollary 5.2.1 are satisfied, hence every solution of equation* (5.3.1) *is almost oscillatory.*

Example 5.3.2. *Consider the neutral differential equation*

$$\left(t^2 \left((x(t) + \frac{1}{3} x(t-2)) \right)'' \right)' + t \max_{[t/8,t]} x^{\frac{1}{3}}(s) = 0, \quad t \geq 1. \tag{5.3.2}$$

Here, $a(t) = t^2$, $p(t) = 1/3$, $\gamma = 1/3$, $q(t) = t$, $\tau(t) = t-2$ *and* $\sigma(t) = t/8$. *By taking* $\beta = 9/7$ *and* $\delta = 2t$, *it is easy to see that all conditions of Corollary 5.2.2 are satisfied and hence every solution of equation* (5.3.2) *is almost oscillatory.*

Example 5.3.3. *Consider the neutral differential equation*

$$\left(e^t \left(t^2 (x(t) + \frac{1}{2} x(t-2))' \right)' \right)' + t e^{2t} \max_{[t-3,t]} x(s) = 0, \quad t \geq 1. \tag{5.3.3}$$

Here, $a(t) = e^t$, $b(t) = t^2$, $p(t) = 1/2$, $\gamma = 1$, $q(t) = te^{2t}$, $\tau(t) = t-2$, $\sigma(t) = t-3$. *Then we get* $B(t) = 1/t$. *By taking* $\delta(t) = t+1$, *it is easy to see that all conditions of Corollary 5.2.4 are satisfied and hence every solution of equation* (5.3.3) *is almost oscillatory.*

We conclude this chapter with the following remark.

Remark 5.3.1. *It would be interesting to obtain results similar to that of in this chapter to the following equation*

$$\big(a(t)\left((b(t)(x(t)+p(t)x(\tau(t))')'\right)\big)' - q(t)\max_{[\sigma(t),t]} x^{\gamma}(s) = 0, \quad t \geq t_0 \geq 0.$$