CHAPTER - V
GAMES IN PRODUCT SPACES

In this chapter we study topological games in product fuzzy topological spaces. For two classes of fuzzy topological spaces $K_1$ and $K_2$ we define $K_1 \times K_2$ as the set of all product spaces $X \times Y$ such that $X \in K_1$, $Y \in K_2$ and all closed subsets of them. Here we explore the possibility of having a winning strategy for Player I in $G' (D(K_1, \times K_2), X \times Y)$ if he has the same in $G' (K_1, X)$ and $G' (K_2, Y)$. Here we make use of concepts like fuzzy rectangles, fuzzy $D$-products etc.

5.1 Preliminaries

5.1.1 Definition [WON2] Let $\{X_i\}_{i \in I}$ be a family of fuzzy topological spaces. Let $X = \prod_{i \in I} X_i$ be the usual Cartesian product and let $P_i$ be the projection from $X$ on to $X_i$ for each $i \in I$. The set $X$ with fuzzy topology having the family $F = \{ P_i^{-1}(B): B \in T_i, i \in I \}$ as a sub base is called the product fuzzy topological space.

5.1.2 Definition Let $X \times Y$ be a fuzzy product space. A fuzzy subset of the form $R = R' \times R''$ where $R'$ and $R''$ are projection of $R$ in to $X$ and $Y$ respectively is called a fuzzy rectangle in $X \times Y$.

5.1.3 Definition Let $X$ be a fuzzy topological space. A fuzzy subset $U$ of $X$ is called a co-zero fuzzy set if there is an $F$-continuous function $C: X \rightarrow [0, 1]$ such that $C^{-1}(0) = 1 - U$. 
5.2 Fuzzy Games in Product Spaces

5.2.1 Definition A product fuzzy topological space $X \times Y$ is called a fuzzy $D$-product if for any disjoint pair $\{E,F\}$ of a closed fuzzy rectangle $E$ and a closed fuzzy set $F$ in $X \times Y$, there exists a $\sigma$-discrete collection $R$ of closed fuzzy rectangles such that $F < \text{Sup} \{R: R \in R\} < (X \times Y) \setminus E$

5.2.2 Theorem Let $X$ and $Y$ be two fuzzy topological spaces such that $Y$ is $\alpha$-compact and player I has a fuzzy winning strategy in $G^* (C, X)$ then player I has a fuzzy winning strategy in $G^* (C, X \times Y)$ where $C$ is the class of all $\alpha$-compact spaces.

Proof

Let $P$ be the projection map from $X \times Y$ on to $X$. Now since $Y$ is $\alpha$-compact, it follows that $P^{-1}(x)$ is $\alpha$-compact for each $x \in X$. Since $P$ is $F$-continuous, it follows that $P$ is an $\alpha$-perfect map. Also since $C$ is an $\alpha$-perfect class, from Theorem 2.4.7 it follows that player I has a fuzzy winning strategy in $G^* (C, X \times Y)$.

Now from the fact that the class of all fuzzy $C$-scattered spaces ($\bar{S} C$) forms an $\alpha$-perfect class and an argument similar to that in proof of Theorem 5.2.2 it follows that

5.2.3 Theorem Let $X$ and $Y$ be two fuzzy topological spaces such that $Y$ is $\alpha$-compact and player I has a fuzzy winning strategy in $G^* (\bar{S} C, X)$, then player I has a fuzzy winning strategy in $G^* (\bar{S} C, X \times Y)$.

5.2.4 Theorem: Let $X \times Y$ be a fuzzy $D$-product. If player I has fuzzy winning strategies in $G^* (K_1, X)$ and $G^* (K_2, X)$ then he has a fuzzy winning strategy in $G^* (D(K_1, \times K_2), X \times Y)$.
Proof

For convenience we use the following notations.

\[ N_0 = \{0, 3, 6, 9, \ldots\}, N_1 = \{1, 4, 7, \ldots\}, N_2 = \{2, 5, 8, \ldots\} \]

so that \( N_0 \cup N_1 \cup N_2 = \omega \cup \{0\} \).

Let \( T = (k_1, k_2, \ldots, k_n) \in \omega^n \) with \( k_n \not\in N_0 \), take \( T^* = (k_1, k_2, \ldots, k_i) \) if \( k_n, k_{n-1}, \ldots, k_{i+1} \in N_1 \) and \( k_i \not\in N_1(\not\in N_2) \) for some \( i < n \).

Now by Theorem 2.3.3, it is enough if we construct a fuzzy winning strategy \( t \) for \( G^*(L, X \times Y) \) where \( L = F[D(K_1 \times K_2)] \). Take \( E_1 = t(X \times Y) = S_1(X) \times S_2(Y) \) where \( S_1 \) and \( S_2 \) are fuzzy stationery winning strategies for player I in \( G^*(K_1, X) \) and \( G^*(K_2, X) \) receptively. Take \( A_0 = \{0\} \), \( R(0) = \{X \times Y\} \) and \( F_0 = R(0) = X \times Y \). Player II chooses some \( F_1 \in \mathcal{L}^{X \times Y} \) with \( F_1 \land E_1 = 0 \).

Assume that we have already constructed an admissible sequence \((E_1, F_1, \ldots, E_m, F_m)\) in \( G^*(L, X \times Y) \) such that \( E_i = t(F_1, F_2, \ldots, F_{i-1}) \) for each \( i \leq m \) and that there exists a discrete collection \( \{R(\alpha): \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \in A_{\mathcal{T}\mathcal{B}_n}\} \) of closed fuzzy rectangles for each \( T \in \omega^n, n \geq 0 \) with \( \Sigma T \leq m-1 \) and \( k \geq 1 \) satisfying

1. \( \alpha \oplus \alpha \in A_{\mathcal{T}\mathcal{B}_n} \) implies \( \alpha \in A_T \)
2. For each \( \alpha \in A_T, R(\alpha) \wedge F_{\Sigma T+1} < \sup \{R(\alpha \oplus \alpha) : \alpha \oplus \alpha \in A_{\Sigma T+1}, k \geq 1\} < R(\alpha) \)
3. For each \( \alpha \oplus \alpha \in A_{\mathcal{T}\mathcal{B}_n} \) where \( T = (k_1, k_2, \ldots, k_n) \in \omega^n, n \geq 0, k \geq 1 \).
   - (i) \( S_1(R(\alpha^*)) \land R(\alpha \oplus \alpha)^* = 0 \) if \( k_n, k \in N_0 \cup N_1 \)
   - (ii) \( S_2(R(\alpha^*)) \land R(\alpha \oplus \alpha)^* = 0 \) if \( k_n, k \in N_0 \cup N_2 \)
   - (iii) \( S_1(R(\alpha^*)) \land R(\alpha \oplus \alpha)^* = 0 \) if \( k_n \in N_2 \) and \( k \in N_0 \cup N_1 \)
   - (iv) \( S_2(R(\alpha^*)) \land R(\alpha \oplus \alpha)^* = 0 \) if \( k_n \in N_1 \) and \( k \in N_0 \cup N_2 \)

where \( \alpha^* = \alpha / \alpha \in A_{\mathcal{T}^*} \) if \( \alpha \in A_{\mathcal{T}^*}, \mathcal{T}^* \in \omega^1 \).

If \( T = 0 \) consider \( k_n = 0 \in N_0 \).

Take \( T = (k_1, k_2, \ldots, k_n) \in \omega^n, n \geq 0 \), with \( \Sigma T = m \). Now \( T = T_\oplus k_n \) and hence \( \Sigma T = m - k_n \leq m - 1 \). Therefore \( R(T) \) is constructed.
For each $a \in A_T$ take $E(a) = S_1(R(a')) \times S_2(R(a''))$ if $k_n \in N_0$

$E(a) = S_1(R(a')) \times S_2(R(a'')) \wedge R(a'')$ if $k_n \in N_1$

$E(a) = S_1(R(a'')) \wedge R(a') \times S_2(R(a'))$ if $k_n \in N_2$

Then $E(a) \in K_1 \times K_2$ and take

$E_{m+1} = t(F_0, F_1, \ldots, F_m)$

$$= \sup_{\alpha \in A_T} \left( \sum_{T \in \omega^n_{n \geq m}} E(\alpha) \wedge F_m \right)$$

and clearly $E_{m+1} \in L$ since $\{R(a): a \in A_T\}$ is discrete in $X \times Y$. Now player II choose an

$F_{m+1} \in L^{X \times Y}$ with $F_{m+1} < F_m$ and $F_{m+1} \wedge E_{m+1} = 0$.

Again take some $T \in \cup_{n \geq 0} \omega^n$ with $\sum T = m$ and $a \in A_T$. Since $R(a)$ is a fuzzy

D-product and $E(a)$ is a closed fuzzy rectangle in $R(a)$ with $E(a) \wedge F_{m+1} = 0$ there exists a

$\sigma$-discrete collection $\{R(\alpha \oplus \alpha): \alpha \in B(a, k), k \geq 1\}$ of closed fuzzy rectangles in $R(a)$.

Now, $R(a) \wedge F_{m+1} < \sup_{\alpha \in B(a, k)} R(\alpha \oplus \alpha) < R(a) \wedge E(a)$.

Now from the fact that $E(a)$ and $R(\alpha \oplus \alpha)$ are disjoint fuzzy rectangles in $X \times Y$, we may

assume that for each $\alpha \in B(a, k)$,

$E(a)' \wedge R(\alpha \oplus \alpha)' = 0$ and $E(a)'' \wedge R(\alpha \oplus \alpha)'' = 0$ implies $k \in N_0$

$E(a)' \wedge R(\alpha \oplus \alpha)' = 0$ and $E(a)'' \wedge R(\alpha \oplus \alpha)'' \neq 0$ implies $k \in N_1$

$E(a)' \wedge R(\alpha \oplus \alpha)' \neq 0$ and $E(a)'' \wedge R(\alpha \oplus \alpha)'' = 0$ implies $k \in N_2$.

Define $A_{T B k} = \{ \alpha \oplus \alpha \mid \alpha \in B(a, k), a \in A_T \}$ for each $k \geq 1$. Now clearly $\{R(b): b \in A_{T B k}\}$ is discrete in $X \times Y$ and (1)-(3) are satisfied.

Now we will prove that $t$ is the required winning strategy by showing that

$\inf_{n \geq 1} F_m = 0$. For, if possible let $\inf_{n \geq 1} F_m(x, y) = \eta$ for some $(x, y) \in X \times Y$ and $\eta \in (0, 1]$. Then
by (2) it follows that we can choose some \((k_1, k_2, \ldots) \in \omega \omega\) and \((\alpha_1, \alpha_2, \ldots)\) such that \(a_n = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in A_{T_n}\) where \(T_n = (k_1, k_2, \ldots, k_n)\) and \(R(a_n)(x, y) = \eta\) for each \(n \geq 1\) for each \(n \geq 1\). Then \(R(a_n)'(x) \geq \eta\) and \(R(a_n)''(y) \geq \eta\) for each \(n \geq 1\).

\[\therefore \inf_{n \geq 1} R(a_n)'(x) \geq \eta\] and \(\inf_{n \geq 1} R(a_n)''(y) \geq \eta.\) (a)

Now assume that \((k_1, k_2, \ldots)\) contains an infinite sequence \((k_{i(1)}, k_{i(2)}, \ldots)\) consisting of all nos \(\in N_0 \cup N_1\). Let \(T(0) = a(0) = 0\) and \(T(j) = T(i(j)) = (k_{i(1)}, \ldots, k_{i(j)})\) and \(a(j) = a(i(j)) = (\alpha_{i(1)}, \ldots, \alpha_{i(j)})\) for each \(j \geq 1\). Now claim \(S_1(R(a(j)))' \land R(a(j+1))' = 0\) for each \(j \geq 0\). For if \(k_{i(j)+1} \in N_0 \cup N_1\), \(T(j+1) = T(j) \oplus k_{i(j)+1}\) follows from \(k_{i(j)+1} = k_{i(j)+1}\) and hence \(a(j+1) = a(j) \oplus a_{i(j)+1} \in A_{T(j+1)}\). Now claim follows from (3)(i). Similar argument holds for \(k_{i(j)+1} \in N_2\) also.

Now from the claim above and (2) it follows that \((S_1(X), R(a(1))'), S(R(a(1)))', R(a(2))', S(R(a(1)))', \ldots)\) is a play of \(G^*(K_1, X)\) and hence \(\inf_{n \geq 1} R(a(j))' = 0\). This is a contradiction to (a). Now a similar argument holds good for the case of \((k_1, k_2, \ldots)\) contains an infinite sequence \((k_{i(1)}, k_{i(2)}, \ldots)\) consisting of all natural numbers \(k_n\) belonging to \(N_0 \cup N_2\) also. This completes the proof.
Chapter VI
APPLICATIONS OF FUZZY TOPOLOGICAL GAMES

In this chapter we discuss some applications of fuzzy topological games in covering properties and dimension theory. We mainly focus our attention on \( \alpha \)-para (meta) compact spaces and shading dimension. Every product space discussed will have a winning strategy in some particular kind of fuzzy topological game. Also a fuzzy version of countable sum theorem for covering dimension in terms of fuzzy topological games is obtained.

6.1. Games and Product in \( \alpha \)-para (meta) Compact Spaces

First we obtain a characterisation of fuzzy regular \( \alpha \)-paracompact spaces, which will be useful in proving the main theorems in this section.

6.1.1 Theorem For a fuzzy regular space \( X \), the following are equivalent

(i) \( X \) is \( \alpha \)-paracompact.
(ii) Every \( \alpha \)-shading of \( X \) by open fuzzy sets has a \( \sigma \)-locally finite \( \alpha \)-shading refinement by open fuzzy sets.
(iii) Every \( \alpha \)-shading refinement of \( X \) by open fuzzy sets has a locally finite \( \alpha \)-shading refinement.
(iv) Every \( \alpha \)-shading of \( X \) by open fuzzy sets has a locally finite \( \alpha \)-shading refinement by closed fuzzy sets.

Proof

(i) \( \Rightarrow \) (ii)

Follows from the fact that every locally finite \( \alpha \)-shading is \( \sigma \)-locally finite.

(ii) \( \Rightarrow \) (iii)

Let \( U \) be an \( \alpha \)-shading of \( X \) by open fuzzy sets. Let \( V \) be the \( \sigma \)-locally finite \( \alpha \)-shading refinement of \( U \) by open fuzzy sets. Therefore \( V = \bigcup_{i=1}^{\infty} V_i \) where each \( V_i = \{ V_{i\beta} : \beta \in \Lambda \} \) is locally finite. Now take \( W_i = \sup_{\beta \in \Lambda} V_i \). Now \( W = \{ W_i : i = 1,2,3,\ldots \} \) is clearly an
α-shading of X. Take $A_1 = W_1$ and $A_i = W_i \cup \bigcap_{i=2}^{n} W_i$ for $i = 1, 2, 3, \ldots$. Now $\{ A_i : i = 1, 2, \ldots \}$ is a locally finite α-shading refinement of $W$. Then by Theorem 1.2.7 it follows that $\{ A_i \cap V_{\beta} : i = 1, 2, \ldots, \beta \in \Lambda \}$ is a locally finite α-shading refinement of $V$ and hence of $U$.

(iii) $\Rightarrow$ (iv)

Let $U$ be an α-shading of $X$ by open fuzzy sets. For any $x \in X$, take some $U_x \in U$ with $U_x(x) > \alpha$. Now since $X$ is fuzzy regular, it is possible to find a fuzzy open nbd $V_x$ of $X$ such that $V_x(x) > \alpha$ and $x \in V_x < \bigcup_x U_x$. Now by (iii) we have $\{ V_x : x \in X \}$ has a locally finite α-shading refinement $\{ A_r : r \in \Gamma \}$ (say). Then by Theorem 1.3.3 $\{ \overline{A}_r : r \in \Gamma \}$ is also locally finite. Now for each $r \in \Gamma$, if $A_r < V_x$ then $\overline{A}_r < V_x < U$ for some $U \in U$. Hence $\{ \overline{A}_r : r \in \Gamma \}$ is the required α-shading refinement by closed fuzzy sets.

(iv) $\Rightarrow$ (i).

Let $U$ be an α-shading of $X$ by open fuzzy sets and $V$ be a locally finite α-shading refinement of $U$ by closed fuzzy sets. For each $x \in X$, let $W_x$ be an open fuzzy set such that $W_x(x) = 1$ and $V_x = 1 \setminus W_x$ holds for all but almost finitely many $i$. This is possible since $V$ is locally finite where $V_x \in V$. Now clearly $W = \{ W_x : x \in X \}$ is an α-shading of $X$ and let $A$ be a locally finite α-shading refinement of $W$ by closed fuzzy sets. Now we take $V^* = X \setminus \text{Sup} \{ A : A \in A, A \cap V = 0 \}$. Clearly each $V^*$ is fuzzy open and contains $V$. Consider $V^* = \{ V^* : V \in V \}$. Now $V^*$ is a locally finite α-shading of $X$. For let $x \in X$, now we can find an open fuzzy set $U$ such that $U(x) = 1$ and $A_i \leq 1-U$ holds for all but at most finitely many $i$. (since $A$ is locally finite). i.e., $A_i \cap U \neq 0$ for almost finitely many $i$. Now if $U \land V^* \neq 0$ for some $V^* \in V^*$, then $V^* \land A_i \neq 0$ for some $i = 1, 2, 3, \ldots, n$. By the definition of $V^*$ it follows that $A_i \land V = 0$ for some $i = 1, 2, \ldots, n$. Now if $A$ is a refinement of $\{ W_x : x \in X \}$ and each $W_x$ meet only finitely many $V \in V$ implies that $A_i$ meets only finitely many members of $V$ and hence we have $U \land V^* = 0$ for all but finitely many $V^* \in V^*$. And hence $V^*$ is locally finite.
Now for every $V \in V$ take $U \in U$ such that $V \subseteq U$ and consider $\{ U \land V^* : V \in V \}$. This is clearly an $\alpha$-shading refinement of $U$ by open fuzzy sets which is locally finite. Hence $X$ is $\alpha$-paracompact.

6.1.2 Theorem If Player I has a fuzzy winning strategy in $G^*(DC, X)$ and $X$ is $\alpha$-paracompact, then $X \times Y$ is $\alpha$-paracompact for every $\alpha$-paracompact space $Y$. Where $DC$ denote the class of all fuzzy topological spaces which have a finite fuzzy closed $\alpha$-shading by members of $C$, where $C$ is the collection of all $\alpha$-compact spaces.

Proof

Let $S$ be a fuzzy stationery winning strategy for player I in $G^*(DC, X)$. Let $G$ be any $\alpha$-shading of $X \times Y$ by open fuzzy sets. Then from the characterization of $\alpha$-paracompactness in theorem 6.1.1 it suffices to prove that $G$ has $\sigma$-locally finite refinement by open fuzzy sets. Takes $U_0 = \{0\}$, $A_0 = \{0\}$ and $R(0) = H(0) = X \times Y$, we shall construct a collection $U_n$ of fuzzy co-zero rectangles and a collection $\{(R(a), H(a))| a = (a_1, a_2, ..., a_n), a \in A_n\}$ of pairs of closed rectangles $R(a)$ and open rectangle $H(a)$ for each $n \geq 1$, satisfying the following conditions

(i) $U_n$ is locally finite in $X \times Y$

(ii) Each $U \times V$ in $U_n$ is contained in some $G \in G$.

(iii) $\{H(a): a \in A_n\}$ is locally finite in $X \times Y$ for each $a \in A_n$ and $n \geq 1$.

(iv) $a. \in A_{n-1}$, where $a. = (a_1, a_2, ..., a_{n-1})$

(v) $R(a) < R(a.) \land H(a)$

(vi) $S(R(a.))' \land R(a)' = 0$

(vii) $R(a) \backslash \text{Sup } U_{n+1} < \text{Sup}\{R(a+\alpha) | a+\alpha \in A_{n+1}\}$.

Where `$'$ and `$\backslash$' represents projections on $X$ and $Y$ axes respectively.

The construction of $U_i$ are similar to that of Yajima [Y1] in crisp case and hence we omit it.

Now take $U = \bigcup_{n \geq 1} U_n$. From (i) and (ii) we get that $U$ is a $\sigma$-locally finite collection of co-zero rectangles and $U \times V \in U$ is contained in some $G \in G$. 
Now from (v) and (vi) we get if \((a_n)\) is a sequence such that \(a_n \in A_n\) and \((a_n) \in A_{n-1}\) for each \(n \geq 1\) where \(a_0 = \emptyset\), then \(\sup_{n \geq 1} R(a_n)' = 0\). For \((S(X), R(a_1)', S(R(a_1))', \ldots, S(R(a_n))')\), \(R(a_n)', \ldots\) is a play of \(G^*(DC, X)\) and player I wins this play and hence \(\sup_{n \geq 1} R(a_n)' = 0\).

Now it is enough if we prove that \(U\) is an \(\alpha\)-shading of \(X \times Y\). If possible let \(U\) be not an \(\alpha\)-shading of \(X \times Y\). Therefore \(U(x, y) \leq \alpha\) for every \(U \in U\). Then by (vii) we can find an infinite sequence \((\alpha_1, \alpha_2, \ldots)\) such that \(a_n = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) and \((x, y) \in R(a_n) \forall n \geq 1\). Now clearly \(R(a_n)'(x) > 0\) and hence \(\sup_{n \geq 1} R(a_n)' \neq 0\). This is a contradiction and hence the proof is complete.

Using the notion of the \(P_\alpha\)-spaces discussed in Chapter IV, we obtain an analogue of theorem 6.1.2 for \(\alpha\)-metacompact spaces as follows.

6.1.3 Theorem Let \(X\) be a fuzzy regular \(\alpha\)-metacompact \(P_\alpha\)-space and Player I has a winning strategy in \(G^*(DC, X)\), then \(X \times Y\) is \(\alpha\)-metacompact for every \(\alpha\)-metacompact space \(Y\). Where \(DC\) denote the class of all \(f\)s which have a discrete fuzzy closed \(\alpha\)-shading by members of \(C\), where \(C\) is the collection of all \(\alpha\)-compact spaces.

Proof

We use the following notations. If \(a = (a_1, a_2, \ldots, a_n)\) then \(a \oplus \zeta = (a_1, a_2, \ldots, a_n, \zeta), a \backslash k = (a_1, a_2, \ldots, a_k)\) and \(\hat{a} = a \backslash n\). Also \(\backslash\) and \(\backslash\) represents the projections on \(X\) and \(Y\) respectively.

Given that Player I has a fuzzy winning strategy in \(G^*(DC, X)\). Therefore by Theorem 2.2.8 it follows that Player I has a stationary winning strategy and let this be \(s\). Let \(p\) be a function defined as in Theorem 4.1.5. We will prove that every \(\alpha\)-shading \(G\) of \(X \times Y\) by open fuzzy sets has a point finite \(\alpha\)-shading refinement by open fuzzy rectangles.
Let $U_0 = \{0\}$, $A_0 = \{0\}$ and $R(0) = H(0) = X \times Y$. For each $n \geq 1$, we shall construct a collection $U_n$ of open fuzzy rectangles and a collection $\{\{R(a), H(a)\}: a = (a_1, a_2, \ldots, a_n) \in A_n\}$ of pairs consisting of fuzzy closed $\times$ open rectangle $R(a)$ and open rectangle $H(a)$ satisfying the following conditions

For each $n \geq 1$

(i) $U_n$ is a point finite collection in $X \times Y$

(ii) For every $U \times V \in U_n$, there is a $G \in G$ such that $U \times V < G$

(iii) $\{H(a): a \in A_n\}$ is point finite in $X \times Y$.

(iv) $\text{Sup} \{U: U \in U_n\} < \text{Sup} \{H(a): a \in A_n\}$

(v) $a_\_ \in A_{n-1}$

(vi) $R(a) < R(a^+) \text{ and } R(a) < H(a) < H(a)$

(vii) $S(R(a^+)) \land R(a) = 0$

(viii) $R(a) \setminus \text{Sup}\{U: U \in U_{n+1}\} < \text{Sup}\{R(a^+, \xi): \{a^+, \xi\} \in A_{n+1}\}$

(ix) $p(R(a^+)^+, \ldots, R(a(n-1)), R(a)^+) \land H(a)^{''} = 0$

Assume that for each $i \leq n$, the collections $U_i$ and $\{R(a), H(a): a \in A_i\}$ have been constructed.

Now for any $a \in A_n$, let $\{C_\gamma: \gamma \in \Gamma(a)\}$ be a discrete collection of $\alpha$-compact sets whose supremum is $S(R(a^+))$. From the fact that $X$ is fuzzy regular $\alpha$-metacompact it follows that there exists point finite collections $\{W_\gamma: \gamma \in \Gamma(a)\}$ and $\{O_\gamma: \gamma \in \Gamma(a)\}$ of open fuzzy sets such that $C_\gamma < W_\gamma < \text{cl} W_\gamma < O_\gamma < H(a) \setminus \text{Sup}\{C_\beta: \beta \in \Gamma(a), \beta \neq \gamma\}$ for each $\gamma \in \Gamma(a)$. Now $Y$ is $\alpha$-metacompact and $R(a)^{''}$ is open in $Y$. Now $R(a)^{''}$ is $\alpha$-metacompact (Since $\alpha$-metacompact is hereditary with respect to open subsets) and hence for each $\gamma \in \Gamma(a)$, there exists a collection $U_\gamma = \{U_{\delta_j} \times V_\delta: j = 1, 2, 3, \ldots, m_\delta \text{ and } \delta \in \Delta(\gamma)\}$ such that

(i) $C_\gamma < U_\delta = \text{Sup}_{\delta \in \Delta(\gamma)} U_{\delta_j} < W_\delta$ for each $\delta \in \Delta(\gamma)$.

(ii) Each $U_{\delta_j} \times V_\delta$ is contained in some $G \in G$.

(iii) $\{V_\delta: \delta \in \Delta(\gamma)\}$ is point finite $\alpha$-shading of $R(a)^{''}$.

Set $U_{n+1} = \cup \{U_\gamma: \gamma \in \Gamma(a) \text{ and } a \in A_n\}$ and $A_{n+1} = \{a \oplus \delta: \delta \in \Delta(\gamma), \gamma \in \Gamma(a), a \in A_n\} \cup \{a^+ \theta: a \in A_n\}$. 


Take any $a+\xi \in A_{n+1}$. Then observe that $\alpha \cap i \in A_i$ for all $i \leq n$.

If $\xi = \delta$ for some $\delta \in \Delta(\gamma)$ and $\gamma \in \Gamma(a)$, put $R(\alpha \oplus \delta) = [\text{cl } W_\gamma \setminus V_\delta] \cap R(\alpha') \times V_\delta$ and $H(\alpha \oplus \delta) = [a \setminus (\text{cl } W_\gamma \setminus V_\delta)] \cap R(\alpha') \times V_\delta$.

If $\xi = \omega$, put $R(a+\omega) = R(a) \setminus \text{Sup}_\gamma W_\gamma \times R(a')$, and $H(a+\omega) = H(a) \setminus p (R(a/1), \ldots, R(a/n), R(a+\omega)) \times V_\delta$.

Then clearly $U_{n+1}$ and $\{R(a), H(a) : a \in A_{n+1}\}$ satisfies conditions (i) --- (ix).

Now take $U = \bigcup_{n \geq 1} U_n$. Now it can be shown that $U$ is an $\alpha$-shading of $X$ and we will prove that $U$ is also point finite. Also by (ii) $U$ is a collection of open fuzzy rectangles in $X \times Y$ and any $U \times V \in U$ is contained in some $G \in G$.

Similar to the proof of claim in Theorem 6.1.2, we get if $\{a_n\}$ is a sequence such that $a_n \in A_n$ and $(a_n)_n = a_n$, for each $n \geq 1$, where $a_0 = 0$, then $\text{Inf}_{n \geq 1} H(a_n) = 0$.

Again we claim that $\text{Inf}_{n \geq 1} (\text{Sup} H_n) = 0$. Where $H_n = \{H(a) : a \in A_n\}$. For if possible let there be an $z_0$ such that $\text{Inf}_{n \geq 1} (\text{Sup} H_n)(z_0) > \eta = \text{sup} \{\eta \geq 0\}$. Take $A_n(z_0) = \{a \in A_n : H(a)(z_0) \geq \eta\}$. By (iii) we get $A_n(z_0)$ is finite and by (v) and (vi), $a \in A_n(z_0) \Rightarrow a \in A_{n+1}(z_0)$. Then by Konings' Lemma [Lemma 2.8 of [Y1]] there exists $(\beta_1, \beta_2, \beta_3, \ldots)$ such that $a_n \in (\beta_1, \beta_2, \ldots, \beta_n) \in A_{n}(z_0)$ for each $n \geq 1$. Then $H(a_n)(z_0) \geq \eta$ for each $n \geq 1$.

Hence $\text{Inf}_{n \geq 1} H(a_n)(z_0) \geq \eta$. This is a contradiction to our claim.

Let $z \in X \times Y$ then by claim above we can find an $m \geq 1$ such that $\text{Sup} H_n(z) = 0$.

Now from (v) and (vi) it follows that $\text{Sup} H_{n+1} < \text{Sup} H_n$ for each $n \geq 1$. Since $\text{Sup} H_n(z) = 0$ for each $n \geq m$, from (iv) we get that $\text{Sup} U_n(z) = 0$ whenever $n > m$. Hence it follows from (i) that $U$ is point finite in $X \times Y$. This completes the proof.

From Theorems 4.1.6, 3.3.6, and 6.1.3 next corollary follows easily.

6.1.4 Corollary If $X$ is a fuzzy regular $\alpha$-metacompact space with a $\sigma$-closure preserving $\alpha$-shading by $\alpha$-compact sets, then $X \times Y$ is $\alpha$-metacompact for every $\alpha$-metacompact space $Y$. 
6.2 Games and Shading Dimension

6.2.1 Notation Through out this section by shading we mean \( \theta \)-shading and assume that every shading is essential.

6.2.2 Definition Let \( X \) be a non empty set and \( \mathcal{U} = \{ U_{\lambda} : \lambda \in \Lambda \} \) be a non empty family of fuzzy subsets of \( X \). Then the order of \( \mathcal{U} \) is the largest integer \( n \) such that there exists a subset \( M \) of \( \Lambda \) having \( n + 1 \) elements such that \( \text{Inf}_{\lambda \in M} U_{\lambda} > 0 \). And if there is no such integer, order is \( \infty \). If the collection is void, its order is defined to be \(-1\).

6.2.3 Definition The shading dimension of a fuzzy topological space \( X \) (\( \text{Shad} \ X \)) is the least integer \( n \) such that every finite open shading of \( X \) has an open shading refinement of order not exceeding \( n \). If there is no such integer, the shading dimension is said to be \( \infty \).

6.2.4 Definition A shading \( \mathcal{U} = \{ U_{\alpha} : \alpha \in \Lambda \} \) is essential if for every \( \beta \in \Lambda \), \( U_{\beta} > 1 \text{Sup}_{\alpha \neq \beta} \{ U_{\alpha} : \alpha \in \Lambda \} \).

6.2.5 Theorem If \( X \) is a fuzzy topological space, then the following are equivalent

(i) \( \text{Shad} \ X \leq n \)

(ii) For every finite open shading \( \{ U_{1}, U_{2}, U_{3}, U_{k} \} \) of \( X \), there is an open shading \( \{ V_{1}, V_{2}, V_{3}, \ldots, V_{k} \} \) of order not exceeding \( n \) such that \( V_{i} < U_{i} \) for each \( i = 1, 2, 3, \ldots k \).

(iii) If \( \{ U_{1}, U_{2}, U_{3}, U_{n+2} \} \) is an open shading of \( X \), there is an open shading \( \{ V_{1}, V_{2}, V_{3}, \ldots, V_{n+2} \} \) such that \( V_{i} < U_{i} \) and \( \text{Inf}_{i \leq n} V_{i} = 0 \).

Proof

(i) \( \Rightarrow \) (ii)

Let \( \text{Shad} \ X \leq n \). Therefore the shading \( \{ U_{1}, U_{2}, U_{3}, U_{k} \} \) has an open shading refinement say \( W \) with order not exceeding \( n \). Now if \( W \in W \), there is some \( i \) such that \( W < U_{i} \) and suppose that each \( W \) is associated with one of the \( U_{i} \) containing it and take \( V_{i} = \text{Sup}_{W < U_{i}} \{ W : W < U_{i} \} \). Clearly each \( V_{i} \) is open and \( V_{i} < U_{i} \) for every \( i \). Since order of \( W \) is not
exceeding \( n \), it follows that for any \( x \in X \), \( W(x) > 0 \) for at most \( n+1 \) members of \( W \) and each \( W \in W \) is associated with a unique \( U_i \). Therefore it follows that \( V(x) > 0 \) for at most \( n+1 \) members of \( \{V_i\} \) and hence \( \{V_i\} \) is a shading of \( X \) with order not exceeding \( n \).

(ii) \( \Rightarrow \) (iii) is clear.

(iii) \( \Rightarrow \) (ii)

Let \( U = \{U_1, U_2, U_3, ..., U_k\} \) be a finite open shading of \( X \). Assume \( k > n+1 \),

We define a collection \( \{G_i; 1 \leq i \leq n+2\} \) as follows
\[
G_i = U_i \quad \text{for} \quad 1 \leq i \leq n+1 \quad \text{and} \\
G_{n+2} = \text{Sup}_{i \geq n+2} U_i
\]

Clearly each \( G_i \) is open and \( \{ G_i \} \) is a shading of \( X \). Then by (iii) there is an open shading \( \{H_i; 1 \leq i \leq n+2\} \) such that \( H_i < U_i \) and \( \text{Inf}_{i \geq n+2} H_i = 0 \). Now take \( W_i = U_i \) if \( 1 \leq i \leq n+1 \) and \( W_i = U_i \land H_{n+2} \) if \( i > n+1 \). Then clearly \( W = \{W_i; 1 \leq i \leq n+2\} \) is an open shading of \( X \) with the property that \( W_i \leq U_i \) and \( \text{Inf}_{i \geq n+2} W_i = 0 \). Now if there exists a subset \( B \) of \( \{1, 2, ..., k\} \) with \( n+2 \) elements such that \( \text{Inf}_{i \in B} W_i = 0 \), we will renumber the family \( W \) to give a family \( P = \{P_1, P_2, P_3, ..., P_k\} \) such that \( \text{Inf}_{i \in B} P_i = 0 \). By proceeding in a manner similar to the construction above, we can obtain an open shading \( W' = \{W'_1, W'_2, W'_3, ..., W'_k\} \) such that \( W'_i < P_i \) and \( \text{Inf}_{i \geq n+2} W'_i = 0 \). By repeating this process for a finite number of times, we will end up with an open shading \( \{V_1, V_2, V_3, ..., V_k\} \) of \( X \) with order not exceeding \( n \) and \( V_i < U_i \).

(ii) \( \Rightarrow \) (i) is obvious.

6.2.6 Theorem If \( A \) is a closed fuzzy subset of a fuzzy topological space \( X \), then
\[ \text{Shad} A \leq \text{Shad} X. \]
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Proof
Let Shad $X \leq n$. Now it is enough to prove that Shad $A \leq n$. For let $U = \{U_1, U_2, U_3, U_k\}$ be a finite open shading of $A$. Then clearly for each $i$, $U_i = A \wedge V_i$ for some $V_i$ open in $X$. Then it follows clearly that $\{V_1, V_2, V_3, V_k, 1/A\}$ is an open shading of $X$. Now since Shad $X \leq n$, this collection has an open shading refinement $W$ with order not exceeding $n$. Now consider $\{W \wedge A: W \in W\}$. This is an open refinement of $U$ with order not exceeding $n$ and hence Shad $X \leq n$.

6.2.7 Definition [HU] A fuzzy topological space $(X,T)$ is normal if for every closed fuzzy set $k$ and open fuzzy set $b$ such that $k \leq b$, there exists a fuzzy set $a$ such that $k \leq \text{int} a \leq c\text{l} a \leq b$.

6.2.8 Definition An open shading $U = \{U_\alpha: \alpha \in \Lambda\}$ is said to be shrinkable if there exists an open shading $V = \{V_\alpha: \alpha \in \Lambda\}$ such that $c\text{l} V_\alpha < U$ for every $\alpha \in \Lambda$. Then $V$ is called a shrinking of $U$.

6.2.9 Definition Two families $\{A_\lambda: \lambda \in \Lambda\}$ and $\{B_\lambda: \lambda \in \Lambda\}$ of fuzzy subsets of a set $X$ is similar if for each $\gamma \subset \Lambda$, $\inf_{\lambda \in \gamma} A_\lambda$ and $\inf_{\lambda \in \gamma} B_\lambda$ are both zero or both non zero.

6.2.10 Proposition A fuzzy topological space $X$ is normal if and only if every point finite shading of $X$ by open fuzzy sets is shrinkable.

Proof
Suppose that $X$ is fuzzy normal. Let $U = \{U_\alpha: \alpha \in \Lambda\}$ be a point finite shading of $X$ by open fuzzy sets. For convenience take $A = \{1, 2, 3, \ldots\}$. Construct $V = \{V_\alpha: \alpha \in \Lambda\}$ by transfinite induction as follows.

Put $F_1 = 1 \setminus \sup_{\alpha \in A} U_\alpha$. Now clearly $F_1 < U_1$. Then by normality [Theorem 1.12 of [M;B1]], there exists an open set $V_1$ such that $F_1 < V_1$ and $c\text{l} V_1 < U_1$. Let $U_\beta$ be constructed for each $\beta < \alpha$ and let $F_\alpha = 1 \setminus [\sup_{\beta < \alpha} V_\beta \vee \sup_{\gamma > \alpha} U_\gamma]$. Now clearly $F_\alpha$ is closed and $F_\alpha < U_\alpha$. Also by normality there exists an open fuzzy subset $V_\alpha$ with $F_\alpha < V_\alpha$.
< \text{cl} V_\alpha < U_\alpha \). Now \( V = \{ V_\alpha: \alpha \in \Lambda \} \) is the required shrinking provided that it is a shading of \( X \). For, let \( x \in X \), since \( U \) is a shading and point finite, it follows that \( U(x)>0 \) for at most finitely many \( U \in U \). Say \( U_{k_1}, U_{k_2}, \ldots, U_{k_n} \). Take \( k = \max\{k_1, k_2, \ldots, k_n\} \).

Now clearly \( x \not\in U_\gamma \) for any \( \gamma > k \). And hence if \( x \not\in U_\beta \) for any \( \beta < k \), then clearly \( x \not\in V_\gamma \) for any \( \gamma > k \). Therefore \( x \in F_\gamma < V_\gamma \). Hence \( x \in V_\gamma \). Thus in any case \( V_\beta (x)>0 \) for some \( \beta \in \Lambda \). Hence \( \{ V_\alpha: \alpha \in \Lambda \} \) is the required shrinking.

Conversely let \( A \) and \( B \) be two closed subsets of \( X \) such that \( A \subseteq \text{cl} B \). Then clearly \( \{ \text{cl} A, \text{cl} B \} \) is a point finite open shading of \( X \). For any shrinking \( \{ U, V \} \) of \( \{ \text{cl} A, \text{cl} B \} \) we have the open fuzzy sets \( \text{cl} A \) and \( \text{cl} B \) containing \( B \) and \( A \) respectively. Hence \( X \) is fuzzy normal. [By theorem 1.12 of [M;B,1]]

6.2.11 Proposition Let \( X \) be a fuzzy normal space and \( \{ G_\alpha: \alpha \in \Lambda \} \) a family of locally finite collection of open fuzzy sets in \( X \) and \( \{ F_\alpha: \alpha \in \Lambda \} \) be a collection of closed fuzzy sets of \( X \) such that \( F_\alpha < G_\alpha \) for each \( \alpha \). Then there is an open collection \( \{ H_\alpha: \alpha \in \Lambda \} \) similar to \( \{ F_\alpha: \alpha \in \Lambda \} \) and \( F_\alpha < H_\alpha < G_\alpha \) for each \( \alpha \).

Proof

Well order \( \Lambda \) and construct the set \( F \) as follows. Take all finite intersections from \( \{ F_\alpha: \alpha \in \Lambda \} \) which does not meet \( F_0 \). Take \( F \) as the supremum of all these and from normality it follows that there exists an open fuzzy set \( H_0 \) such that \( F_0 < H_0 < \text{cl} H_0 < (\text{cl} F) \wedge G_0 \). Now \( \{ \text{cl} H_0, F_1, F_2, \ldots \} \) is similar to \( \{ F_\alpha: \alpha \in \Lambda \} \). Continuing trans-finitely we get a collection \( \{ H_\alpha: \alpha \in \Lambda \} \) which has the required property.

6.2.12 Proposition (A characterisation for shading dimension for fuzzy normal spaces)
The following are equivalent for a fuzzy normal space.

(i) \( \text{Shad} \ X \leq n \)

(ii) Every finite open shading of \( X \) can be refined by a finite closed fuzzy shading of order not exceeding \( n+1 \).

(iii) Every finite shading \( \{ G_1, G_2, \ldots, G_k \} \) of \( X \) can be refined by a finite closed fuzzy shading \( \{ H_1, H_2, \ldots, H_k \} \) of order not exceeding \( n+1 \) such that \( H_i < G_i \) for each \( i \).
Proof follows from theorem 6.2.6 and Proposition 6.2.10 above.

6.2.13 Lemma If E is a closed fuzzy subset of a fuzzy normal space X with Shad E ≤ n, then for each finite open shading \{U_i: i ≤ k\} of X, there exists a finite shading V =\{V_i: i ≤ k\} of open fuzzy sets and an open set G containing E such that

(i) \( V_i < U_i \) for each \( i ≤ k \).

(ii) \( \text{Ord}(x, V) ≤ n+1 \) for each \( x ∈ cl G \)

Proof

From the characterisation in Proposition 6.2.12, clearly there exists a a closed collection \( F =\{F_i: i ≤ k\} \) which is a shading of E and \( F_i < U_i, \forall E \) for each \( i \) with order \( \text{Ord}(x, F) ≤ n+1 \) for each \( x ∈ X \). Now proceeding in a similar manner as in theorem 7.14 of Engeling [E] in the crisp case, there exists a finite open collection of fuzzy sets \( W =\{W_i: i ≤ k\} \) such that \( F_i < U_i < W_i \) for each \( i ≤ k \) and \( \text{Ord}(x, W) ≤ n+1 \) for each \( x ∈ X \).

Take an open set G in X such that \( E < G < cl G < \bigcup_{i≤k} W_i \). (This is possible by normality). Put \( V_i = W_i ∨ (U_i \setminus cl G) \). Then \( V =\{V_i: i ≤ k\} \) and G has the required properties.

6.2.14 Notation Shad\( _n \) denote the family of all fuzzy topological spaces with shading dimension \( ≤ n \).

6.2.15 Remark By Theorem 6.2.6, it follows that \( X ∈ Shad_n \Rightarrow I^X_C ∈ Shad_n \).

6.2.16 Main Theorem Let X be a fuzzy normal space and Player I has a winning strategy in \( G^∗(Shad_n,X) \), then Shad X ≤ n.

Proof

Let \( \{U_j: j ≤ k\} \) be a shading of X by open fuzzy sets and \( S \) be a fuzzy stationary winning strategy for Player I in \( G^∗(Shad_n,X) \).
For each $i$ construct an open shading $U_i = \{ U_{i,j} : j \leq k \}$ of $X$ and open fuzzy sets $G_i$ with the following properties for each $i \geq 1$.

(i) $\text{cl } U_{i,j} < U_{i-1,j} < U_j : j = 1, 2, 3, \ldots, k$.

(ii) $\text{Ord}(x, U_i) \leq n+1$ for each $x \in \text{cl } G_i$.

(iii) $S(\cap G_{i-1}) \cup G_{i-1} < G_i$.

Take $U_{0,j} = U_j$ for each $j \leq k$ and $G_0 = 0$. Let $G_1$ and $U_i$ are constructed for each $i \leq m$. Let $E = S(1 \setminus G_m)$. Then clearly $1 \setminus G_m$ is closed and hence $\text{Shad } E \leq n$. Then by lemma 6.2.13 we have an open shading $U_{m+1} = \{ U_{m+1,j} : j \leq k \}$ of $X$ and an open set $G \supset E$ with $U_{m+1,j} < U_{m,j}$ for each $j \leq k$ and $\text{Ord}(x, U_{m+1}) \leq n+1$ for each $x \in \text{cl } G$. Also $U_{m+1,j} < \text{cl } U_{m+1,j} < U_{m,j}$, since $X$ is fuzzy normal. Let $G_{m+1} = G \cup G_m$. Then $U_{m+1}$ and $G_{m+1}$ satisfy (i) --- (iii).

Now take $F_j = \inf_{i \geq 1} \text{cl } U_{i,j}$ for each $j \leq k$. Take an $x \in X$, now since each $U_i$ is a finite shading of $X$, we can choose $j(x) \leq k$ such that $x \in U_{i,j(x)}$ for infinitely many $i$. But by (i) we have $x \in \inf_{i \geq 1} \text{cl } U_{i,j(x)} = F_j(x)$. Thus the collection $F = \{ F_j : j \leq k \}$ is a shading of $X$ by closed fuzzy sets such that $F_j < U_j$ for each $j \leq k$. Now $\{ G_i : i \geq 1 \}$ is a shading of $X$. Take some $i \geq 1$ such that $x \in G_{i,0}$. Now by (ii) we have $\text{Ord}(x, F) \leq \text{Ord}(x, U_{k,0}) \leq n+1$. Therefore it follows from Proposition 6.2.12 that $\text{Shad } X \leq n$. 