4. STABILITY OF RANDOM EXTREMES

4.1. Introduction

Let us recall from Section 1.4, equations (1.4.4) and (1.4.5) that a r.v \( X \) with d.f \( F(x) \) having support \([0, \infty)\), is maximum stable w.r.t an independent r.v \( N \) with PGF \( Q(s) \) (\( F(x) \) is \( N \)-max stable) if

\[
Q(F(x)) = F(cx) \quad \text{for all } x \geq 0 \text{ and some } 0 < c < 1. \tag{4.1.1}
\]

Similarly \( F(x) \) is minimum stable w.r.t \( N \) (\( F(x) \) is \( N \)-min stable) if

\[
Q[F(cx)] = F(x) \quad \text{for all } x \geq 0 \text{ and some } c > 0. \tag{4.1.2}
\]

Here \( \overline{F}(x) = 1 - F(x) \). Also \( N \)-extremes refer to both \( N \)-max and/or \( N \)-min.

We had also noticed (p.20) that the following problems are worth studying. (i) Stability of \( N \)-extremes of exponential laws, (ii) Discussion of distributions other than geometric(1) or its generalizations for \( N \), (iii) Identifying \( N \) that imparts \( N \)-max and/or \( N \)-min stability for a given \( F(x) \), and (iv) Extending these notions to lattice laws, which means that we describe distributions of the same type in the lattice set up so that equations analogous to (4.1.1) and (4.1.2) hold. In this chapter we focus attention on these problems. Also for notational convenience we may use \( R \) instead of \( \overline{F} \).

Sreehari (1995) has considered the connection between the LT of \( N \) and the d.f \( F(x) \) (in fact a generalization of \( N \)-max stability). The method

\[\text{This chapter is based on Satheesh and Nair (2000a) and (2000b).}\]
Connecting the PGF of $N$ and the d.f $F(x)$, presented in the following sections, was done independently and motivated by our study of stability of random sums. It has the following advantages over the discussions in Bhaehari (1995). We are using PGFs consequent of which (i) we can use Lemma 2.2.1 to exploit the connection between the parameters of $N$ and $F(x)$ in a transparent manner and the range of the parameters can be found out and (ii) the discussion is directly relevant in the context of stability of extremes with random $(N)$ sample size.

In Section 4.2 we discuss the problems (i) and (ii) giving rise to a non-geometric(1) law for $N$ viz. the Sibuya($\nu$) law, and also characterize the general distribution that is max stable w.r.t Sibuya. Identifying $N$ in $N$-extreme stability is taken up in Section 4.3. Showing that we can suitably define lattice analogue of distributions of the same type we extend the notions of $N$-extreme stability to the lattice domain that is done in Section 4.4. The chapter ends with an example showing that in the case of lattice laws distributions of the same type in the contexts of $N$-sum and $N$-extremes are different. We end this section by defining the semi Pareto law of Pillai (1991) that has a main role in the discussions to follow.

Definition 4.1.1 A d.f $F(x)$, $x \geq 0$, is said to follow semi Pareto($\rho, \alpha$) law if,

$$F(x) = 1 - [1 + \psi(x)]^{-1}, \quad \psi(0) = 0$$

and $\psi(x)$ satisfies the functional equation (a variation of equation (1.5.1))

$$\psi(x) = \frac{1}{\rho} \psi(\rho^{1/\alpha} x)$$

for all $x > 0$ and some $0 < \rho < 1$, and $\alpha > 0$. (4.1.3)
4.2. Stability of Extremes - Continuous Case

We know that (Voorn (1987)) under N-max stability $c \in (0, 1)$. Since $\tilde{F}(x)$ is a decreasing function, under N-min stability also we have $c \in (0, 1)$. The value of $c$ will be discussed in the sequel. Now we have the following result concerning the stability of exponential law.

**Theorem 4.2.1** If $F(x) = 1 - e^{-x}$, $x > 0$, then it is N-max stable iff $N$ follows Sibuya($v$) distribution with PGF

$$Q(s) = 1 - (1-s)^v, \ 0 < v < 1,$$

and $c = v$. It is N-min stable iff $N$ is degenerate at $k$, an integer greater than unity and $c = 1/k$.

**Proof.** When $N$ follows Sibuya($v$) law,

$$Q[F(x)] = 1 - e^{vx} = F(vx).$$

Conversely,

$$1 - e^{vx} = 1 - [1 - (1 - e^{-v})]^v$$

shows that

$$Q(s) = 1 - (1 - s)^v$$

and hence $N$ is Sibuya with $v = c$.

For N-min stability we have,

$$F(cx) = e^{cv} \text{ and } e^{-x} = [e^{cx}]^{1/c},$$

proving that $N$ must be degenerate at $c^{-1} = k$, an integer greater than unity. Conversely, when $Q(s) = s^k, k = 1/c$, we have

$$Q(e^{cx}) = e^{-x}$$

and the proof is complete.
Theorem 4.2.1 suggests the conditions for N-max stability of exponential laws and also a distribution for N other than geometric(1, p). Next, we identify the most general form of $F(x)$ that is max stable w.r.t Sibuya($\nu$) and min stable w.r.t degenerate.

Theorem 4.2.2 A distribution is max stable w.r.t. Sibuya($\nu$) law iff it has semi Weibull($p, \alpha$) law with d.f.

$$F(x) = 1 - e^{-\psi(x)}, \ x \geq 0,$$

where $\psi(x)$ satisfies the functional equation (4.1.3). Further $p = \nu = c^\alpha$.

**Proof.** Setting $\psi(x) = -\ln \overline{F}(x)$, we have

$$Q[F(x)] = 1 - [1 - F(x)]^\nu = 1 - e^{-\nu \psi(x)}.$$

Under max stability w.r.t Sibuya($\nu$) we have

$$1 - e^{-\nu \psi(x)} = 1 - e^{-\nu \psi(c x)} \text{ or } \psi(x) = \frac{1}{\nu} \psi(c x) \text{ for all } x > 0.$$

Hence $F(x)$ has semi Weibull($p, \alpha$) law with $p = \nu = c^\alpha$.

Conversely, if $F(x)$ has semi Weibull($p, \alpha$) law and $N$ has a Sibuya($\nu$) law with $p = \nu = c^\alpha$, then

$$Q[F(x)] = 1 - e^{-\nu \psi(x)} = 1 - e^{-\nu \psi(c x)}.$$

This completes the proof. \( \square \)

**Note.** Notice that a Sibuya($\nu$)-sum of Bernoulli law results in $\nu$-Bernoulli law of example 3.2.3.
\[ F(x) = 1 - [1 + \psi(x)]^{1/k}, k \geq 0 \text{ integer.} \]

Under min stability of \( F(x) \) w.r.t Harris \((a,k)\) law we have

\[
\frac{[1 + \psi(cx)]^{-1/k}}{[a - (a - 1)/(1 + \psi(cx))]^{1/k}} = \frac{1}{[1 + \psi(x)]^{1/k}}.
\]

But L.H.S equals \([1 + a\psi(cx)]^{-1/k}\) and hence \( F(x) \) follows a generalized semi Pareto law with \( p = \frac{1}{a} = c^a \) and \( \beta = \frac{1}{k} \).

Conversely, with

\[ F(x) = 1 - [1 + \psi(x)]^\beta, \ x > 0 \text{ and } \beta > 0 \]

and \( N \) having a Harris \((a,k)\) law with \( a = 1/p \) and \( \beta = \frac{1}{k} \) we have

\[ Q[R(cx)] = R(x) \]

which completes the proof. \( \square \)

Notice that when \( \beta = 1 \) the generalized semi Pareto law becomes the semi Pareto law of Pillai (1991). Also notice that the extended geometric(1) law with parameters \( \lambda \) and \( k \) of Voorn (1987) is a reparametrization of Harris \((a,k)\) law by setting \( 1 - \lambda^k = 1/a \).

4.3. Identifying \( N \) that imparts stability for a given \( F(x) \)

Results (except Theorem 4.2.1) in the previous Section characterizes the distribution of \( F(x) \) for a given \( N \). The sufficiency part in these results state that certain \( F(x) \) is \( N \)-max (min) stable for the given \( N \). Here we look for the stronger assertion that for a certain \( F(x) \) this \( N \) is necessary as well. That is, characterizing the distribution of \( N \) for a given \( F(x) \). Here again we assume that \( F(x) \) is absolutely continuous so that \( F^{-1}(x) \) exists and is unique.
From (4.1.1) we have for all $x \geq 0$ and some $0 < c < 1$,

$$Q[F(x)] = F(cx) = F_c(x).$$

In particular, when $x = F^{-1}(s)$ for $0 < s < 1$, we have

$$F_c[F^{-1}(s)] = Q(s) \text{ under N-max stability.}$$  \hspace{1cm} (4.3.1)

Similarly, from (4.1.2) using $R$ for $F$, when $x = R_c^{-1}(s)$ for $0 < s < 1$,

$$R[R_c^{-1}(s)] = Q(s) \text{ under N-min stability.}$$  \hspace{1cm} (4.3.2)

(4.3.1) and (4.3.2) can be viewed as definitions of N-max and N-min stability of $F(x)$. We now reproduce Lemma 2.2.1 here (with a different proof) to identify the range of parameters in the distributions of $N$ and $F(x)$.

**Lemma 4.3.1** If $Q(s)$ is a PGF, then $Q(s^t)$ is a PGF iff $t > 0$ is an integer.

**Proof:** Let $N$ be the r.v with PGF $Q(s)$. Let $X$ be a r.v independent of $N$ such that $P\{X=k\} = 1$, $k$ being a positive integer so that its PGF is $P(s) = s^k$. If $X_1$, $X_2$, ..., are independent and identically distributed as $X$, then

$$S_N = X_1 + ... + X_N$$

has the PGF $Q(P(s)) = Q(s^k)$. Conversely, consider the function $Q(s^t)$, where $n < t < n+1$, $n=0, 1, 2, ...$. The first $(n+1)$ derivatives of $Q(s^t)$ are positive and the derivatives from $(n+2)$th onwards are alternatively positive and negative. Thus the function $Q(s^t)$ is not absolutely monotone when $t$ is not a positive integer and hence cannot be a PGF.

**Note.** The converses of the following results are identical to the sufficiency part of the results in the previous section.
Theorem 4.3.2 The semi Weibull($p, \alpha$) law is $N$-max stable iff $N$ follows Sibuya($p$) and $N$-min stable iff $N$ is degenerate at $\frac{1}{p} > 1$ integer. In both the cases $c = p^{1/\alpha}$.

Proof. We have,

$$F(x) = 1 - e^{-\psi(x)} = 1 - \exp\left(-\frac{1}{p} \psi(p^{1/\alpha} x)\right)$$

and

$$F^{-1}(s) = p^{-1/\alpha} \psi^{-1}\{\ln(1-s)^\alpha\}.$$

Hence

$$F_e \{F^{-1}(s)\} = 1 - (1-s)^\rho$$

when $c = p^{1/\alpha}$.

Therefore when $F(x)$ is semi Weibull($p, \alpha$) $N$ must have Sibuya($p$) distribution under $N$-max stability.

Now,

$$R_e^{-1}(s) = \frac{1}{c} \psi^{-1}\left[\log\left(\frac{s}{c}\right)\right]$$

so that

$$R[R_e^{-1}(s)] = s^{1/\rho}$$

when $c = p^{1/\alpha}$.

But $s^{1/\rho}$ is a PGF only when $\frac{1}{p} > 1$ is an integer. Therefore when $F(x)$ is semi Weibull($p, \alpha$) $N$ must have a degenerate distribution under $N$-min stability. Converses of both the statements are easy, hence the proof.

Theorem 4.3.3 The generalized semi Pareto($p, \alpha, \beta$) law of Theorem 4.2.4 is $N$-min stable iff $\beta = \frac{1}{k}$, $c = p^{1/\alpha}$ and $N$ follows Harris($\alpha, k$) law, with $\alpha = \frac{1}{p}$.

Proof. We have,
\[ R(x) = \left[ 1 + \frac{1}{p} \psi(p^{1/\alpha} x) \right]^{-\beta} \] and
\[ R_c^{-1}(s) = \frac{1}{c} \psi^{-1}(u), \quad \text{where} \quad u = \frac{1 - s^{1/\beta}}{s^{1/\beta}}. \]

When \( c = p^{1/\alpha} \) we have
\[ R[R_c^{-1}(s)] = \frac{s}{[a - (a-1)s^{1/\beta}]^\beta} \quad \text{where} \quad a = \frac{1}{p}. \]

This is a PGF only when \( \beta^{-1} = k \) a positive integer and hence \( N \) follows \( \text{Harris}(a,k) \), thus completing the proof, as the converse is easy.

**Corollary 4.3.4** Putting \( k = 1 \) we have: A semi Pareto(\( p, \alpha \)) law is \( N \)-min stable iff \( N \) has geometric(1,\( p \)) distribution, \( c = p^{1/\alpha} \).

Next we identify \( N \) under \( N \)-max stability of the extended log-logistic law of Voorn (1987) where he had shown that the extended log-logistic law is \( \text{max} \) stable w.r.t the extended geometric(1) law under the assumption that the sequence of distributions for \( N \) takes the value one with probability tending to one. However, in this discussion, we do not make any such assumptions on the distributions of \( N \) and we prove:

**Theorem 4.3.5** \( F(x) = [1 + x^{-\alpha}]^{-1/k}, \quad k \geq 1 \) integer and \( \alpha > 0 \) is \( N \)-max stable iff \( N \) follows \( \text{Harris}(c^{-\alpha},k) \).

**Proof.** We have
\[ F^{-1}(s) = [s^{-k} - 1]^{-1/\alpha} \] and
\[ F_c[F^{-1}(s)] = [1 + c^{-\alpha} (s^{-k} - 1)]^{-1/k} \]
\[ \frac{s}{[c^{-a} - (c^{-a} - 1)s^a]^{1/a}} \]

which is the PGF of Harris\((c^{-a},k)\) law. As the converse is easy we have proved the assertion. \(\square\)

4.4. Random Extreme Stability for Non-negative Lattice Distributions

As in the case of Chapter 3, the main requirement for extending the notions of stability of extremes to the lattice domain is to be able to conceive distributions of the same type in the context in a manner analogous to its continuous counterpart. Here we show that we can have scale families of lattice distributions that fit in to the scheme of things. Satheesh and Sandhya (1997) observed that the d.f of a mixture of geometric\((0)\) laws has the general form

\[ F(k) = \Pr\{X < k\} = 1 - m(k), k = 0, 1, \ldots, \quad (4.4.1) \]

where \(\{m(k)\}\) is the moment sequence of the mixing distribution. Further \(\{m(k)\}\) is also the sequence of realizations of a LT \(m(s), s > 0\), at the non-negative integral values of \(s\). Since \(m(\alpha s)\) also is a LT for a constant \(\alpha > 0\), we can define another d.f. by

\[ G(k) = \Pr\{Y < k\} = 1 - m(\alpha k), \quad k = 0, 1, \ldots. \quad (4.4.2) \]

Writing \(F_c(k) = 1 - m(ck)\), for \(c > 0\), we have:

\[ G(k) = F_{\alpha}(k) \text{ for all } k = 0, 1, \ldots. \quad (4.4.3) \]
Thus we have \( \{F_c(k) : c > 0\} \), a parametric family of d.fs (of non-negative lattice laws) and \( F(k) \) is a member of it. The existence of such a family of d.fs justifies the following definition.

**Definition 4.4.1** Two lattice laws \( F(k) \) and \( G(k) \) are of the same type if equation (4.4.3) is satisfied for some \( \alpha > 0 \).

Further, by analogy with the continuous case (equations (4.1.1) and (4.1.2)) we can now propose:

**Definition 4.4.2** Let \( F(k) \) be the d.f of a non-negative lattice r.v \( X \), and \( N \) a positive integer valued r.v. independent of \( X \) with PGF \( Q(s) \). Then \( F(k) \) is N-max stable if

\[
Q[F(k)] = F_c(k) \quad \text{for all} \quad k = 0, 1, 2, \ldots \quad \text{and some} \quad c \in (0, 1),
\]

and \( F(k) \) is N-min stable if

\[
Q[\overline{F}_c(k)] = \overline{F}(k) \quad \text{for all} \quad k = 0, 1, 2, \ldots \quad \text{and some} \quad c \in (0, 1),
\]

where \( \overline{F}(k) = P\{X \geq k\} \).

**Note.** Notice that Definition 4.4.1 appears different from Definition 3.3.2 but is quite similar to the one in the continuous case. We will clarify this point at the end of this chapter.

The following properties of mixtures of geometric(0) laws suggest their potential for applications in different contexts (see Satheesh and Sandhya (1997)). They are log-convex, compound geometric(1) (and hence GID), infinitely divisible and have decreasing hazard rate.
Sandhya and Satheesh (1996) observed that if \( F(x) \) is a mixture of exponential laws then
\[
F(x) = 1 - \phi(x), \quad x \geq 0 \tag{4.4.4}
\]
where \( \phi(x) \) is the LT of the mixing distribution. Notice the similarity of (4.4.4) with (4.4.1) the only difference being in the support of the distributions. Further in the definitions of semi Weibull, semi Pareto, and generalized semi Pareto laws the restriction of \( \alpha \) to \( 0 < \alpha \leq 1 \) make them mixtures of exponential laws. This is because under the restriction their survival functions are LTs (by virtue of the description of equation (1.5.1)). Accordingly we can conceive discrete analogs of semi Weibull, semi Pareto, and generalized semi Pareto laws as mixtures of geometric(0) laws. Now let us define discrete semi Pareto laws.

**Definition.4.4.3** A r.v \( X \) has discrete semi Pareto law (DSP(\( a, b, \alpha \))) if its d.f, in the support of \( \{0, 1, 2, \ldots\} \), has the form
\[
P\{X < k\} = F(k) = 1 - \frac{1}{1 + \psi(k)}, \quad k = 0, 1, 2, \ldots
\]
where \( \psi(k) \) satisfies \( \psi(k) = a \psi(bk) \) for all \( k = 0, 1, 2, \ldots \) and for some \( 0 < b < 1 < a \), satisfying \( ab^{\alpha} = 1 \) for a unique \( \alpha \in (0, 1] \) (the condition is same as that in equation (1.5.1) or (4.1.3)).

A r.v \( X \) in the same support has discrete Pareto distribution (DP(\( \lambda, \alpha \))) if its d.f is
\[
P\{X < k\} = F(k) = 1 - \frac{1}{1 + \lambda k^{\alpha}}, \quad \lambda > 0, \text{ and } 0 < \alpha \leq 1.
\]
The survival sequence of DSP\((a, b, \alpha)\) is the sequence of realizations of the LT of a semi Mittag-Leffler law introduced in Sandhya (1991) and that of DP\((\lambda, \alpha)\) corresponds to the realizations of the LT of a Mittag-Leffler law discussed in Pillai (1990). Here the construction conforms to that of a mixture of geometric(0) laws studied in Satheesh and Sandhya (1997). The mixing distribution in these cases are that of \(Y = \exp\{-X\}, X\) being semi Mittag-Leffler and Mittag-Leffler respectively.

**Theorem 4.4.1** A non-negative lattice distribution is geometric(1,\(p\))-max (min) stable iff it is DSP\((a, b, \alpha)\) with \(a = 1/p\) and \(b^\alpha = p\).

**Proof.** (i) For geometric(1,\(p\))-max stability of \(F(k)\) we should have, for all integer \(k \geq 0\),

\[
\frac{pF(k)}{1-qF(k)} = F_c(k), \text{ for some } c > 0
\]  

(4.4.5)

writing \(F(k) = 1 - \frac{1}{1 + \psi(k)}\), this is equivalent to

\[
\frac{p\psi(k)}{1 + p\psi(k)} = \frac{\psi(k)}{1 + \psi(ck)}, \text{ or } \psi(k) = \frac{1}{p} \psi(ck).
\]

This means that \(F(k)\) is DSP\((a, b, \alpha)\) with \(a = 1/p, b = c\) and \(0 < \alpha \leq 1\) is defined by \(b^\alpha = p\).

Conversely, suppose that \(F(k)\) is DSP\((a, b, \alpha)\) with \(a = 1/p, b^\alpha = p\). Then,

\[
\frac{p\psi(k)}{1 + p\psi(k)} = \frac{p\alpha\psi(bk)}{1 + p\alpha\psi(bk)} = \frac{\psi(bk)}{1 + \psi(bk)}
\]

and hence (4.4.5) is satisfied with \(c = b = p^{1/\alpha}\).
(ii) For geometric(1,p)-min stability we consider the requirement for all non-negative integer $k$,

$$\frac{p\bar{F}_c(k)}{1-q\bar{F}_c(k)} = \bar{F}(k), \text{ for some } c > 0$$  \hspace{1cm} (4.4.6)

writing

$$F(k) = 1 - \frac{1}{1 + \psi(k)},$$
we should have

$$\frac{p / [1 + \psi(ck)]}{1 - q / [1 + \psi(ck)]} = \frac{1}{1 + \psi(ck)}, \text{ or } \frac{1}{p} \psi(ck) = \psi(k).$$

Hence $F(k)$ is DSP$(a,b,\alpha)$ with $a = 1/p$, $b = c$ and $0 < \alpha \leq 1$ is defined by $b^\alpha = p$. Conversely, by retracing the steps we see that if $F(k)$ is DSP$(a,b,\alpha)$ with $a = 1/p$, $b^\alpha = p$, (4.4.6) is satisfied with $c = b$. This completes the proof.

**Corollary 4.4.2** The only non-negative lattice distribution that is geometric(1,p)-max (min) stable for two values of $p$, say $p_1$ and $p_2$ such that $\ln p_1 / \ln p_2$ is irrational is DP$(\lambda, \alpha)$.

**Proof.** By Theorem 4.4.1 we know that the distribution must be DSP$(a,b,\alpha)$. Further if $\psi(k) = a\psi(bk)$ for two different values of $a$ say $a_1$ and $a_2$ such that $\ln a_1 / \ln a_2$ is irrational, then by the description of equation (1.5.1) $\psi(k) = \lambda k^\alpha$ for some $\lambda > 0$ constant and the result follows.

We may conceive the definitions of semi Weibull and generalized semi Pareto laws in an analogous manner as is done in Definition 4.4.3 and
Accordingly the following results. Proofs being similar to those in Section 4.2, only statements of the theorems are presented.

**Theorem 4.4.3** If $F(k)$ is geometric($0,p$) then it is N-max stable iff $N$ has Sibuya($v$) distribution and $c = v$. It is N-min stable iff $N$ is degenerate at $k > 1$ integer and $c = 1/k$.

**Theorem 4.4.4** A non-negative lattice law is max stable w.r.t Sibuya($v$) iff it is discrete semi Weibull $(p,\alpha)$, $0 < \alpha < 1$, where $p = v = c^\alpha$.

**Theorem 4.4.5** A non-negative lattice law is min stable w.r.t a degenerate law at $k > 1$ integer, iff it is discrete semi Weibull $(p,\alpha)$, $0 < \alpha < 1$, where $p = \frac{1}{k} = c^\alpha$.

**Theorem 4.4.6** A non-negative lattice law is min stable w.r.t a Harris($a,k$) law iff it is discrete generalized semi Pareto$(p,\alpha,\beta)$ distribution, $0 < \alpha < 1$, where $p = \frac{1}{a} = c^\alpha$ and $\beta = 1/k$.

Having conceived the idea of distributions of the same type for lattice laws in the context of stability of N-extremes it is interesting to know whether this is equivalent to the Definition 3.3.2 of distributions of the same $D$-type (for lattice laws) in the context of stability of N-sums. Notice that in the case of continuous distributions both are equivalent though we are using the description in terms of d.fs for stability of N-extremes and that in terms of CFs (or LTs) for stability of N-sums.
Recall that two lattice laws $F(k)$ and $G(k)$ (with PGFs $Q_1(s)$ and $Q_2(s)$) are of the same type in terms of d.fs, if equation (4.4.3) is satisfied for some $a > 0$. They are of the same D-type if $Q_1(1-s) = Q_2(1-cs)$, for all $0<s<1$, or equivalently, $Q_1(u) = Q_2(1-c+cu)$ for all $0<u<1$, and some $0<c<1$. Now consider the following example.

**Example 4.4.1** Let $X$ has a geometric$(0,p)$ law. Then its d.f is

$$F(k) = 1 - q^k, \ k = 0,1,2, \ldots \text{ and } q = 1 - p$$

and PGF is

$$Q_X(s) = p/(1 -qs).$$

Now in accordance with Definition 4.4.1 consider the r.v $Y$ with d.f

$$G(k) = 1 - q^{ck} \text{ for some } 0<c<1.$$

Setting $q = \frac{1}{4}$ and $c = \frac{1}{2}$, we have $q^c = \sqrt[4]{\frac{1}{4}} = \frac{1}{2}$. Further;

$$Q_X(s) = 3/(4 - s) \text{ and } Q_Y(s) = 1/(2 - s) \text{ and }$$

$$Q_X(1 - \frac{1}{2} + s/2) = 6/(7 - s) \text{ and } Q_Y(\frac{1}{2} + s/2) = 2/(3 - s).$$

Thus neither $Q_X(s) = Q_Y(\frac{1}{2} + s/2)$ nor $Q_X(s) = Q_Y(\frac{1}{2} + s/2)$ considering both the possibilities. Hence the two definitions are not equivalent.

In the next chapter we will consider the uniqueness of geometric$(1)$ laws in the context of stability of extremes motivated by another look at the characterizations of semi Pareto laws by Pillai (1991) amd Pillai and Sandhya (1996). The discussion also has relevance in a parameterization scheme introduced by Marshall and Olkin.


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