Chapter 4

Third Order Half-linear Neutral Differential Equations with "Maxima"-II

4. Third Order Half-linear Neutral Differential Equations with "Maxima"-II

4.1 Introduction

In this chapter, we continue the study on the oscillation of all solutions of the third order half-linear neutral differential equations of the form

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t)\max_{[\sigma(t),t]} x^{\alpha}(s) = 0, \quad t \geq t_0 \geq 0, \tag{4.1.1}$$

subject to the following conditions:

(C_1) $\tau(t) \leq t$ and $\sigma(t) < t$ are continuous functions in $[t_0, \infty)$ with $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C_2) α is a ratio of odd positive integers and $p(t) \in C^3([t_0,\infty),\ R)$ with $0 \leq p(t) \leq p < 1$, and $q(t) \in C([t_0,\infty),\ R_+)$ with $q(t)$ is not identically zero on any ray of the form $[t_*,\infty)$ for any $t_* \geq t_0$;

(C_3) $a(t) \in C^1([t_0,\infty), R)$, is positive and nonincreasing and $\int_{t_0}^{t} a^{-\frac{1}{\alpha}}(s)ds \to \infty$ as $t \to \infty$.

The results presented in this chapter extend that of in [34] for the third order neutral differential equation without "maxima". Here we follow the same strategy as in the previous chapter, but with new estimates in Lemmas 4.2.2, 4.2.3 and 4.2.4.

In Section 4.2, we obtain some criteria for the oscillation of all solutions of equation (4.1.1), and in Section 4.3, we present some examples to illustrate the main results.

4.2 Oscillation Results

In this section, we obtain a oscillatory criterion for equation (4.1.1). For a solution $x(t)$ of (4.1.1) we define the corresponding function $z(t)$ by

$$z(t) = x(t) + p(t)x(\tau(t)). \tag{4.2.1}$$

Throughout this chapter without loss of generality we can deal with the positive solution of equation (4.1.1), since the proof for the negative solution is similar.

To obtain sufficient condition for the oscillation of solutions of equation (4.1.1), we need the following lemmas.

Lemma 4.2.1. *The function $x(t)$ is a negative solution of equation* (4.1.1) *if and only if $-x(t)$ is a positive solution of the equation*

$$\left(a(t)\left((x(t) + p(t)x(\tau(t)))''\right)^{\alpha}\right)' + q(t) \min_{[\sigma(t),t]} x^{\alpha}(s) = 0. \tag{4.2.2}$$

Proof. The assertion can be verified easily. □

Lemma 4.2.2. *Assume that $u(t) > 0$, $u^{'}(t) > 0$, $\left(a(t)(u^{'}(t))^{\alpha}\right)^{'} \leq 0$ on $[t_0, \infty)$. Then for each $l \in (0,1)$ there exists $T_l \geq t_0$ such that*

$$\frac{u(\tau(t))}{A(\tau(t))} \geq l\frac{u(t)}{A(t)} \quad \textit{for all} \quad t \geq T_l,$$

where

$$A(t) = \int_t^{\infty} \frac{1}{a^{1/\alpha}(s)} ds.$$

Proof. Since $a(t)(u'(t))^{\alpha}$ is non-increasing, so is $a^{1/\alpha}(t)(u'(t))$. Then by the definition of $A(t)$, we have

$$\begin{aligned} u(t) - u(\tau(t)) &= \int_{\tau(t)}^{t} a^{1/\alpha}(s)(u'(s))\frac{1}{a^{1/\alpha}(s)}ds \\ &\leq a^{1/\alpha}(\tau(t))u'(t)(\tau(t))(A(t) - A(\tau(t))). \end{aligned} \tag{4.2.3}$$

Also

$$u(\tau(t)) \geq u(\tau(t)) - u(t_0) \geq a^{1/\alpha}(\tau(t))u'(\tau(t))(A(\tau(t)) - A(t_0)).$$

Since $\lim_{t\to\infty} \frac{A(\tau)-A(t_0)}{A(\tau)} = 1$, for each $l \in (0,1)$ there exists $T_l \geq t_0$ such that $(A(\tau(t))) - A(t_0) > lA(\tau(t))$ for $t \geq T_l$. From the above inequality,

$$\frac{u(\tau(t))}{u'(\tau(t))} \geq la^{1/\alpha}(\tau(t))A(\tau(t)), \quad t \geq T_l. \tag{4.2.4}$$

Combining inequalities (4.2.3) and (4.2.4), we obtain

$$\frac{u(t)}{u(\tau(t))} \leq 1 + \frac{A(t) - A(\tau(t))}{lA(\tau(t))} \leq \frac{A(t)}{lA(\tau(t))},$$

which completes the proof. □

Lemma 4.2.3. *Assume that* $z(t) > 0,\ z'(t) > 0,\ z''(t) > 0,\ (a(t)(z''(t))^\alpha)' \leq 0$ *on* (T_l, ∞). *Then*

$$\frac{z(t)}{z'(t)} \geq \frac{a^{1/\alpha}(t)A(t)}{2} \quad \textit{for all } t \geq T_l.$$

Proof. Since $a(t)(z''(t))^\alpha$ is positive and non-increasing, so is $a^{1/\alpha}(t)z''(t)$. From $z'(t) > 0,\ a(t) > 0$, we have

$$z'(t) \geq z'(t) - z'(\tau(t)) \geq \int_{\tau(t)}^{t} \frac{a^{1/\alpha}(s)z''(s)}{a^{1/\alpha}(s)} \geq a^{1/\alpha}(t)A(t)z''(t).$$

Since $A'(t) = a^{-1/\alpha}(t)$,

$$A'(t)z'(t) \geq A(t)z''(t), \quad \text{for all} \quad t \geq T_l.$$

Integrating both sides of the above inequality, and using that $A(T_l)z'(T_l) > 0$, we obtain

$$\int_{T_l}^{t} A'(s)z'(s)ds \geq A(t)z'(t) - \int_{T_l}^{t} A'(s)z'(s)ds.$$

Therefore,

$$\int_{T_l}^{t} A'(s)z'(s)ds \geq \frac{1}{2}A(t)z'(t), \tag{4.2.5}$$

since $a(t)$ is non-increasing, we have $A(t) > 0,\ A'(t) > 0,\ A''(t) \geq 0$ and

$$(A'(t)z(t))' = A'(t)z'(t) + A''(t)z(t) \geq A'(t)z'(t).$$

Integrating on both sides of the above equality, and using the fact $A'(T_l)z(T_l) > 0$ and the inequality (4.2.9), we obtain

$$A'(t)z(t) \geq \frac{1}{2}A(t)z'(t), \quad t \geq T_l,$$

which implies the desired result. □

Lemma 4.2.4. *Assume that* $z'(t) > 0,\ z''(t) > 0,\ (a(t)(z''(t))^{\alpha})' \leq 0$ *on* $[T_\ell, \infty)$. *Then*

$$\frac{A(t)z''(t)}{A'(t)z'(t)} \leq 1, \quad \textit{for} \ \ t \geq T_\ell.$$

Proof. The proof is found in [34]. □

For simplicity we introduce the following notations:

$$\begin{aligned} P &= \liminf_{t\to\infty} A^{\alpha}(t) \int_t^{\infty} P_\ell(s)ds, \\ Q &= \limsup_{t\to\infty} \frac{1}{A(t)} \int_{t_0}^{t} A^{\alpha+1}(s)P_\ell(s)ds \end{aligned}$$

where

$$P_\ell(s) = \ell^{\alpha} \max_{[\sigma(t),t]}(1-p(s))^{\alpha}q(s)a(\tau(t))\left(\frac{A(\tau(s))}{A(s)}\right)^{\alpha}\left(\frac{A(\tau(s))}{2}\right)^{\alpha} \tag{4.2.6}$$

with $\ell \in (0,1)$ arbitrarily chosen and T_ℓ large enough. Moreover for $z(t)$ satisfying Case (I) of Lemma 3.2.1, we define

$$w(t) = a(t)\left(\frac{z''(t)}{z(t)}\right)^{\alpha}, \tag{4.2.7}$$

$$r = \liminf_{t\to\infty} \ A^{\alpha}(t)w(t),$$

and

$$R = \limsup_{t\to\infty} \ \ A^{\alpha}(t)w(t). \tag{4.2.8}$$

Now, we present the main results.

Theorem 4.2.1. *Assume that condition* (3.2.6) *holds. If*

$$P > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{4.2.9}$$

then every solution of equation (4.1.1) *is almost oscillatory.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of equation (4.1.1). We can assume without loss of generality that $x(t)$ is positive and the corresponding function $z(t)$ satisfies case(I) of Lemma 3.2.1. First note that

$$x(t) = z(t) - p(t)x(\tau(t)) \geq (1-p(t))z(t) \tag{4.2.10}$$

or

$$\max_{[\sigma(t),t]} x^\alpha(s) \geq z^\alpha(t) \max_{[\sigma(t),t]} (1-p(s))^\alpha.$$

Using the above inequality in equation (4.1.1) we obtain

$$(a(t)(z''(t))^\alpha)' + q(t) \max_{[\sigma(t),t]} (1-p(s))^\alpha z^\alpha(t) \leq 0. \tag{4.2.11}$$

From the definition of $w(t)$ we see that $w(t) > 0$ and from (4.1.1) we have

$$w'(t) \leq \frac{-q(t)z^\alpha(\tau(t)) \max_{[\sigma(t),t]}(1-p(s))^\alpha}{(z'(t))^\alpha} - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t). \tag{4.2.12}$$

From Lemma 4.2.2 with $u(t) = z'(t)$, we have ,

$$\frac{1}{z'(t)} \geq \ell \frac{A(\tau(t))}{A(t)} \frac{1}{z'(\sigma(t))}, \quad t \geq T_\ell$$

where, ℓ is the same as in $P_\ell(t)$. Now (4.2.12) becomes

$$w'(t) \leq -q(t)\ell^\alpha \left(\frac{A(\tau(t))}{A(t)}\right)^\alpha \frac{z^\alpha(\tau(t))}{(z'(\tau(t)))^\alpha} \max_{[\tau(s),s]} (1-p(s))^\alpha - \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t).$$

Using the fact from Lemma 3.2.4 that $z(t) \geq \frac{a^{1/\alpha}\ A(t)}{2} z'(t)$, we have

$$w'(t) + P_\ell(t) + \frac{\alpha}{a^{1/\alpha}(t)} w^{\frac{\alpha+1}{\alpha}}(t) \leq 0. \tag{4.2.13}$$

Since $P_\ell(t) > 0$ and $w(t) > 0$ for $t \geq T_\ell$, it follows that $w'(t) \leq 0$ and

$$-\left(\frac{w'(t)}{\alpha w^{\frac{\alpha+1}{\alpha}}(t)}\right) > \frac{1}{a^{1/\alpha}(t)} \quad \text{for} \quad t \geq T_\ell. \tag{4.2.14}$$

This implies that

$$\left(\frac{1}{w^{1/\alpha}(t)}\right)' > \frac{1}{a^{1/\alpha}(t)}. \tag{4.2.15}$$

Integrating the last inequality from T_ℓ to t, and using that $w^{-1/\alpha}(T_\ell) > 0$, we obtain

$$w(t) < \frac{1}{\left(\int_{T_\ell}^{t} a^{-1/\alpha}(s)ds\right)^{\alpha}} \tag{4.2.16}$$

using $t \to \infty$ and using (C_3), we get $\lim_{t\to\infty} w(t) = 0$. On the otherhand, from the definition of $w(t)$, and Lemma 4.2.1, we see that

$$A^{\alpha}(t)w(t) = a(t)\left(\frac{A(t)z''(t)}{z'(t)}\right)^{\alpha} = \left(\frac{A(t)z''(t)}{A'(t)z'(t)}\right)^{\alpha} \leq 1^{\alpha} \leq 1. \tag{4.2.17}$$

Then

$$0 \leq r \leq R \leq 1. \tag{4.2.18}$$

Now, let $\varepsilon > 0$, then from the definitions of P and r we can pick $t_2 \in [T_\ell, \infty)$ sufficiently large that

$$A^{\alpha} \int_t^{\infty} P_\ell(s)ds \geq P - \varepsilon,$$

and

$$A^{\alpha}(t)w(t) \geq r - \varepsilon \quad \text{for} \quad t \geq t_2.$$

Integrating (4.2.13) from t to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w(t) \geq \int_t^{\infty} P_\ell(s)ds + \alpha \int_t^{\infty} \frac{w^{1+1/\alpha}(s)}{a^{1/\alpha}(s)}ds, \quad \text{for} \quad t \geq t_2. \tag{4.2.19}$$

Assume $P = \infty$, then from (4.2.19), we have

$$A^{\alpha}(t)w(t) \geq \; A^{\alpha}(t) \int_t^{\infty} P_\ell(s)ds.$$

Taking limit infimum on both sides as $t \to \infty$, we get in view of (4.2.18) that $1 \geq r \geq \infty$. This is a contradiction. Next assume that $P < \infty$. Now from (4.2.19) and the fact $a'(t) \leq 0$, we have

$$\begin{aligned} A^{\alpha}(t)w(t) &\geq A^{\alpha}(t)\int_t^{\infty} P_{\ell}(s)ds + \alpha\ A^{\alpha}(t)\int_t^{\infty} \frac{A^{\alpha+1}(s)w^{\frac{\alpha+1}{\alpha}}(s)}{A^{\alpha+1}(s)a^{1/\alpha}(s)}ds \\ &\geq (P-\varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}} A^{\alpha} \int_t^{\infty} \frac{\alpha A'(s)}{A^{\alpha+1}(s)}ds \end{aligned}$$

and so

$$A^{\alpha}(t)w(t) \geq (P-\varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}}.$$

Taking limit infimum on both sides as $t \to \infty$, we get,

$$r \geq (P-\varepsilon) + (r-\varepsilon)^{1+\frac{1}{\alpha}}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$P \leq r - r^{1+\frac{1}{\alpha}}.$$

Using the inequality,

$$Bu - Du^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}$$

with $B = D = 1$, we get

$$P \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}$$

which contradicts (4.2.9). This completes the proof. □

From Theorem 4.2.1 we have the following corollary.

Corollary 4.2.1. *Assume that (3.2.6) holds. If*

$$\liminf_{t\to\infty}\ A^{\alpha}(t)\int_t^{\infty} q(s) \max_{[\sigma(t),t]}(1-p(s))^{\alpha} a(\tau(s)) \frac{A(\tau(s))^{2\alpha}}{A^{\alpha}}ds > \frac{(2\alpha)^{\alpha}}{\ell^{\alpha}} \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{4.2.20}$$

then every solution of equation (4.1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. Proof is similar to Corollary 3.2.1 and hence the details are omitted. □

Theorem 4.2.2. *Assume that condition* (3.2.6) *holds. If*

$$P + Q > 1, \tag{4.2.21}$$

then every solution of equation (4.1.1) *is almost oscillatory.*

Proof. Suppose $x(t)$ is a nonoscillatory solution of equation (4.1.1). Then we can assume without loss of generality that $x(t)$ is positive. Let the corresponding function $z(t)$ satisfies Case(I) of Lemma 3.2.1. Proceeding as in the proof of Theorem 3.2.1, we obtain (4.2.13). Now multiplying (4.2.13) by $A^{\alpha+1}(t)$, and integrating from t_2 to t, we get

$$\int_{t_2}^{t} A^{\alpha+1}(s)w'(s)ds \leq -\int_{t_2}^{t} A^{\alpha+1}(s)P_\ell(s)ds - \alpha\int_{t_2}^{t} \frac{(\ A^{\alpha}(s)w(s))^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}(s)}ds. \tag{4.2.22}$$

Using integration by parts, we obtain

$$\begin{aligned} A^{\alpha+1}(t)w(t) \ &\leq \ A^{\alpha+1}(t_2)w(t_2) - \int_{t_2}^{t} A^{\alpha+1}(s)P_\ell(s)ds \\ &\qquad - \alpha\int_{t_2}^{t} \frac{(A^{\alpha}(s)w(s))^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}(s)}ds + \int_{t_2}^{t} w(s)(\ A^{\alpha+1}(s))'ds. \end{aligned}$$

Hence

$$\begin{aligned} A^{\alpha+1}(t)w(t) \ &\leq \ A^{\alpha+1}(t_2)w(t_2) - \int_{t_2}^{t} A^{\alpha+1}(s)P_\ell(s)ds \\ &\qquad + \int_{t_2}^{t} \left[\frac{(\alpha+1)A^{\alpha}(s)w(s)}{a^{1/\alpha}(s)} - \frac{\alpha(\ A^{\alpha}(s)w(s))^{\frac{\alpha+1}{\alpha}}}{a^{1/\alpha}(s)}\right] ds. \end{aligned}$$

Using the inequality

$$Bu - Du^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}.\frac{B^{\alpha+1}}{D^{\alpha}},$$

with

$$u(t) = \ A^{\alpha}(t)w(t),$$

and positive constants,

$$D = \frac{\alpha}{a^{1/\alpha}(t)}$$

and

$$B = \frac{\alpha+1}{a^{1/\alpha}(t)},$$

we get,

$$A^{\alpha+1}(t)w(t) \leq A^{\alpha+1}(t_2)w(t_2) - \int_{t_2}^{t} A^{\alpha+1}(s)P_\ell(s)ds + A(t) - A(t_2). \qquad (4.2.23)$$

It follows that

$$A^{\alpha}(t)w(t) \leq \frac{1}{A(t)}A^{\alpha+1}(t_2)w(t_2) - \frac{1}{A(t)}\int_{t_2}^{t} A^{\alpha+1}(s)P_\ell(s)ds + 1 - \frac{A(t_2)}{A(t)}. \quad (4.2.24)$$

Taking limit sup on both sides as $t \to \infty$, we obtain $R \leq -Q + 1$. Combining this with the inequality (4.2.18) we get

$$P + Q \leq 1. \qquad (4.2.25)$$

This is a contradiction to (4.2.21). If $z(t)$ satisfies Case (II) of Lemma 3.2.1, then proof follows from Lemma 3.2.6. This completes the proof. □

From Theorem 4.2.2 we have the following corollary.

Corollary 4.2.2. *Assume that (3.2.6) holds and $a'(t) \leq 0$ for all $t \geq t_0$. If*

$$Q = \limsup_{t\to\infty} \frac{1}{A(t)} \int_{t_0}^{t} A^{\alpha+1}(s)P_\ell(s)ds > 1, \qquad (4.2.26)$$

then every solution of equation (4.1.1) *is almost oscillatory.*

Proof. Proof is easy and hence it is omitted. □

4.3 Examples

In this section, we present some examples to illustrate the main results.

Example 4.3.1. *Consider the differential equation*

$$\left(\frac{1}{t^3}\left((x(t)+\frac{1}{3}x(t/2))''\right)^3\right)' + \frac{1500}{27t^{10}}\max_{[t/2,t]} x^3(s) = 0, \quad t \geq 1. \tag{4.3.1}$$

Here, $a(t) = \frac{1}{t^3}$, $p(t) = 1/3$, $\tau(t) = \sigma(t) = t/2$, $q(t) = \frac{1500}{27t^{10}}$, *and* $\alpha = 3$. *One can easily verify that all conditions of Theorem 4.2.1 are satisfied and hence every solution of equation* (4.3.1) *is almost oscillatory. Infact,* $x(t) = \frac{1}{t}$ *is one such almost oscillatory solution of equation* (4.3.1).

Example 4.3.2. *Consider the differential equation*

$$\left(\frac{1}{t}\left((x(t)+\frac{1}{3}x(t/3))\right)''\right)' + \frac{8}{3t^2}\max_{[t/3,t]} x(s) = 0, \quad t \geq 1. \tag{4.3.2}$$

Here, $a(t) = \frac{1}{t}$, $p(t) = 1/3$, $\tau(t) = \sigma(t) = t/3$, $q(t) = \frac{8}{3t^2}$, *and* $\alpha = 1$. *One can easily verify that all conditions of Theorem 4.2.2 are satisfied and hence every solution of equation* (4.3.2) *is almost oscillatory. Infact,* $x(t) = \frac{1}{t}$ *is one such almost oscillatory solution of equation* (4.3.2).

We conclude this chapter with the following remark.

Remark 4.3.1. *It would be interesting to obtain results similar to that of in this chapter to the following equation*

$$\left(a(t)\left((x(t)+p(t)x(\tau(t)))''\right)^\alpha\right)' - q(t)\max_{[\sigma(t),t]} x^\alpha(s) = 0, \quad t \geq t_0 \geq 0.$$