CHAPTER – III
CLOSURE PRESERVING SHADING FAMILIES

In this chapter we study closure preserving shading families and weakly \( \sigma \)-discrete families and a complete characterisation of spaces with closure preserving shading families by fuzzy sets with finite support are obtained. This characterisation involves the concept of fuzzy \( K \)-scattered spaces and hereditarily metacompact spaces. Some close relationships of \( K \)-scattered \( \alpha \)-metacompact spaces and countably \( \alpha \)-compact \( \alpha \)-metacompact spaces with the Game \( G^* (DK, X) \) are investigated.

3.1 Closure Preserving Shading Families

3.1.1 Definition A fuzzy topological space \( X \) said to be weakly \( \sigma \)-discrete if \( X \) is the supremum of a countable number of discrete subsets \( \{ X_n : n \geq 1 \} \) such that \( \forall \{ X_i : 1 \leq i \leq n \} \) is a closed fuzzy set in \( X \) for each \( n \geq 1 \), where a subset \( D \) of a fuzzy topological space \( X \) is discrete if \( D \) is a discrete space given the subspace topology.

3.1.2 Lemma Let \( (X, T) \) be a fuzzy topological space and let \( U = \{ U_x : x \in X \} \) be an \( \alpha \)-shading of \( X \) by open fuzzy sets such that \( U_x(x) > \alpha \) for each \( x \in X \) and \( U_k < U_y \) whenever \( U_y(x) > \alpha \). Then the collection \( F = \{ F_x : x \in X \} \) is a closure preserving \( \alpha \)-shading of \( X \) where \( F_x \) is defined as follows : \( F_x = \{ y \in X : U_f(x) > \alpha \text{ and } F_x(y) = U_f(y) \} \) for each \( x \) in \( X \).

Some Results mentioned in this Chapter are included in a paper titled "Fuzzy Topological Games, \( \alpha \)-Metacompactness and \( \alpha \)-Perfect maps" to appear in Glasnik Mathematicki Vol 35 (55) (2000), 63-72.
Proof

First we will show that $F$ is an $\alpha$-shading of $X$. Let $x \in X$, then clearly $U_x(x) > \alpha$. Now from the definition of $F_x$ it follows that $F_x(x) = U_x(x) > \alpha$. Therefore $F$ is an $\alpha$-shading of $X$.

Again to prove $F$ is closure preserving, clearly we have $\bigvee_{y \in Y} cl F_y \leq cl \{ \bigvee_{y \in Y} F_y \}$. Now let $cl \{ \bigvee_{y \in Y} F_y \}(x) > \alpha$ where $\alpha \in (0,1]$. Clearly $U_x(x) > \alpha$ where $U_x \in U$. Then there is a $y_0$ in $Y$ such that $F_{y_0} \wedge U_x \neq \emptyset$. Let $z$ be a point in $F_{y_0} \wedge U_x$ with $[F_{y_0} \wedge U_x](z) > \alpha$. Then clearly $U_z(z) > \alpha$. Also $U_z(z) > \alpha$. Therefore $U_z < U_x$. Also $F_{y_0}(z) > \alpha$. Therefore $U_{y_0} > \alpha$ by definition. But $U_z < U_x$. Therefore $U_{y_0} > \alpha$. This implies $F_{y_0}(x) > \alpha$. Therefore $cl \{ F_{y_0}(z) \} > \alpha$. Thus $\bigvee_{y \in Y} [cl F_y] > \alpha$. This completes the proof.

3.1.3 Lemma The following are equivalent for a fuzzy topological space $X$.

(a) $X$ has a closure preserving $\alpha$-shading by fuzzy sets having finite support.
(b) $X$ has an $\alpha$-shading $U = \{ U_x : x \in X \}$ by open fuzzy sets such that
   (i) $U_x(x) > \alpha$ for each $x \in X$.
   (ii) $U_x < U_y$ whenever $U_y(x) > \alpha$ and
   (iii) $U$ is point finite in $X$.

Proof

Let $F = \{ F_\lambda : \lambda \in \Lambda \}$ be a closure preserving $\alpha$-shading by fuzzy sets having finite support. Now we define $U_x = \inf \{ F_\lambda : F_\lambda(x) > \alpha \}$. Now clearly $U_x(x) > \alpha$ and hence $U = \{ U_x : x \in X \}$ is an $\alpha$-shading.

Now $U_y = \inf \{ F_\lambda : F_\lambda(y) > \alpha \}$. Now if possible let $U_y(x) < \alpha$. Clearly $U_x(x) > \alpha$ and hence $U_y(x) < U_x(x)$ and hence (ii).
Again we will prove that $U$ is a point finite collection in $X$. If possible let for any $x \in X$, $U_x(x) > \alpha$ for infinitely many $U \in U$. Thus we can choose an infinite number of points $\{x_\lambda : \lambda \in \Lambda\}$ of $X$ such that $U_{x_\lambda}(x) > \alpha$ for each $\lambda \in \Lambda$. Now since $F$ is an $\alpha$-shading we have an $F_{\lambda} \in F$ such that $F_{\lambda}(x_{\lambda}) > \alpha$ for each $\lambda \in \Lambda$. This is a contradiction since $F_{\lambda}$ is a fuzzy set with finite support.

Converse part follows from Lemma 3.1.2.

3.1.4 Lemma Let $X$ be a fuzzy topological space with a point finite $\alpha$-shading $U = \{U_x : x \in X\}$ by open fuzzy sets with $U_x(x) > \alpha$ for every $x \in X$ and $U_x < U_y$ whenever $U_y(x) > \alpha$, then $X$ has a countable pair wise disjoint $\alpha$-shading $\{X_n : n \in \mathbb{N}\}$ such that each $X_n$ is discrete and $\bigvee_{i=1}^{n} X_i$ is fuzzy closed in $X$ for each $n \in \mathbb{N}$.

Proof

Let $U$ be a point finite $\alpha$-shading of $X$ by open fuzzy sets, we define $\alpha$-Ord $(x, U) = \text{Card}\{U \in U : U(x) > \alpha\}$ and let $X_n$ be the collection of all fuzzy points with $\alpha$-Ord $(x, U) = n$ and values defined by $X_n(x) = \sup\{U(x) : U \in U \text{ and } U(x) > \alpha\}$. This is possible since $U$ is point finite. Clearly $X_n$’s are pair wise disjoint. Now we will prove that $K_n = \bigvee_{i=1}^{n} X_i$ is fuzzy closed for each $n \geq 1$. For, if possible let $K_n$ have a cluster point $t_\eta$ which does not belong to $K_n$. Then every nbhd of $t_\eta$ contains some point of $K_n$. Now $U_i(t) > \alpha$ for at most finitely many $U_i \in U$. Now $U_i$ is the smallest of all such $U$’s. Consider the neighbourhood $U_i$ of $t_\eta$. Then $U_i$ contains some point of $K_n$ say $s_\lambda$. Now clearly $U_i$ is a nbhd of $s_\lambda$ and hence $U_i < U_i$. Now $t_\eta \in X_k$ for some $k > n$ and $s_\lambda \in X_p$ for some $p \leq n$. Since $t_\eta \in X_k$, it follows that $U_i(t) > \alpha$ for $i = 1, 2, \ldots, k$ and $U_i < U_i$ for $i = 1, 2, \ldots, k$ where $U_i(s) > \alpha$ and $U_i < U_i$. Thus $U_i(s) > \alpha$ for $i = 1, 2, \ldots, k$. Thus $s_\lambda \in$
$X_k$ for some $k>n$. This is a contradiction since $X_i$ are disjoint. Thus $\bigvee_{i=1}^{n} X_i$ is closed for each $n \geq l$.

Now let $n \geq l$ and $x \in X_n$. Now $\alpha \text{-Ord}(x, U) = n$. Therefore we can find $x_1, x_2, \ldots, x_n \in X$ such that $U_{\alpha}(x) > \alpha$ for $i = 1, 2, \ldots, n$. Let $\eta$ be a fuzzy point in $U_k \cap X_n$ where $\eta = U_j(y)$. Clearly $\eta > \alpha$. Now $\eta \in U_k$ implies $\alpha < \eta \in U_\alpha(y)$. Thus $U_\alpha(y) > \alpha$. Also $U_j(y) > \alpha$. Therefore $U_j \subseteq U_k$ for $i = 1, 2, \ldots, n$. Now $\alpha < U_j(y) < U_{\alpha}(y)$. Thus $U_j(y) > \alpha$ and $y \in X_n$ implies that $y = x_i$ for some $i$.

Now consider the set of fuzzy singletons with support $\{x_i : i = 1, 2 \ldots, n\}$. Now clearly $U_k \cap X_n \subseteq \{p_1, p_2, \ldots, p_n\}$, where $p_i$ are fuzzy singletons with support $x_i$. Now since $X$ is assumed to be fuzzy $T_i$, singletons are fuzzy closed and hence it follows that $X_n$'s are discrete for each $n \geq l$.

3.1.5 Proposition If a fuzzy topological space $X$ has a closure preserving $\alpha$-shading by fuzzy closed and $\alpha$-compact sets, then $X$ is $\alpha$-metacompact.

Proof
From the characterization of $\alpha$-metacompactness, it is enough if we prove that every directed $\alpha$-shading by open fuzzy sets of $X$ has a closure preserving closed refinement.

Let $U$ be a directed $\alpha$-shading of $X$ by open fuzzy sets. Let $C$ be any closure preserving $\alpha$-shading by closed $\alpha$-compact fuzzy sets. Now $U$ is an $\alpha$-shading of $C$ for any $C \in C$. Now since $C$ is $\alpha$-compact, it has a finite $\alpha$-sub shading say $\{U_1, U_2, \ldots, U_k\}$. Now $C < U_1 \vee U_2 \vee \ldots \vee U_k < U$ for some $U \in U$ since $U$ is directed.
Thus $C < U$ for some $U \in U$. Therefore $\mathcal{C}$ is an $\alpha$-shading refinement of $U$. This completes the proof.

## 3.2 Fuzzy K-Scattered Spaces

Let $X$ be a fuzzy topological space and $F \in \mathcal{F}^K$. Then we define the $K$-derivative of $F$ of order $1$, $F^{(1)}$, as the collection of all fuzzy points in $X$ whose support set is given by $\text{Supp } F^{(1)} = \{ x \in X : F(x) > 0 \text{ and } \exists \text{ no fuzzy nbd } g \in K \text{ with } g(x) > 0 \text{ and } g < F \}$. Where $K$ is a collection of spaces such that we always assume that $K$ is non empty and $X \in K$ implies $\mathcal{F}^X \subseteq K$. The value of $x$ at $F^{(1)}$ is the same as that at $F$. That is $F^{(1)}(x) = F(x)$ for all $x \in \text{Supp } F^{(1)}$. Inductively we define $F^{(\lambda+1)} = (F^{(\lambda)})^{(1)}$ for each ordinal $\lambda$. If $\lambda$ is a limit ordinal $F^{(\lambda)} = \bigwedge_{\mu<\lambda} F^{(\mu)}$.

Now take $\zeta(X) = \inf \{ \lambda : X^{(\lambda)} = 0 \}$ if it exists
$$= \infty \text{ otherwise.}$$

### 3.2.1 Definition
A fuzzy topological space $X$ is said to be $K$-scattered if $\zeta(X) = \eta$ for some ordinal $\eta$. Or equivalently for every $0 \neq F \in \mathcal{F}^X$ there exists a point $x \in F$ and a fuzzy nbd $N$ of $X$ with $N(x) > 0$ where $N < F$ and $N \in K$.

### 3.2.2 Definition
Let $U$ be an $\alpha$-shading of a fuzzy topological space $X$. We say that $U$ is $\alpha$-disjoint if $U \wedge V < \alpha$ for all $U, V \in U$ and $U \neq V$.

### 3.2.3 Definition
An $\alpha$-disjoint $\alpha$-shading $\{ L_\lambda : \lambda < \eta \}$ of a fuzzy topological space $X$ is called a fuzzy $K$-scattered partition if $L_\lambda(x) \leq N(x)$ for all $x \in X$ and for some $N \in K$ and $\bigwedge_{\mu \leq \lambda} L_\mu$ is open in $X$ for each $\lambda < \eta$.

### 3.2.4 Lemma
A $K$-scattered fuzzy topological space has a $K$-scattered partition.
Proof

Let $X$ be a $K$-scattered fuzzy topological space. Therefore by definition, we have $\zeta(X) = \eta$ for some ordinal $\eta$. Let $\lambda < \eta$ and take $Y_\lambda = X^{(\lambda)} \setminus X^{(\lambda+1)}$. Therefore clearly it follows that each $x \in Y_\lambda$ has a fuzzy neighbourhood $N_x$ in $Y_\lambda$ with $N_x(x) > 0$ and $N_x \in K$.

Well order $Y_\lambda$ by $<_\lambda$. For each $x \in Y_\lambda$, take $L_x = [\operatorname{Int} N_x] \setminus [\bigvee_{y < x} \operatorname{Int} N_y : y < x]$. Clearly each of $L_x$ is open and $L_x < N_x \forall x$ and $N_x \in K$. Also since each $L_x$ is fuzzy open, so is their arbitrary union. Thus $\bigvee_{x \in Y_\lambda} \{L_x : x \in Y_\lambda \text{ and } \lambda < \zeta(X)\}$ is open. Therefore it follows that $\{L_x : x \in Y_\lambda \text{ and } \lambda < \zeta(X)\}$ is a $K$-scattered partition of $X$.

3.2.5 Definition Let $A$ be a fuzzy set. Then a fuzzy point $p \in A$ is called an isolated point if it has a fuzzy neighbourhood $U$ such that $U(x) = 0$ for all $x \in A$ with $x \neq p$.

3.2.6 Definition A fuzzy topological space $X$ is said to be scattered if each fuzzy subset $A$ of $X$ has an isolated point in $X$.

3.2.7 Remark Clearly $I$-scattered and scattered spaces coincide. Where $I$ is the class consisting of all one point spaces and the empty space.

3.2.8 Remark The converse of the Lemma 3.2.4 is in general not true. This follows from the next example.

3.2.9 Example Take $X = \mathbb{R}^2$. Define a fuzzy topology $T$ on $\mathbb{R}^2$ by declaring each point in $\mathbb{R}^2$ with rational co-ordinates as fuzzy open singletons (we denote it by $Q^*$) together with the sets of the form $\{z\} \cup \{Q^* \cap U\}$, where $z \in \chi_U$ and $U$ is the usual crisp open set in $\mathbb{R}^2$.

Let $A$ be any fuzzy subset of $X$. If $A \cap Q^* \neq 0$, from the definition of $Q^*$ it follows that $A$ has an isolated point. Now if $A \cap Q^* = 0$, then $A < (Q^*)'$. Then every fuzzy point $p \in A$ is contained in $\{p\} \cup \{Q^*\}$. This cannot contain any other point of $A$, $A < (Q^*)'$. Hence $(X,T)$ is $I$-scattered.
Now for every $\alpha$-disjoint $\alpha$-shading of $X$, any member of this shading cannot be contained in some $N \in I$. Thus $X$ has no $I$-scattered partition even though it is $I$-scattered.

3.2.10 Definition [G;K;M] A fuzzy topological space $X$ is said to be fuzzy regular if and only if for every fuzzy point $p$ in $X$ and for every open fuzzy set $U$ containing $p$, there exists an open fuzzy set $W$ such that $p \leq W \leq W \leq U$.

The converse of the Lemma 3.2.4 is true only when $X$ is fuzzy regular.

3.2.11 Lemma A regular fuzzy topological space with a $K$-scattered partition is $K$-scattered.

Proof Let $X$ be a fuzzy regular space with a $K$-scattered partition \{$L_\lambda : \lambda < \eta$\}. Let $F$ be any fuzzy closed set in $X$. Let $\delta = \text{Min} \{ \lambda : L_\lambda \cap F \neq \emptyset \}$. And take some $x \in L_\lambda \cap F$. Now clearly from the definition of the $K$-scattered partition, it follows that $\bigvee_{\lambda \leq \delta} L_\lambda$ is an open fuzzy set containing $x$. Since $X$ is fuzzy regular, it is possible to find an open fuzzy set $U$ such that $x \in U < \overline{U} < \bigvee_{\lambda \leq \delta} L_\lambda$. Now $L_\delta < N$ for some $N \in K$. Thus for each closed fuzzy set $F$ there exists $x \in F$ such that $x \in F < \overline{U} \cap F < L_\delta \cap F < N \cap F \in K$. Thus $X$ is $K$-scattered.

3.2.12 Proposition A fuzzy topological space $X$ is hereditarily $\alpha$-metacompact if and only if every $\alpha$-disjoint $\alpha$-shading \{$L_\lambda : \lambda < \eta$\} of $X$ such that $\bigvee_{\mu(\lambda)} L_\mu$ is open in $X$ for each $\lambda < \eta$ has a point finite expansion \{$U_\lambda : \lambda < \eta$\} of open fuzzy sets (i.e., $L_\lambda < U_\lambda$ for each $\lambda < \eta$.)
Proof

Let $X$ be hereditarily $\alpha$-metacompact and let $\left\{ L_\lambda : \lambda < \eta \right\}$ be an $\alpha$-disjoint $\alpha$-shading of $X$ such that $\bigvee_{\mu \in \lambda} L_\mu$ is open for each $\lambda < \eta$. We give the proof by the method of induction. Now clearly the statement holds for $\eta = 1$. Let $\eta$ be a fixed ordinal. Assume that the statement is true for all $\xi < \eta$. If $\eta$ is not a limit ordinal, take $\eta = \xi + 1$ Now $\left\{ L_\mu : \mu < \xi \right\}$ is an $\alpha$-disjoint $\alpha$-shading of $\bigvee_{\mu \in \lambda} L_\mu$. Since $\bigvee_{\mu \in \lambda} L_\mu$ is open, from hereditarily $\alpha$-metacompactness and induction hypothesis it follows that $\left\{ L_\lambda : \lambda < \xi \right\}$ has a point finite expansion $\left\{ U_\lambda : \lambda < \xi \right\}$ by open fuzzy sets. Now put $U_\xi = X$. Then clearly $\left\{ U_\lambda : \lambda < \eta \right\}$ is a point finite expansion of $\left\{ L_\lambda : \lambda < \eta \right\}$ by open fuzzy sets.

Now if $\eta$ is a limit ordinal, we define $S_\delta = \bigvee_{\lambda \leq \delta} L_\lambda$ for every $\lambda < \eta$. Now clearly $\left\{ S_\delta : \delta < \eta \right\}$ is an $\alpha$-shading of $X$ by open fuzzy sets. Now since $X$ is $\alpha$-metacompact, there is an $\alpha$-shading refinement $\left\{ U_\delta : \delta < \eta \right\}$ by open fuzzy sets such that $U_\delta < S_\delta$ for each $\delta < \eta$. Now consider the collection $\left\{ L_\lambda \wedge U_\delta : \lambda < \delta \right\}$. This is an $\alpha$-disjoint $\alpha$-shading of $U_\delta$ of length $\delta < \eta$. Hence from hereditarily $\alpha$-metacompactness and induction hypothesis it follows that $\left\{ L_\lambda \wedge U_\delta : \lambda < \delta \right\}$ has a point finite expansion $\left\{ W_\lambda, \delta : \lambda < \delta \right\}$ such that $W_\lambda, \delta < U_\delta$ for each $\lambda < \delta$. Take $W_\lambda = \vee_{\lambda < \delta < \eta} W_\lambda, \delta$ for each $\lambda < \eta$. Now clearly $\left\{ W_\lambda : \lambda < \eta \right\}$ is point finite expansion of $\left\{ L_\lambda : \lambda < \eta \right\}$ by open fuzzy sets.

Conversely to prove every open subspace of $X$ is $\alpha$-metacompact, let $O$ be an open subspace of $X$. Let $U = \left\{ U_\lambda : \lambda < \eta \right\}$ be an $\alpha$-shading of $O$ by open fuzzy sets. We define $\left\{ L_\lambda : \lambda < \eta \right\}$ as follows. $\text{Supp}(L_\lambda) = \left\{ x \in X : U_\lambda(x) > \lambda \right\}$ and $L_\lambda(x) = U_\lambda(x)$ for all $x \in X$. Also take $L_\eta = \left[ \text{Sup} \{ U : U \in U \} \right]$. Now consider $\left\{ L_\lambda : \lambda \leq \eta \right\}$. Also this is an $\alpha$-disjoint $\alpha$-shading of $X$ such that $\bigvee_{\mu \in \lambda} L_\mu$ is open for each $\lambda < \eta$ and hence has a point finite expansion $\left\{ W_\lambda : \lambda \leq \eta \right\}$ by open fuzzy sets such that $W_\lambda < U_\lambda$ for each $\lambda < \eta$. Now clearly $W = \left\{ W_\lambda : \lambda < \eta \right\}$ is a point finite refinement of $U$ by open fuzzy sets. Hence $O$ is $\alpha$-metacompact.
3.2.13 **Definition** A class of fuzzy topological spaces \( K \) is said to be finitely additive if every space with a finite \( \alpha \)-shading by closed fuzzy sets of \( K \) belongs to \( K \).

3.2.14 **Lemma** Let \( K \) be a finitely additive family of fuzzy topological spaces and suppose that each \( X \) belongs to \( K \) has a countable \( \alpha \)-shading \( \{X_n: n \geq 1\} \) and for each \( n \geq 1 \) there exists a point finite \( \alpha \)-shading \( U_n \) of \( X \) such that \( X_n \setminus \text{Sup} \{ U_n \setminus V \} \in K \) for each finite \( V \subset U_n \). Then \( X \) has a closure preserving \( \alpha \)-shading by closed fuzzy sets which belongs to \( K \).

**Proof**

Consider the collection \( W = \cup_{n \geq 1} W_n \) where \( W_n \) are defined as follows. Take

- \( W_1 = U_1 \)
- \( W_2 = \{ W_{2, U}: U \in U_2 \} \) where
- \( \text{Supp}(W_{2, U}) = \{ x \in X: U(x) > \alpha \} \setminus \{ x \in X: X_1(x) > \alpha \} \) and \( W_{2, U}(x) = U(x) \) for all \( x \in X \).

Proceeding like this we get

- \( W_{n+1} = \{ W_{n+1, U}: U \in U_{n+1} \} \) where
- \( \text{Supp}(W_{n+1, U}) = \{ x \in X: U(x) > \alpha \} \setminus \{ x \in X: [X_1 \vee X_2 \vee X_3 \vee \ldots \vee X_n](x) > \alpha \} \)

Now clearly \( W \) is an \( \alpha \)-shading of \( X \) by open fuzzy sets and is also point finite. For, let \( x \in X \). Let \( k \) be the smallest integer such that \( X_k(x) > \alpha \). Now clearly \( W(x) = 0 \) for all \( W \in W_n \) for \( m = k+1, k+2, \ldots \). Also since \( U_i \) are point finite, if follows that each of \( W_n, W_1, W_2, \ldots, W_k \) has at most finitely many members with membership values of \( x \) greater than zero. Thus \( W \) is also point finite.

Now we know that if an \( \alpha \)-shading, \( U \) of a fuzzy topological space \( X \) is interior preserving then the collection \( F = \{ X \setminus \text{Sup} \{ U: U(x) = 0 \}: x \in X \} \) is a closure preserving \( \alpha \)-shading of \( X \) by closed fuzzy sets. Now consider the collection \( K = \{ K_x: x \in X \} \) where \( K_x = X \setminus \text{Sup} \{ W: W(x) = 0 \} \). Now since \( W \) is point finite, it is interior preserving and
hence \( K \) is a closure preserving \( \alpha \)-shading of \( X \) by closed fuzzy sets. Take an \( x \in X \). Let 
\[
n_x = \min \{ n : X_n(x) > \alpha \}.
\]
Now if \( y \in K_x \), then clearly \( n_y \leq n_x \). Also 
\[
K_x < \bigvee_{n \leq n_x} \left[ X_n \setminus \sup \{ U : U(x) = 0 \} \right] \in K.
\]
Since \( K_x \) is closed it follows that \( K_x \in K \).

**3.2.15 Theorem** Let \( K \) be a finitely additive class of fuzzy topological space. If \( X \) is a hereditarily \( \alpha \)-metacompact space with countable \( \alpha \)-shading by closed and \( K \)-scattered fuzzy sets, then \( X \) has a closure preserving \( \alpha \)-shading by closed fuzzy sets which belong to \( K \).

**Proof**

Take a countable \( \alpha \)-shading \( \{ X_n : n \geq 1 \} \) of \( X \) by closed fuzzy sets where each \( X_n \) is \( K \)-scattered. Let \( n \geq 1 \). Then \( X_n \) has a \( K \)-scattered partition say \( \{ L_{n,\lambda} : 0 < \lambda < \eta_n \} \). Also take \( L_{n,\lambda} = X_n' \). Now take \( \mathbf{L_n} = \{ L_{n,\lambda} : 0 \leq \lambda < \eta_n \} \). Clearly \( \mathbf{L_n} \) is an \( \alpha \)-shading of \( X \). Since \( \mathbf{L_n} \) is a \( K \)-scattered partition of \( X_n \) together with \( X_n' \), it is disjoint and hence has an open expansion say \( \mathbf{U_n} = \{ U_{n,\lambda} : 0 \leq \lambda < \eta_n \} \). Let \( \mathbf{V} \) be any finite sub collection of \( \mathbf{U_n} \). Then from the fact \( \mathbf{L_n} \) is an \( \alpha \)-shading of \( X \) and \( L_{n,\lambda} < U_{n,\lambda} \) for all \( 0 \leq \lambda < \eta_n \), it follows that 
\[
F_n(\mathbf{V} ) = X_n \setminus \bigvee \{ \mathbf{U_n} \setminus \mathbf{V} \}
\]
\[
< \bigvee \{ L_{n,\lambda} \in \mathbf{L_n} : U_{n,\lambda} \in \mathbf{V} \ , \ \lambda > 0 \}
\]
\[
\in K
\]

Thus \( F_n(\mathbf{V} ) \) is contained in some member of \( K \) and \( F_n(\mathbf{V} ) \in E^X_n \). It follows that \( F_n(\mathbf{V} ) \in K \). Thus by Lemma 3.2.14, \( X \) has a closure preserving \( \alpha \)-shading by members of \( K \) which are closed fuzzy sets.

Now we give a complete characterisation of spaces with closure preserving \( \alpha \)-shading by fuzzy sets with finite support.

**3.2.16 Theorem** The following are equivalent for a fuzzy topological Space \( X \).

(a) \( X \) has a closure preserving \( \alpha \)-shading by fuzzy sets with finite support.
(b) $X$ is hereditarily $\alpha$-metacompact and has a $\sigma$-closure preserving $\alpha$-shading by fuzzy sets with finite support.

(c) $X$ is hereditarily $\alpha$-metacompact and weakly $\sigma$-discrete.

(d) $X$ has a countable $\alpha$-shading by fuzzy $I$-scattered subsets and is hereditarily $\alpha$-metacompact.

**Proof**

(a) $\Rightarrow$ (b)

If $X$ has a closure preserving $\alpha$-shading by fuzzy sets with finite support, then every subspace of $X$ also should have such a shading. Then from Proposition 3.1.5 it follows that $X$ is hereditarily $\alpha$-metacompact.

(a) $\Rightarrow$ (c)

This follows from Lemma 3.1.3 and Lemma 3.1.4.

(b) $\Rightarrow$ (c)

Given that $X$ has a $\sigma$-closure preserving $\alpha$-shading by fuzzy sets with finite support. Thus there is an $\alpha$-shading $\{X_n: n \in \mathbb{N}\}$ of $X$ such that each of $X_n$ has a c-p $\alpha$-shading by fuzzy sets with finite support. Thus each of $X_n$ is weakly $\sigma$-discrete. (by (a) $\Rightarrow$ (c)). Thus each of $X_n$ is the supremum of a countable number of discrete subsets $\{X_{n,k}: n,k \in \mathbb{N}\}$ such that $\vee_{k \leq m} X_{n,k}$ is closed in $X_n$ for each $m \in \mathbb{N}$. Since a countable collection of countable sets is countable, it follows that $\{X_{n,k}: n,k \in \mathbb{N}\}$ is a countable $\alpha$-shading which satisfies (c).

(c) $\Rightarrow$ (d)

Consider the set $\{Y_n: n \geq l\}$ where $Y_n = \vee_{k \leq n} X_k$ for each $n \geq l$. Then clearly $\{Y_n: n \geq l\}$ is a countable closed $\alpha$-shading of $X$. Now each $X_n$ is discrete and hence is $I$-scattered. Also the union of two $I$-scattered spaces is also $I$-scattered. Therefore it follows that each $Y_n$ is scattered.

(d) $\Rightarrow$ (a)

Follows from Theorem 3.2.15.
3.2.17 Lemma Let $X$ be a fuzzy topological spaces and $U = \{ U_\lambda : \lambda \in \Lambda \}$ be a point finite $\alpha$-shading by open fuzzy sets. Let $B_n = \{ x \in X : \alpha - \text{Ord}(x, U) \leq n \}$. Then $\{ B_n : n \geq 0 \}$ is an $\alpha$-shading of $X$ by closed fuzzy sets. If $n > 0$ and $F$ is a closed fuzzy set with $F \subseteq B_n$ and $F \cap B_{n-1} = \emptyset$, then $F$ has a discrete $\alpha$-shading by closed fuzzy sets where each member is contained in some $U \in U$.

Proof

For any $x \in X$ with $B_n(x) = \emptyset$ for some $n$, by the definition of $B_n$ it follows that there is some $\Lambda' \subseteq \Lambda$ with $n+1$ numbers such that $U_\lambda(x) > \alpha$ for all $\lambda \in \Lambda'$. Now since each $U_\lambda$ is fuzzy open, so is $\{ U_\lambda : \lambda \in \Lambda \}$. This is an open fuzzy nbd of $x$ disjoint from $B_n$. Therefore it follows that $I \setminus B_n$ is fuzzy open and so $B_n$ are closed fuzzy sets.

Also given that $U$ is a point finite $\alpha$-shading of $X$. Therefore there exists at most finitely many $U \in U$ with $U(x) > \alpha$ for any $x \in X$. Then clearly $B_n(x) \geq \alpha$ for some $n$. Thus $\{ B_n : n \geq 0 \}$ is an $\alpha$-shading of $X$.

Take $F$ as in the statement of the Lemma. Let $\Omega$ be the set of all subsets of $\Lambda$ which have $n$ elements and for each $\gamma \in \Omega$ define $V_\gamma = \bigwedge \{ U_\lambda : \lambda \in \gamma \}$. Now clearly $V_\gamma \wedge F < U_\lambda$ for each $\lambda$ in $\gamma$ and the collection $\{ V_\gamma \wedge F : \gamma \in \Omega \}$ is disjoint and hence a discrete $\alpha$-shading of $X$ by closed fuzzy sets.

3.2.18 Corollary Let $U = \{ U_\lambda : \lambda < \eta \}$ be a point finite $\alpha$-shading of a fuzzy topological spaces $X$ by open fuzzy sets and $X_n = \{ x \in X : \alpha - \text{Ord}(x, U) \leq n \}$ for each $n \geq 1$. Then $\{ X_n : n \geq 1 \}$ is a countable $\alpha$-shading of $X$ by closed fuzzy sets and $B_n = \{ B(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) : \lambda_1 < \lambda_2 < \lambda_3, ..., \eta \}$ is a discrete clopen $\alpha$-shading of $X_n \setminus X_{n-1}$ for each $n \geq 1$ where $B(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = \bigwedge_{1 \leq n} U_{\lambda_i} \wedge (X_n \setminus X_{n-1})$.

Proof

Take $F = X_n \setminus X_{n-1}$ in Lemma 3.2.17 the corollary follows.
3.2.19 Definition An $\alpha$-disjoint $\alpha$-shading $\{L_\alpha : \lambda < \eta\}$ of a fuzzy topological space is a $K$-scattered partition if $L_\alpha(x) \leq N(x)$ for all $x \in X$. and for some $N \in K$ and $\vee\{L_\mu : \mu < \eta\}$ is fuzzy open in $X$ for each $\lambda < \mu$.

3.2.20 Theorem Let $K$ be a finitely additive class of fuzzy topological spaces. If a hereditarily $\alpha$-metacompact space $X$ is $K$-scattered then Player I has a winning strategy in $G^\ast(DK, X)$.

Proof

Since $X$ is fuzzy $K$-scattered, $X$ has a fuzzy $K$-scattered partition. Say $V = \{V_\lambda : \lambda < \eta\}$. Now from proposition 3.2.11, it follows that there exists a point finite fuzzy open expansion $U = \{U_\lambda : \lambda < \eta\}$ of $V$. Now since $V$ is an $\alpha$-shading of $X$, it follows that $U$ is also an $\alpha$-shading of $X$. Let $X_n$ and $B_n$, $n \geq 1$ be taken as in Corollary 3.2.18. For each $F \in \mathcal{I}^X$, take $k(F) = \text{Min}\{k \geq 1 : F \cap X_k \neq 0\}$ and $B(F) = \{B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) : B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \subseteq B_k \text{ and } k = k(F)\}$ and $B(0) = \{0\}$. Now by Corollary 3.2.18 it follows that each member of $B(F)$ is fuzzy closed in $X$ and $B(F)$ is discrete in $X$.

We have $B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) = \bigwedge_{i \leq k} U_{\lambda_i} \wedge (X_k \setminus X_{k-1})$. Thus $B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) < \bigvee_{i \leq k} U_{\lambda_i} < \bigvee_{i \leq k} V_{\lambda_i}$. Also since each $B(F)$ is fuzzy closed and $K$ is finitely additive

$\cup B(F) = \cup \{B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) \wedge F : B(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k) \subseteq B_k \text{ and } k = k(F)\}$. Also by Corollary 3.2.18, $B_k$ is a discrete $\alpha$-shading of $X_k \setminus X_{k-1}$ by closed fuzzy sets. Hence $(X_k \setminus X_{k-1}) \wedge F \in DK \cap \mathcal{I}^X$ where $k = k(F)$.

Now we define a fuzzy stationery winning strategy $S$ of Player I for $G^\ast(DK, X)$ as follows

$S : \mathcal{I}^X \rightarrow DK \cap \mathcal{I}^X$, where $S(F) = (X_{k(F)} \setminus X_{k(F)-1}) \wedge F$

Consider the play $(S(\lambda), F_1, S(F_1), F_2, \ldots)$ of $G^\ast(DK, X)$. We have clearly $S(Fn) < Fn$ and hence $S$ is stationery. Now we want to prove $S$ is winning, that is $\text{Inf} F_n = 0$. Now since $\{X_n : n \geq 1\}$ is an $\alpha$-shading of $X$ and $F_n \wedge X_n = 0$ for all $k = 1, 2, 3, \ldots$ It follows that it is enough to prove $F_n \wedge X_n = 0$ for all $n \geq 0$. We will prove this by
induction. Let \( F_n \cap X_n = 0 \) and assume that \( F_n \cap X_{n+1} \neq 0 \). Therefore by definition of \( k(F_n) \) we get \( k(F_n) = n+1 \).

Now \( S(F_n) \cap F_{n+1} = ((X_{n+1} \setminus X_n) \cap F_n) \cap F_{n+1} \)
\[ = (X_{n+1} \setminus X_n) \cap F_{n+1} \]
\[ = 0 \]

Now clearly \( X_n \cap F_n = 0 \) and \( F_{n+1} < F_n \). Hence \( F_n \cap X_{n+1} = 0 \). Therefore it follows that \( F_{n+1} \cap X_{n+1} = 0 \). Thus the proof is complete by induction.

### 3.3 Countably \( \alpha \)-compact Spaces and the Game \( G^{*}(DK, X) \)

We know that most of the properties of countably compact spaces in general topology are discussed in terms of cluster points and accumulation points. So we define \( \alpha \)-cluster points and \( \alpha \)-accumulation points in fuzzy topological spaces in a language which is closely related to shading families and in this framework we obtain a characterisation for countable compactness in fuzzy topological spaces and later use this to obtain some relations of countably \( \alpha \)-compact spaces and the fuzzy topological Game \( G^{*}(DK, X) \).

#### 3.3.1 Definition. Let \( \alpha \in [0,1] \). An \( \alpha \)-cluster point (resp. \( \alpha^{*} \)-cluster point) of a set \( A \) in a fuzzy topological space \( X \) is a fuzzy point \( \chi_{\lambda} \) such that each fuzzy neighbourhood \( U \) of \( \chi_{\lambda} \) with \( U(x) \geq \alpha \) (resp. \( U(x) \leq \alpha \)) contains some fuzzy point of \( A \) with distinct support.

#### 3.3.2 Definition. A sequence \( (\chi_{\lambda}^{n}) \) of fuzzy points with distinct support in a fuzzy topological space \( X \) \( \alpha \)-accumulates at \( \chi_{\lambda} \) (resp. \( \alpha^{*} \)-accumulates) at \( \chi_{\lambda} \) if and only if for every fuzzy neighbourhood \( U \) of \( \chi_{\lambda} \) with \( U(\chi) \geq \alpha \) (resp. \( U(\chi) \leq \alpha \)) and for every \( n \in N \), there is an \( m \geq n \) such that \( \chi_{\lambda_{m}}^{n} < U \) and \( (\lambda_{m}) \) accumulates at \( \lambda \) in the crisp sense in \([0,1] \).
3.3.3 Theorem. The following are equivalent in a fuzzy topological space.

(i) $X$ is countably $\alpha$-compact.

(ii) Every fuzzy subset of $X$ with countably infinite support has at least one $\alpha$-cluster point.

(iii) Every sequence of fuzzy points in $X$ with distinct support has an $\alpha$-accumulation point.

Proof:

(i) $\Rightarrow$ (ii)

If possible let $A$ be a fuzzy subset of $X$ with countably infinite support and has no $\alpha$-cluster point. Then it follows that every fuzzy point $x^i_\lambda$ in $A$ has a fuzzy neighbourhood $U_i$ with $U_i (x^i) > \alpha$ which contains no other fuzzy point of $A$ with distinct support. Now $\text{Supp}(A)$ clearly closed and $X \setminus \text{Supp}(A)$ is open. Now consider the collection $\bigcup \{ U_i : i \in \mathbb{N} \}$. This is clearly a countable $\alpha$-shading of $X$ by open fuzzy sets which has no finite $\alpha$-subshading.

(ii) $\Rightarrow$ (iii)

Let $(x^n_\lambda_{im})$ be a sequence of fuzzy points in $X$ with distinct support. Then there are two possibilities.

(a) Cardinality of the support of the range set is countably infinite. Then by (ii) this has at least one $\alpha$-cluster point say $x_\lambda$. Now every fuzzy nbd $U$ of $x_\lambda$ with $U(x) > \alpha$ contains infinitely many points of the sequence other than $x_\lambda$. Clearly this $x_\lambda$ is an $\alpha$-accumulation point of the sequence. For, For any $n \in \mathbb{N}$ the set $\{ x^n_\lambda_{im} : 1 \leq m \leq n \}$ is finite. Therefore it follows that for any neighbourhood $U$ of $x_\lambda$ with $U(x) > \alpha$ and for any $n \in \mathbb{N}$, there is an $m \geq n$ such that $x^n_\lambda_{im} < U$ and $(\lambda_n)$ accumulates at $\lambda$. 

(b) If cardinality of range set is finite, then there should be some fuzzy point $x_\lambda$ with $x_{\lambda n} = x_\lambda$ for infinitely many $n \in N$. Then clearly this $x_\lambda$ is an $\alpha$-accumulation point.

(iii) $\Rightarrow$ (i)

Let $X$ be not countably $\alpha$-compact. Let $U=\{U_1, U_2, U_3, \ldots\}$ be a countable $\alpha$-shading of $X$ by open fuzzy sets which has no finite $\alpha$-subshading. Therefore $\{U_1, U_2, U_3, \ldots, U_k\}$ cannot $\alpha$-shade $X$ for any finite $k$. Therefore corresponding to each finite $k$ we can find an $x^k \in X$ such that $U_j(x^k) > \alpha$ for some $j > k$ and $U_i(x^k) \leq \alpha$ for $1 \leq i \leq k$. Let $U_j(x^k) = \eta_k$ where $\eta_k \in (\alpha, 1]$. Now the sequence $(x^k_{\eta_k})$ has no $\alpha$-accumulation point. For, if possible let $x_\eta$ be an $\alpha$-accumulation point of $(x^k_{\eta_k})$. Now since $U$ is an $\alpha$-shading of $X$, we can find a minimum $l \in \mathbb{N}$ such that $U_l(x) > \alpha$ and $U_i(x) \leq \alpha$ for all $1 \leq i \leq l$. Now take $n = l + 1$ and consider the neighbourhood $U_l$ of $x$. Then for any $m \geq n$ we have $x^m_{\eta_m} < U_l$. For corresponding to any $m$, we can find some $U_j$ such that $U_j(x^m) > \alpha$ for some $j > m$ and $U_i(x^m) \leq \alpha$ for $1 \leq i \leq m$. Here $m \geq n = l + 1$. Therefore $l < m$ and it follows that $U_l(x^m) \leq \alpha$. But $\eta_m \in (\alpha, 1]$. Thus $x^m_{\eta_m} < U_l$ which is a contradiction. This completes the proof.

3.3.4 Definition [M;B]. A family $\{a_s : s \in S\}$ of fuzzy sets in a fuzzy topological space $(X, \tau)$ is said to be discrete if for each $x \in X$, there exists an open fuzzy set $g$ of $X$ with $g(x) = 1$ such that $a_s \leq I - g$ holds for all but at most one $s$ in $S$.

3.3.5 Theorem. If $C$ is a closure preserving $\alpha$-shading of a fuzzy topological space $X$ by fuzzy closed and countably $\alpha$-compact sets and if $K$ is a class of fts with $C \subset K$, then Player I has a fuzzy stationary winning strategy in $G^*(DK, X)$.

Proof

Corresponding to each fuzzy closed set $F$ in $X$, consider the collection $\{C \wedge F : C \in C\}$ and let $D(F)$ be the maximal disjoint sub-collection of this. This is possible since $C$ is
an $\alpha$-shading of $X$. Clearly $D(F)$ is closure preserving and disjoint and hence it is discrete. 

Now define $S: \mathcal{F} \xrightarrow{\text{into}} \mathcal{F} \cap DK$ by $F \rightarrow \vee D(F)$. We will show that $S$ is a fuzzy stationary winning strategy for Player I in $G^*(DK,X)$.

Let $\{F_n : n \in \mathbb{N}\}$ be a decreasing $(F_1 \supseteq F_2 \supseteq F_3 \ldots)$ sequence with $S(X) \land F_1 = 0$ and $S(F_n) \land F_{n+1} = 0$. If possible let if $\inf F_n \neq 0$. Then there exists $C_0 \in \mathcal{C}$ such that $C_0$ has non empty meet with each of $F_n$. Now $C_0 \land F_n \not\in D(F_n)$ for each $n \geq 1$. For, if $C_0 \land F_n \in D(F_n)$ for some $n$, then

1. $C_0 \land F_n = (C_0 \land F_n) \land F_{n+1}$
2. $= [\vee D(F_n)] \land F_{n+1}$
3. $= S(F_n) \land F_{n+1}$
4. $= 0$. This is a contradiction. Therefore $C_0 \land F_n \not\in D(F_n)$ for each for every $n \geq 1$.

Fix some $n \geq 1$. $D(F_n)$ is maximal and disjoint. Also $C_0 \land F_n \not\in D(F_n)$. Therefore we can take some $C_n \in \mathcal{C}$ such that $C_n \land F_n \in D(F_n)$ and $(C_n \land C_0) \land F_n \neq 0$. For each $n \geq 1$, take some $x' \in X$ such that $[(C_0 \land F_n) \land C_n] (x') \geq \alpha$ where $\alpha \in (0,1]$. Let $\min (C_0(x), F_n(x), C_n(x)) = \lambda_n$. Now clearly we have $[S(F_n)](x') > \alpha$. Also $S(F_n) \land F_{n+1} = 0$. Therefore $F_{n+1}(x') = 0$. Now consider the sequence $(x'_{\lambda_n})$ in $C_0$. Now $C_0$ is countably $\alpha$-compact. There fore it has an $\alpha$-cluster point say $x_\alpha$ in $C_0$. This follows from Theorem 3.3.3.

Now we have $\inf_{n \geq 1} F_n(x) > \alpha$. For, if $F_n(x) \leq \alpha$ for some $n$, then we can choose some $m \geq n$ with $\lambda_m > F_m(x_m)$. But $F_m < F_n$. Therefore $F_m(x_m) < F_n(x_m)$. Now $\lambda_m \leq F_m(x_m) < F_n(x_m)$. Therefore $\lambda_m < F_n(x_m)$. This is a contradiction.

Now claim $\sup_{n \geq 1} C_n(x) = 0$. For, let $C_n(x) > 0$ for some $n$. Now $C_0 \land F_n \in D(F_n)$ and $F_{n+1}(x) > \alpha$. Then $(C_n \land F_n \land F_{n+1})(x) = (S(F_n) \land F_{n+1})(x) = 0$. Therefore $C_n(x) = 0$. This is a contradiction.
Since $C$ is closure preserving, we have $\text{cl}\{x^\alpha_{\lambda_n} : n \geq l\}(x) > \alpha$. Also $\text{cl}\{x^\alpha_{\lambda_n} : n \geq l\} < \text{cl}\sup_{n \geq l} C_n = \sup_{n \geq l} C_n$. Therefore $\sup_{n \geq l} C_n(x) > \alpha$, where $\alpha \in (0,1]$. This is a contradiction to $\sup_{n \geq l} C_n(x) = 0$. This completes the proof.

From Theorem 3.3.5 and Theorem 2.3.4, next corollary follows clearly.

3.3.6 Corollary If a fuzzy topological space $X$ has a $\sigma$-closure preserving $\alpha$-shading by $\alpha$-compact closed fuzzy sets, then player I has a winning strategy in $G^\ast(DC,X)$.
Chapter - IV
FUZZY P-SPACES AND THE GAME $G_\alpha(X)$

The concept of $P$-spaces was introduced by K. Morita and as a generalization of this in this chapter we define fuzzy $P$-spaces ($P_\alpha$-spaces) and some results regarding them are obtained. Again a characterization of $P_\alpha$-spaces in terms of a particular type of fuzzy topological game $G_\alpha(X)$ is also obtained.

4.1 Fuzzy $P$-Spaces

4.1.1 Definition A collection $\{U_i: i = 1, 2, 3, \ldots\}$ of fuzzy subsets of a set $X$ is called an increasing family if $U_i < U_{i+1}$ for every $i = 1, 2, 3, \ldots$.

4.1.2 Definition A fuzzy topological space $X$ is said to be a $P_\alpha$-space if for every increasing family $U = \{U(a_1, a_2, \ldots, a_i): a_1, a_2, \ldots, a_i \in A, i = 1, 2, 3, \ldots\}$ of open fuzzy sets in $X$, there exists a precise refinement $F = \{F(a_1, a_2, \ldots, a_i): a_1, a_2, \ldots, a_i \in A, i = 1, 2, 3, \ldots\}$ by closed fuzzy sets satisfying the condition that if $U$ is an $\alpha$-shading of $X$, then $F$ is also an $\alpha$-shading of $X$ where $\alpha \in [0, 1)$.

4.1.3 Theorem A fuzzy topological space $X$ is a $P_\alpha$-space if and only if there exists a crisp function $p: \cup G^n \rightarrow \hat{G}^X$ such that

(i) If $(G_1, G_2, G_3, \ldots, G_n) \in G^n$, then $p(G_1, G_2, G_3, \ldots, G_n) < \text{Sup}\{G_k: 1 \leq k \leq n\}$

(ii) If $\{G_1, G_2, G_3, \ldots\}$ is an $\alpha$-shading of $X$, then so is $\{p(G_1), p(G_1, G_2), p(G_1, G_2, G_3), \ldots\}$. Where $G$ represent the family of all open fuzzy subsets of $X$.

Some Results mentioned in this Chapter are accepted for publication in the paper titled Fuzzy P-Spaces in the Journal Discourse(s) No.3 Vol. 1 (2000)
Proof

Let $X$ be a $P_\alpha$-space. Let $(G_1,G_2,G_3,\ldots) \in G^n$ and take $a_i = G_i$ in the definition of $P_\alpha$-spaces and define $U(a_1,a_2,\ldots,a_n) = U(G_1,G_2,G_3,\ldots G_n) = \sup \{G_i : 1 \leq k \leq n\}$. Then clearly $U(G_1,G_2,G_3,\ldots G_n) < U(G_1,G_2,G_3,\ldots G_{n+1})$. Then from the definition of $P_\alpha$-spaces the remaining follows.

Conversely let $U= \{U(a_1,a_2,\ldots,a_i) : a_i \in A, i = 1,2,3,\ldots\}$ be an increasing family of open fuzzy sets in $X$. Now corresponding to each $U(a_1,a_2,a_3,\ldots,a_j)$ in $U$, we define

$$F(a_1,a_2,\ldots,a_i) = p(U(a_1), U(a_1,a_2), U(a_1,a_2,a_3),\ldots U(a_1,a_2,a_3,\ldots,a_j))$$

$$< \sup \{U(a_1,a_2,\ldots,a_i) : 1 \leq i \leq n\}$$

$$= U(a_1,a_2,\ldots,a_i)$$

since $U$ is increasing.

Now if $U$ is an $\alpha$-shading of $X$, for every $x \in X$, there exists a $U(a_1,a_2,a_3,\ldots,a_k)$ such that $U(a_1,a_2,a_3,\ldots,a_k)(x)>\alpha$. Now clearly by definition, we have $F(a_1,a_2,a_3,\ldots,a_k)(x) > \alpha$ and hence $\{F(a_1,a_2,a_3,\ldots,a_k) : a_k \in A, i = 1,2,3,\ldots\}$ is an $\alpha$-shading of $X$. Hence $X$ is a $P_\alpha$-space.

From the definition of $P_\alpha$-Spaces and Theorem 4.1.3 next theorem follows clearly.

4.1.4 Theorem A fuzzy topological space $X$ is a $P_\alpha$-space if and only if there is a crisp function defined from the family of all increasing finite sequences of open fuzzy sets $G$ to the collection of all closed fuzzy sets $t$ with $p(G_1,G_2,G_3,\ldots G_n) < G_n$ where $(G_1,G_2,G_3,\ldots G_n) \in G^n$ and if $G_n < G_{n+1}$ for each $n \in N$ and if $\{G_i,G_2,G_3,\ldots G_n\}$ is an $\alpha$-shading then so is $\{p(G_1), p(G_1,G_2), p(G_1,G_2,G_3), \ldots\}$.

4.1.5 Theorem A fuzzy topological space $X$ is a $P_\alpha$-space if and only if there exists a crisp function $p : \cup (\mathcal{F}^n) \to \mathcal{F}^X$ such that
(i) For each \((F_0, F_1, \ldots, F_n) \in (\mathcal{F}_X)^n\), \(n \geq 0\)

\[
p(F_0, F_1, \ldots, F_n) \wedge \text{Inf}_{i \geq n} F_i = 0
\]

(ii) For each \((F_0, F_1, \ldots) \in (\mathcal{F}_X)\) with \(\text{Inf}_{n \geq 0} F_n = 0\),

the collection \(\{p(F_0, F_1, \ldots, F_n) : n \geq 0\}\) is an \(\alpha\)-shading of \(X\).

**Proof**

Let \((F_1, \ldots, F_n) \in (\mathcal{F}_X)^n\) then \(F_i \uparrow F_1 \vee F_2 \vee F_3 \vee \ldots\) is an increasing family of open sets. Take

\[
U(a_j) = F_{i_j}, U(a_{1,2}) = F_1 \vee F_2 \vee \ldots \vee U(a_1, a_2, \ldots, a_n) = F_1 \vee F_2 \vee \ldots \vee F_n.
\]

Now since \(X\) is a \(P_\alpha\)-space, there exists a collection \(\{F(a_1), F(a_1, a_2), \ldots\}\) such that \(F(a_1, a_2, \ldots, a_i) < U(a_1, a_2, \ldots, a_i)\) for each \(i = 1, 2, 3, \ldots\)

Now define \(p(F_1, \ldots, F_n) = 0\) if \(\text{Inf}_{i \geq n} F_i \neq 0\)

\[
p(F_1, F_2, \ldots, a_1, a_2, \ldots, a_n)
\]

otherwise

Clearly \(p\) has properties (i) and (ii)

Conversely let \((G_1, G_2, \ldots, G_n) \in G^n\). Then \(F_i = G_i \uparrow, F_1 = G_2 \vee \ldots F_n = G_n \downarrow\) are all closed and hence there exists a function \(p' : (\mathcal{F}_X)^n \to \mathcal{F}_X\) such that

\[
p'(F_1, \ldots, F_n) \wedge \text{Inf}_{i \geq n} F_i = 0.
\]

Take \(p(G_1, G_2, \ldots, G_n) = p'(F_1, \ldots, F_n)\) in Theorem 4.1.3, then

\[
p(G_1, G_2, \ldots, G_n) \wedge \text{Inf}_{i \geq n} G_i = 0
\]

Therefore \(p(G_1, G_2, \ldots, G_n) < (\text{Inf}_{i \geq n} F_i) \uparrow\)

\[
= \text{Sup}_{i \geq n} F_i \uparrow
\]

\[
= \text{Sup}_{i \geq n} G_i
\]

and hence \(p\) satisfies (i) and (ii) of

Theorem 4.3.1 and hence \(X\) is a \(P_\alpha\)-space.

**4.1.6 Theorem** If a fuzzy topological space \(X\) has a \(\sigma\)-closure preserving fuzzy closed \(\alpha\)-shading by countably \(\alpha\)-compact sets, then \(X\) is a \(P_\alpha\)-Space.
Proof

Let $F = \bigcup \{ F_n : n \in N \}$ be an $\alpha$-shading of $X$ such that each $F_n$ is closure preserving and countably $\alpha$-compact. Let $\{ U(a_1, a_2, \ldots, a_n) : a_i \in A, i = 1, 2, 3, \ldots \}$ be an increasing sequence of open fuzzy sets. Now corresponding to each $U(a_1, a_2, \ldots, a_n)$ we define $F(a_1, a_2, \ldots, a_n) = \sup \{ F : F < U(a_1, a_2, \ldots, a_n), F \in \bigcup_{i=1}^{n} F_i \}$. Since $\bigcup_{i=1}^{n} F_i$ is closure preserving it follows that $F(a_1, a_2, \ldots, a_n)$ is fuzzy closed and $F(a_1, a_2, \ldots, a_n) < U(a_1, a_2, \ldots, a_n)$ for each $n \geq 1$.

Again let $\{ U(a_1, a_2, \ldots, a_i) : i = 1, 2, 3, \ldots \}$ be an $\alpha$-shading of $X$. Let $x \in X$. Now since $F$ is an $\alpha$-shading of $X$, there exists an $F_0 \in F$ such that $F_0(x) > \alpha$. Let $F_0 \in F_k$ for some $k$. Since $F_0$ is countably $\alpha$-compact, and $U(a_1, a_2, \ldots)$'s are increasing we can find out some $j \in N$ such that $j \geq k$ and $F_0 < U(a_1, a_2, \ldots, a_j)$.

Now $F(a_1, a_2, \ldots, a_j)(x) = \sup_{F \in \bigcup_{i=1}^{n} F_i} \{ F(x) : F \in \bigcup_{i=1}^{n} F_i \} \geq F_0(x) > \alpha$

Thus $\{ F(a_1, a_2, \ldots, a_i) : a_i \in A, i = 1, 2, 3, \ldots \}$ is also an $\alpha$-shading of $X$. This completes the proof.

4.2 A Characterisation of $P_\alpha$-spaces using the Game $G_\alpha(X)$

In this section we describe a game associated with $P_\alpha$-spaces. Here $G_\alpha(X)$ denote the following infinite positional fuzzy topological game. Let $G$ and $F$ denote the collection of all open (resp. closed) fuzzy subsets of a fuzzy topological space $X$. There are two players Player I and Player II. Players alternatively choose fuzzy subsets of $X$ so that each player knows $X$ and first $k$ elements when he is choosing the $(k+1)^{th}$ element.

We say that a sequence $(G_1, F_1, \ldots, G_n, F_n)$ is a play for $G_\alpha(X)$ if for each $n \geq 1$, we have
(i) \(G_n \in G\) is a choice of Player I.

(ii) \(F_n \in F\) and \(F_n \leq \text{Sup} \{G_k : 1 \leq k \leq n\}\) is a choice of Player II.

Player I wins the play \((G_1, F_1, \ldots, G_n, F_n)\) if \(\{G_n : n \in N\}\) is an \(\alpha\)-shading of \(X\) and \(\{F_n : n \in N\}\) is not. And Player II wins if \(\{F_n : n \in N\}\) or both \(\{G_n : n \in N\}\) and \(\{F_n : n \in N\}\) are \(\alpha\)-shadings of \(X\).

A strategy for Player I is a crisp function \(S: \{0\} \cup \mathbb{F}^n \rightarrow G\) and that of Player II is \(t: \mathbb{F}^n \rightarrow F\) such that \(t (G_1, G_2, \ldots, G_n) \leq \text{Sup} \{G_i : 1 \leq i \leq n\}\) for each \((G_1, G_2, \ldots, G_n) \in G^n\) and \(n \geq 1\).

Now clearly for each pair of strategies \((s, t)\) there exists a unique Play \((G_1, F_1, \ldots, G_n, F_n)\) of \(G_\alpha (X)\) defined as follows.

Take \(G_1 = s(0), F_1 = t(G_1), G_2 = s(F_1), F_2 = t(G_1, G_2)\) and so on.

A strategy \(s\) (resp. \(t\)) is winning for Player I (resp. Player II) if he wins every play of \(G_\alpha (X)\) using it.

From Theorem 4.1.6 and definition of \(G_\alpha (X)\), we get the following game theoretic characterization of \(P_\alpha\)-spaces.

**4.2.1 Theorem** A fuzzy topological space is a \(P_\alpha\)-space if and only if Player II has a winning strategy in \(G_\alpha (X)\).
4.3 Remarks

Just like the applications of $P$-spaces in general topology, $P_\alpha$-spaces help the study of covering properties in fuzzy topological spaces. In fact the results regarding the product of $\alpha$-metacompact spaces is discussed in Chapter VI and there it is shown that the product of two $\alpha$-metacompact spaces need not be $\alpha$-metacompact and if we impose some conditions on one of these spaces, we can make the product $\alpha$-metacompact and this is done in terms of $P_\alpha$-spaces.