CHAPTER – II
THE FUZZY TOPOLOGICAL GAME $G^*(K, X)$

2.1 Introduction

A pursuit evasion game $G(K, X)$ in which the pursuer and the evader choose certain subsets of a topological space in a certain way is defined and studied by Telgarsky [T2]. In this chapter we generalise the concept of topological games in to a fuzzy topological space and some results related to them are obtained. Just like in the case of $G(K, X)$, the fuzzy topological game $G^*(K, X)$ has plenty of applications in fuzzy topology especially in fuzzy metacompactness etc, which will be discussed in the succeeding chapters.

2.2 The Fuzzy Topological Game

2.2.1 Notation By $K$ we denote a non empty family of fuzzy topological spaces, where all spaces are assumed to be $T_i$. That is all fuzzy singletons are fuzzy closed. $\mathcal{F}$ denote the family of all fuzzy closed subsets of $X$. Also $X \in K$ implies $\mathcal{F} \subseteq K$. $DK \ (FK)$ denote the class of all fuzzy topological spaces which have a discrete (finite) fuzzy closed $\alpha$-shading by members of $K$.

2.2.2 Definition Let $K$ be a class of fuzzy topological spaces and let $X \in K$. Then the fuzzy topological game $G^*(K, X)$ is defined as follows. There are two players Player I and Player II. They alternatively choose consecutive terms of the sequence $(E_1, F_1, E_2, F_2, \ldots)$ of fuzzy subsets of $X$. When each player chooses his term he knows $K, X$ and their previous choices. A sequence $(E_1, F_1, E_2, F_2, \ldots)$ is a play for $G^*(K, X)$ if it satisfies the following conditions for each $n \geq 1$.

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(1) $E_n$ is a choice of Player I.

(2) $F_n$ is a choice of Player II.

(3) $E_n \in \mathcal{E} \cap K$.

(4) $F_n \in \mathcal{E}$.

(5) $E_n \lor F_n < F_{n-1}$ where $F_0 = X$.

(6) $E_n \land F_n = 0$.

Player I wins the play if $\inf_{n \geq 1} F_n = 0$. Otherwise Player II wins the game.

2.2.3 Definition A finite sequence $(E_i, F_i, E_2, F_2, \ldots, E_m, F_m)$ is admissible if it satisfies conditions (1) -- (6) for each $n \leq m$.

2.2.4 Definition Let $S'$ be a crisp function defined as follows:

$$S': \bigcup_{n \geq 1} (\mathcal{E})^n \xrightarrow{\text{into}} \mathcal{E} \cap K$$

Let $S_I = \{X\}$.

$S_2 = \{F \in \mathcal{E} : (S'(X), F) \text{ is admissible for } G^*(K, X)\}$. Continuing like this inductively we get $S_n = \{(E_1, F_1, E_2, F_2, \ldots, F_n) : (E_1, F_1, E_2, F_2, \ldots, E_n, F_n) \text{ is admissible for } G^*(K, X) \text{ where } F_0 = X \text{ and } E_i = S'(E_1, F_1, E_2, F_2, \ldots, F_{i-1}) \text{ for each } i \leq n\}$. Then the restriction $S'$ of $S' \cup \bigcup_{n > 1} S_n$ is called a fuzzy strategy for Player I in $G^*(K, X)$.

2.2.5 Definition If Player I wins every play $(E_i, F_i, E_2, F_2, \ldots, E_n, F_n, \ldots)$ such that $E_n = S(F_1, F_2, \ldots, F_{n-1})$, then we say that $S$ is a fuzzy winning strategy.
2.2.6 Definition $S: K \rightarrow K$ is called a fuzzy stationery strategy for Player I in $G^*(K, X)$ if $S(F) < F$ for each $F \in K$. We say that $S$ is a fuzzy stationary winning strategy if he wins every play $(S(X), F_1, S(F_1), F_2, \ldots)$.

From definitions above, we get

2.2.7 Result A function $S: K \rightarrow K$ is a fuzzy stationary winning strategy if and only if it satisfies

(i) For each $F \in K$, $S(F) < F$

(ii) If $\{F_n: n \geq 1\}$ satisfies $S(X) \wedge F_i = 0$ and $S(F_n) \wedge F_{n+1} = 0$ for each $n \geq 1$ then

$$\text{Inf}_{n \geq 1} F_n = 0.$$  

2.2.8 Theorem Player I has a fuzzy winning strategy in $G^*(K, X)$ if and only if he has a fuzzy stationary winning strategy in it.

Proof is similar to that of Yajima [Y_1] and for completeness we are including it.

Proof:

Sufficiency part follows clearly. Conversely let $S$ be a fuzzy winning strategy of Player I for $G^*(K, X)$. Well order $K \setminus \{0\}$ by $\prec$. Let $H$ be any non empty closed fuzzy subset of $X$.

Claim-(1) Now we will prove that there is some $F(H) = (F_1, F_2, F_3, \ldots, F_m) \in (K)^m$ satisfying

(i) $S(F_0, F_1, \ldots, F_{i}) \wedge H = 0$ for $0 \leq i \leq m-1$.

(ii) $S(F_0, F_1, \ldots, F_{m}) \wedge H \neq 0$

(iii) $F_{i+1} = \text{Min}\{F \in K : H \leq F \leq F_i \text{ and } F \wedge S(F_1, F_2, \ldots F_i) = 0 \}$ for $0 \leq i \leq m-1$ where $F_0 = X$ and $F(H) = 0$ may occur.

To prove the above claim assume the contrary. Then we can inductively choose some $(F_1, F_2, \ldots) \in (K)^\omega$ such that $S(F_1, F_2, \ldots F_k) \wedge H = 0$ and $F_k = \text{Min}\{F \in K : H \leq F \leq F_{k-1} \text{ and } S(F_1, F_2, \ldots F_{k-1}) \wedge H = 0 \}$ for each $k \geq 1$. 


Now \((E_1,F_1,E_2,F_2,\ldots)\) where \(E_k = S(F_1,F_2,\ldots,F_{k-1})\) is a play for each \(k \geq 1\) for \(G^*(K,X)\) and by definition of fuzzy strategy, we have \(\inf_{k \geq 1} F_k = 0\). Also \(H \leq F_k\) for all \(k \geq 1\). Therefore \(H \leq \inf_{k \geq 1} F_k = 0\). This is a contradiction to \(H \neq 0\). Thus claim- (1) holds.

Take \(S^*(0) = 0\) and \(S^*(H) = S(F_1,F_2,\ldots,F_m) \land H\) where \(F(H) = (F_1,F_2,\ldots,F_m)\) for each \(H \in E \setminus \{0\}\). Then \(S^*\) is a function from \(E\) into \(E \cap K\) such that \(S^*(H) \leq H\) for each \(H \in E\). We will prove that \(S^*\) is a fuzzy stationary winning strategy for Player I in \(G^*(K,X)\).

Let \((E_1,H_1,E_2,H_2,\ldots)\) be a play such that \(E_i = S^*(X)\) and \(E_n = S^*(H_{n-1})\) for \(n \geq 2\). We show that \(\inf_{n \geq 1} H_n = 0\). For \(n \leq m\), take \(F(H) \land F_1,F_2,\ldots,F_n\) and \(|F(H)| = m\)

Claim-(2)

We will show that there are some \((F_1,F_2,\ldots) \in (E)^n\) and a sequence \(k(1) < k(2) < \ldots\) such that \(k \geq k(n)\) implies \((F_1,F_2,\ldots,F_n) = F(H_k) \land F_1,F_2,\ldots,F_n\) for each \(n \geq 1\).

Take \(F_0 = X\) and assume that \((F_1,F_2,\ldots,F_n) \in (E)^n\) and \{\(k(i) : i \leq n\}\} has been already chosen. First we will prove that \(|F(H_k)| > n\) for each \(k > k(n)\). Let \(k > k(n)\), then by induction we have \(F(H_k) \land F_1,F_2,\ldots,F_n\).

If \(S(F_0,F_2,\ldots,F_n) \land H_{k(n)} = 0\), then from \(H_k < H_{k(n)}\) it follows that \(S(F_0,F_2,\ldots,F_n) \land H_k = 0\). Otherwise if \(S(F_0,F_2,\ldots,F_n) \land H_k \neq 0\), by (ii) of Claim-(1) above we have \(F(H_{k(n)}) = (F_0,F_2,\ldots,F_n)\) so that \(S^*(H_{k(n)}) = S(F_0,F_2,\ldots,F_n) \land H_{k(n)}\).

Hence \(S(F_0,F_2,\ldots,F_n) \land H_k = S^*(H_{k(n)}) \land H_k\)

\[< E_{k(n)+1} \land H_{k(n)+1}\]

\[= 0\]

Thus in both cases \(S(F_0,F_2,\ldots,F_n)\) is disjoint from \(H_k\). By the choice of \(F(H_k)\) this means \(|F(H_k)| > n\).
Let $F_{n+1}(k)$ be the $(n+1)^{th}$ term of $F(H_k)$ for $k > k(n)$. This exists since we have already proved that $|F(H_k)| > n$. Now take $F_{n+1} = \min \{ F_{n+1}(k) : k > k(n) \}$. Choose some $k(n+1) > k(n)$ such that $F_{n+1} = F_{n+1}(k(n+1))$. Let $k > k(n+1)$.

Clearly $F_{n+1} \leq F_{n+1}(k)$. Also $F(H_k)/n = F(H_{k(n+1)})/n = (F_1, F_2, ..., F_n)$ and $H_k < H_{k(n+1)}$.

By (ii) of claim-(1) above we obtain $F_{n+1}(k) \leq F_{n+1}(k(n+1)) = F_{n+1}$. Hence $F_{n+1} = F_{n+1}(k)$ whenever $k \geq k(n+1)$. This means $(F_1, F_2, ..., F_{n+1}) = F(H_k)/n$ for each $k > k(n+1)$. Thus claim - (2) holds.

Now consider $(E_1, F_1, E_2, F_2, ..., E_n, F_n)$ such that $E_i = S(F_0, F_i, F_2, ..., F_{i-1})$ for $1 \leq i \leq n$ and $F_o = X$. This is an admissible sequence in $G^*(K, X)$. By the definition of fuzzy winning strategy we have $\inf_{n \geq 1} F_n = 0$. Also by claim-(2), each $F_n$ is in terms of some $F(H_k)$. Then from (ii) of claim- (1), it follows that $H_k < F_n$ for each $F_n$. Therefore we have $\inf_{n \geq 1} H_n \leq \inf_{n \geq 1} F_n$. But $\inf_{n \geq 1} F_n = 0$. Therefore it follows that $\inf_{n \geq 1} H_n = 0$. Thus $S^*$ is a fuzzy stationary winning strategy for Player I in $G^*(K, X)$.

2.2.9 Proposition Let $K_1$ and $K_2$ be two classes of fuzzy topological spaces with $K_1 \subseteq K_2$ and if Player I has a fuzzy winning strategy in $G^*(K_1, X)$, then he has a fuzzy winning strategy in $G^*(K_2, X)$.

Proof

From Theorem 2.2.8 it follows that Player I has a fuzzy stationary winning strategy in $G^*(K_1, X)$, say $S$. From theorem 2.2.8 it suffices to prove that Player I has a fuzzy stationary winning strategy in $G^*(K_2, X)$. Now $S: F^* \xrightarrow{\text{into}} F^* \cap K_1$. Then by Result 2.2.7 we have $S(F) < F$ where $F \in F^*$ where and if $\{F_n : n \geq N \} \subseteq F^*$ satisfies $S(X) \land F_1 = 0$ and $S(F_n) \land F_{n+1} = 0$ for all $n \geq 1$, then $\inf_{n \geq 1} F_n = 0$.

Now define $S^*: F^* \xrightarrow{\text{into}} F^* \cap K_2$ by $F \rightarrow S(F) \land K_2$. Now we will show that $S^*$ is a fuzzy winning strategy for $G^*(K_2, X)$. 

Now $S^*(F) = S(F) \land K_2$, 
\[ \leq S(F) \]
\[ \leq F \]

Therefore $S^*$ is a stationary strategy for Player I in $G^*(K_2, \lambda)$.

Now to prove that $S^*$ is winning, we want to prove that Player I wins every play of the form $(S^*(X), F_1, S^*(F_1), \ldots)$. For that we want to prove that $\inf_{n \geq 1} F_n = 0$. Now we have $S^*(X) \land F_1 = [S(X) \land K_2] \land F_1$
\[ = S(X) \land K_2 \land F_1 \]
\[ = 0 \] Since $S$ is a stationary winning strategy of Player I in $G^* (K_1, \lambda)$.

Also $S^*(F_n) \land F_{n+1} = S(F_n) \land K_2 \land F_{n+1}$
\[ = 0 \]

By Result 2.2.7 it follows that $\inf_{n \geq 1} F_n = 0$. Therefore $S^*$ is a fuzzy stationary winning strategy for Player I in $G^* (K_2, \lambda)$.

2.2.10 Proposition Let $Y$ be a fuzzy closed subspace of a fuzzy topological space $X$. If Player I has a fuzzy winning strategy in $G^* (K, \lambda)$, then he has a winning strategy in $G^* (K, Y)$.

Proof

Let $S: \mathcal{L} \xrightarrow{\text{int}} \mathcal{L} \land K$ be a fuzzy stationary winning strategy of $G^* (K, \lambda)$.

Now define $S^*: \mathcal{L} \xrightarrow{\text{int}} \mathcal{L} \land K$ by $F' \mapsto S(F) \land Y$ where $F' = F \land Y$ and $F \in \mathcal{L}$

Now $S^*(F') = S(F) \land Y$
\[ < F \land Y \]
\[ = F' \]

Thus $S^*$ is a fuzzy stationary strategy of Player I in $G^* (K, Y)$.

Let $\{ F_n': n \geq 1 \} \subset \mathcal{L}$ where $F_n' = F_n \land Y$ for some $F_n \in \mathcal{L}$

Now $S^*(Y) \land F_1' = [S(X) \land Y] \land F_1'$
\[ = [S(X) \land Y] \land [F_1 \land Y] \]
\[\begin{align*}
S(X) \land Y \land F_1 &= 0 \quad \text{since } S \text{ is winning} \\
S^*(F_{n^{'}}) \land F_{n+1^{'}} &= 0 \quad \text{also follows clearly. Therefore from Result 2.2.7, it follows that}
\end{align*}\]

\[\begin{align*}
\inf_{n \geq 1} F_n = 0. \quad \text{Therefore it follows that } \inf_{n \geq 1} F_n = 0. \quad \text{Thus proving } S^* \text{ is a fuzzy stationary winning strategy of Player I in } G^* (K, Y).
\end{align*}\]

### 2.3 Finite and Countable Unions

Clearly we have \(K \subseteq FK\) and \(X \in FK\) implies \(\ell K \subseteq FK\).

#### 2.3.1 Proposition

If Player I has a fuzzy winning strategy in \(G^* (FK, X)\), then he has a fuzzy winning strategy in \(G^* (K, X)\).

**Proof**

Let \(S\) be a fuzzy winning strategy for Player I in \(G^* (FK, X)\). We will try to define a fuzzy strategy \(t\) for \(G^* (K, X)\). Now take \(E_0 = X\), \(E_1 = S(E_0)\) and \(F_0 = E_0\). Now \(E_1 \in \ell K \cap FK\). Therefore \(E_1 = \bigvee \{H_{1,m} : m \leq k_1\}\) where \(\{H_{1,m} : m \leq k_1\} \subseteq \ell K \cap K\). We set \(F_1 = H_{1,0}\) and \(t(F_0) = F_1\). Also take \(F_2 \in \ell K\) in such a way that \(F_1 \land F_2 = 0\) and also set \(F_3 = F_2 \land H_{1,1}\) and \(t(F_0, F_1, F_2) = F_3\). Continuing like this we get an admissible sequence \((F_0, F_1, \ldots, F_{2k_1})\) for \(G^* (K, X)\). Take \(F_{2k_1+1} = t(F_0, F_1, \ldots, F_{2k_1}) = F_{2k_1} \land H_{1,k_1}\). Take \(F_{2k_1+2} \in \ell K\) with \(F_{2k_1+2} \leq F_{2k_1}\) and \(F_{2k_1+2} \land F_{2k_1+1} = 0\). Take \(E_2 = F_{2k_1+2}\). Now clearly \(E_1 \land E_2 = 0\) and set \(E_3 = S(E_0, E_1, E_2)\). Since \(E_3 \in \ell K \cap FK\), we have \(E_3 = \bigvee \{H_{3,m} : m \leq k_3\}\) where each \(H_{3,m} \in \ell K \cap K\).

Continuing like this we get the Play \((E_0, E_1, E_2, \ldots)\) of \(G^* (FK, X)\) and \((F_0, F_1, F_2, \ldots)\) of \(G^* (K, X)\). Since \(S\) is a fuzzy winning strategy for \(G^* (FK, X)\), \(\inf_{n \geq 1} E_{2n} = 0\).

Now \(\{E_{2n} : n \in \mathbb{N}\} \subseteq \{F_{2n} : n \in \mathbb{N}\}\). Therefore it follows that \(\inf_{n \geq 1} F_{2n} = 0\). Therefore \(t\) is a fuzzy winning strategy for Player I in \(G^* (K, X)\).
2.3.2 Remark From $K \subseteq FK$ and Proposition 2.2.9 it follows that if Player I has a fuzzy winning strategy in $G^*(K, X)$, then he has a fuzzy winning strategy in $G^*(FK, X)$.

From Remark 2.3.2 and Proposition 2.3.1 we get

2.3.3 Theorem Player I has a fuzzy winning strategy in $G^*(K, X)$ if and only if he has the same in $G^*(FK, X)$.

2.3.4 Proposition If a fuzzy topological space $X$ has a fuzzy closed countable $\alpha$-shading $\{X_n : n \in N\}$ such that Player I has a fuzzy winning strategy in $G^*(K, X_n)$ for each $n \in N$ then he has a fuzzy winning strategy in $G^*(K, X)$.

Proof

Let $S_n$ be a fuzzy stationery winning strategy for Player I in $G^*(K, X_n)$ for each $n \in N$. Now it is enough if we prove that Player I has a fuzzy winning strategy in $G^*(FK, X)$. Now we take $S(X)=S_I(X)$ and assume that $(E_i, F_1, F_2, \ldots, F_n, F_n)$ is an admissible sequence in $G^*(FK, X)$ such that $E_i = S(F_i, F_2, F_3, \ldots, F_n, F_n)$ for each $i \leq n$ where $F_0 = X$. Take $E_{n+1} = S(F_1, F_2, F_3, \ldots, F_n) = \sup_{k \leq n+1} S_k (F_n \land X_k)$

Consider the Play $(E_1, F_1, F_2, \ldots)$ in $G^*(FK, X)$ such that $E_n = S(F_1, F_2, F_3, \ldots, F_{n-1})$ for all $n \geq 1$. Now take an $m \geq 1$. By definition of Play we have $E_{n+1} \land F_{n+1} = 0$.-------- (1)

Here $E_{n+1} = \sup_{k \leq n+1} S_k (F_n \land X_k)$

$\geq S_m (F_n \land X_m)$

Also $F_{n+1} \land X_m \leq F_{n+1}$. Therefore from (1) it follows that

$[S_m (F_n \land X_m)] \land [F_{n+1} \land X_m] = 0$ for each $n \geq m$. Now since $S_m$ is a stationary winning strategy for Player I in $G^*(K, X_m)$, we have

$S_m (F_n \land X_m) \leq F_n \land X_m$ for each $n \geq m$. 
Therefore $[F_n \wedge X_m] \wedge [F_{n+1} \wedge X_m] = 0$ for each $n \geq m$. Thus $\bigwedge_{n \geq m} [F_n \wedge X_m] = 0$

We also have $F_{n+1} < F_n$ and hence it follows that $\inf_{n \geq 1} F_n = 0$. Thus Player I has a winning strategy in $G^*(FK, X)$, hence the proof is complete by Theorem 2.3.3.

**2.3.5 Theorem** Let $X$ be a fuzzy topological space with a fuzzy subset $E$ such that $E \in \mathcal{F}_c \cap E$. If Player I has a fuzzy winning strategy in $G^*(K, F)$ for each $F \in \mathcal{F}$ with $E \wedge F = 0$, then Player I has a fuzzy winning strategy in $G^*(K, X)$

**Proof**

For each $F \in \mathcal{F}$ with $E \wedge F = 0$, Let $S_F$ be a fuzzy stationary winning strategy for Player I in $G^*(K, F)$. Now we will find out a fuzzy winning strategy $S$ for Player I in $G^*(K, X)$

Define $S(X) = E$ and $(E_1, F_1, E_2, F_2, \ldots, E_n, F_n)$ be an admissible sequence in $G^*(K, X)$ such that $E_i = S(F_0, F_1, F_2, \ldots, F_{i-1})$ for each $i \leq n$ where $F_0 = X$. Take $E_{n+1} = S(F_0, F_1, F_2, \ldots, F_n) = S_{F_1}(F_n)$. Consider the play $(E_1, F_1, E_2, F_2, \ldots)$. Now clearly $E_{n+1} \wedge F_{n+1} = 0$. That is $S_{F_1}(F_n) \wedge F_{n+1} = 0$. Also $S_{F_1}(X) \wedge F_1 = E_1 \wedge F_1 = 0$ Since $S_{F_1}$ is a stationery winning strategy, it follows that $\inf_{n \geq 1} F_n = 0$. Thus Player I has a fuzzy winning strategy in $G^*(K, X)$.

**2.4 Games and Mappings**

**2.4.1 Theorem** Let $X$ and $Y$ be two fuzzy topological spaces and $K_1$ and $K_2$ be two classes of fuzzy topological spaces such that $X \in K_1$ and $Y \in K_2$. If $f$ is an $F$-continuous function from $X$ to $Y$ which maps all $E \in \mathcal{F} \cap K_1$ to $f(E) \in \mathcal{F} \cap K_2$ and if player $I$ has a fuzzy winning strategy in $G^*(K_1, X)$, then Player I has a fuzzy winning strategy in $G^*(K_2, Y)$. 
Proof

Let $S$ be a fuzzy stationary winning strategy for Player I in $G^*(K_1, X)$. Thus player I wins every play of the form $(S(X), F_1, S(F_1), \ldots)$. Now we will define a stationary winning strategy $t$ for Player I in $G^*(K_2, Y)$. Now consider the play $(t(Y), P_1, t(P_1), P_2, \ldots)$ where $P_n = t(F_n)$ and $t : I^Y \rightarrow I^Y \cap K_2$ is defined by $t(P_n) = f[S(F_n)]$. Now $t$ is a stationary winning strategy for $G^*(K_2, Y)$.

For, $t(F_n) = f[S(F_n)]$

$< f(F_n)$

$= P_n$. Therefore $t$ is a fuzzy stationary strategy.

Now $t(P_n) \wedge P_{n+1} = f[S(F_n)] \wedge f(F_{n+1})$

$= f[S(F_n) \wedge F_{n+1}]$

$= f(0)$

$= 0$

Also $t(Y) \wedge P_1 = f[S(X)] \wedge P_1$

$= f[S(X)] \wedge f(F_1)$

$= f[S(X) \wedge F_1]$

$= f(0)$

$= 0$

Therefore it follows from Result 2.2.7 that $\inf_{n \geq 1} F_n = 0$ and hence $t$ is a stationary winning strategy for Player I in $G^*(K_2, Y)$.

2.4.2 Theorem Let $f: X \rightarrow Y$ be an $F$-continuous $F$-closed mapping such that $f^{-1}(E) \in L \cap K_1$ whenever $E \in L \cap K_2$. Then if Player I has a fuzzy winning strategy in $G^*(K_2, Y)$, then Player I has a fuzzy winning strategy in $G^*(K_1, X)$.

Proof

Let $S$ be a fuzzy stationary winning strategy for Player I in $G^*(K_2, Y)$. Therefore Player I wins every play of the form $(S(Y), F_1, S(F_1), \ldots)$. Now we will define a function
$t: \mathcal{F} \longrightarrow \mathcal{F} \cap K_I$ as follows. Now $f: X \dashv \vdash Y$ is $F$-closed and hence we take $P_n = f^{-1}(F_n)$ where $P_n \in \mathcal{F}$ and $t(P_n) = f^{-1}[S(F_n)]$ for all $P_n \in \mathcal{F}$.

Now $t(P_n) = f^{-1}[S(F_n)]$

$= f^{-1}(F_n) \quad (= P_n)$. Thus $t$ is a fuzzy stationary strategy.

Now consider the play $(t(X), P_I, t(P_I), \ldots)$

$t(P_n) \wedge P_{n+1} = f^{-1}[S(F_n)] \wedge P_n$

$= f^{-1}[S(F_n)] \wedge f^{-1}(F_{n+1})$

$= f^{-1}[S(F_n) \wedge F_{n+1}]

= f^{-1}(0)

= 0$.

Also $t(X) \wedge P_I = f^{-1}[S(X)] \wedge P_I$

$= f^{-1}[S(X)] \wedge f^{-1}(F_I)$

$= f^{-1}[S(X) \wedge F_I]

= f^{-1}(0)

= 0$.

Therefore from Result 2.2.7 it follows that $\text{Inf } P_n = 0$ and hence $t$ is a winning strategy also. Thus $t$ is a fuzzy winning strategy for Player I in $G^*(K_I, X)$. This completes the proof.

As an immediate consequence of Theorem 2.4.1 and Theorem 2.4.2 we get the following two Theorems.

2.4.3 Theorem Let $X$ and $Y$ are two fuzzy topological spaces and $f: X \dashv \vdash Y$ be an $F$-continuous function and $f^{-1}(E) \in \mathcal{F} \cap K_I$ whenever $E \in \mathcal{F} \cap K_2$. If Player II has a fuzzy winning strategy in $G^*(K_I, X)$, then Player II has a fuzzy winning strategy in $G^*(K_2, Y)$.

2.4.4 Theorem Let $f: X \dashv \vdash Y$ be an $F$-continuous $F$-closed mapping such that $f^{-1}(E) \in \mathcal{F} \cap K_2$ whenever $E \in \mathcal{F} \cap K_1$. If Player II has a fuzzy winning strategy in $G^*(K_2, Y)$, then Player II has a fuzzy winning strategy in $G^*(K_1, X)$. 
2.4.5 Definition [M,B₂] Let \( 0 \leq \alpha < 1 \) (resp. \( 0 < \alpha \leq 1 \)). An \( F \)-closed \( F \)-continuous function \( f \) from a fuzzy topological space \( X \) to a fuzzy topological space \( Y \) is said to be \( \alpha \)-perfect (resp. \( \alpha^* \)-perfect) if and only if \( f^{-1}(y) \) is \( \alpha \)-compact (resp. \( \alpha^* \)-compact) for each \( y \in Y \).

2.4.6 Definition A class \( K \) of fuzzy topological spaces is said to be \( \alpha \)-perfect if \( X \in K \) is equivalent to \( Y \in K \), provided that there exists an \( \alpha \)-perfect map from \( X \) onto \( Y \).

From Theorems 2.4.1, 2.4.2, 2.4.3 and 2.4.4 next theorem follows immediately.

2.4.7 Theorem Let \( K \) be an \( \alpha \)-perfect class of fuzzy topological spaces and if there is an \( \alpha \)-perfect map from \( X \) on to \( Y \), Then

(i) If Player I has a fuzzy winning strategy in \( G^* (K, X) \), then he has the same in \( G^* (K_1, Y) \).

(ii) If Player II has a fuzzy winning strategy in \( G^* (K, X) \), then he has the same in \( G^* (K, Y) \).