CHAPTER III

ABSOLUTE CONVERGENCE OF FOURIER SERIES OF RESTRICTED \text{Lip}(\alpha, p)\ FUNCTIONS

In this chapter we have studied the absolute convergence of Fourier series of functions belonging to the class \text{Lip}(\alpha, p). It is easy to see that the conclusions of Bernstein's theorem 1.1 and Schaefer's theorem 1.6 remain valid if in their hypotheses the condition that \( f \in \text{Lip} \alpha \) is replaced by the weaker condition that \( f \in \text{Lip}(\alpha, 2) \).

In the year 1942, it was proved by Min-Teh Cheng\textsuperscript{1)} that if \( 0 < \alpha \leq 1, 1 < p \leq 2, h > 0 \) and

\[
\int_0^{2\pi} |f(x+h) - f(x)|^p dx = O\left( h^{-1} \right),
\]

then the series

\[
(1) \quad \sum_{n=1}^{\infty} \left( |a_n| + |b_n| \right) \log n
\]

converges for \( T < \alpha + p^{-1} \). Moreover the series (1) may not converge for \( T = \alpha + p^{-1} - 1 \).

In this chapter we have investigated some stronger

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1) Cheng\textsuperscript{[6]}
conditions to be imposed on the function $f$ so that the series (1) may converge for $T = \zeta + p^{-1}$. Moreover the proof given by us is simpler than that of Chong.

Our theorems are:

Given a periodic function $f \in L^p$, $1 < p \leq 2$, we define the mean modulus of continuity of the function $f$, by

$$
\omega_p^{(m)}(f, \delta) = \sup_{0 \leq t \leq \delta} \left( \frac{1}{2\pi} \int_0^{2\pi} | \Delta_m f(x,t)|^p \right)^{1/p}
$$

**Theorem 11.** If $0 < \alpha \leq 1$, $1 < p \leq 2$, $n > 0$, $\eta > 0$ and

$$
\omega_p^{(m)}(f, n) = O \left( \frac{1}{\eta} (\log h^{-1})^{-\frac{(4+\beta)}{p}} (\log \log h^{-1})^{-\frac{(4+\beta)}{p}} \right),
$$

then the series (1) converges for $T = \zeta + (1/p) - 1$.

**Theorem 12.** If $0 < \alpha \leq 1$, $1 < p \leq 2$, $n > 0$, $\eta > 0$ and

$$
\omega_p^{(m)}(f, n) = O \left( \frac{1}{\eta} (\log h^{-1})^{-\frac{(4+\beta)}{p}} (\log \log h^{-1})^{-\frac{(4+\beta)}{p}} \right),
$$

then the series

$$
\sum_{n=2}^{\infty} (|a_n|^\beta + |b_n|^\beta) \log^2 n
$$

covers for $\beta = \frac{p(\zeta + 1)}{(1 + \eta \rho)}$, where

$$
\delta = 1 + \frac{p(1-\zeta)}{\beta}.
$$

**Proof of Theorem 11.** Let $f(x) \sim \sum_{n=\infty}^{\infty} c_n \sin x$, ...
where the complex Fourier coefficients $a_n$ of $f$ are given by the relation

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

we obtain

$$\Delta_n(x,h) \sim \sum_{n=0}^{\infty} c_n(e^{inh} - e^{-inh}) e^{inx}.$$

By Hausdorff-Young inequality, we have

$$\left( \sum_{n=-\infty}^{\infty} |c_n(e^{inh} - e^{-inh})| \right)^{\frac{1}{p'}} \leq \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \Delta_n(x,h)^{1/p} \right)^{1/p},$$

where $(1/p) + (1/p') = 1$.

Therefore

$$2^{m+1} \sum_{n=-\infty}^{\infty} |c_n|^p |\sin n h| \leq \left( \omega^m_p(f,h) \right)^{p'},$$

$$2^{m'+1} \sum_{n=-\infty}^{\infty} |c_n|^p' |\sin nh| \leq \left( \omega^{m'}_{p'}(f,h) \right)^{p'}.$$

Choosing $h = (\pi/2\Pi)$ where $\Pi$ is a positive integer, we get

$$2^{m'+1} \sum_{n=-\infty}^{N} |c_n|^p' |\sin(n\pi/2\Pi)| \leq \left( \omega^{m'}_{p'}(f, \frac{\pi}{2\Pi}) \right)^{p'}.$$

Since

$$\sin(n\pi/2\Pi) \geq 1/\sqrt{2} \quad \text{for } (\Pi/2) \leq n \leq \Pi,$$

we get
\[ 2^{mp'+1\cdot (mp'/2)} \left| c_n \right|^p' \leq \left( \Omega_p^{(m)}(f, \frac{r}{2^p}) \right)^{p'} . \]

Taking \( N = 2^n \), where \( r \) is an integer greater or equal to \( r_0 \geq (\log 9)^{-2} + 3 \), we get from the last inequality

\[ \sum_{2^{n+1}}^{2^n} \left| c_n \right|^p' \leq \frac{1}{2^{m+p'/2}} \left( \Omega_p^{(m)}(f, \frac{r}{2^p}) \right)^{p'} . \]

Now applying Hölder’s inequality we get

\[ \sum_{2^{n+1}}^{2^n} \left| c_n \right|^p' \leq \left( \sum_{2^{n+1}}^{2^n} \left| c_n \right|^{p'} \right)^{1/p'} \left( \sum_{2^{n+1}}^{2^n} 1 \right)^{1-1/p'} \]

\[ \sum_{2^{n+1}}^{2^n} \left| c_n \right| \leq A \left( \Omega_p^{(m)}(f, \frac{r}{2^p}) \right) \cdot \varepsilon (r-1)(1-\frac{1}{p'}) . \]

Hence in virtue of the hypothesis of the theorem we get

\[ \sum_{2^{n+1}}^{2^n} \left| c_n \right| \leq A \left( \frac{r/2^p}{2\cdot 2^{\frac{p}{p}}} \right)^\frac{r}{2^p} \frac{2^p}{\left( \log(2^p/n) \right)^{p+1} \left( \log \log(2^p/n) \right)^{1+p}} . \]

Therefore

\[ \sum_{2^{n+1}}^{2^n} \left| c_n \right| \log^T n \leq \log^T(2^p) \sum_{2^{n+1}}^{2^n} \left| c_n \right| \]

\[ \leq \left( \frac{\log(2^p)}{\left( \log(2^p/n) \right)^{p+1} \left( \log \log(2^p/n) \right)^{1+p}} \right)^{1+\varepsilon} \]

\[ = 0 \left( \frac{1}{r \log n} \right)^{1+\varepsilon} , \text{ for } T = \alpha + p^{-1} - 1 . \]
Hence
\[ \sum_{n=2}^{\infty} |c_n| \log^T n = \sum_{n=2}^{\infty} \sum_{2^{n-1}}^{2^n} |c_n| \log^T n \]
\[ = O \left( \sum_{n=2}^{\infty} \frac{1}{r \log^{1+\varepsilon} r} \right) \]
\[ = O(1). \]

This proves theorem 11.

**Proof of Theorem 12.** Using Holder's inequality we obtain
\[ \sum_{2^{n-1}}^{2^n} |c_n|^\theta \leq \left( \sum_{2^{n-1}}^{2^n} |c_n|^p \right)^{\theta/p} \left( \sum_{2^{n-1}}^{2^n} 1 \right)^{1-\frac{\theta}{p'}}. \]

Hence it follows from (3) that
\[ \sum_{2^{n-1}}^{2^n} |c_n|^\theta \leq A \left( \omega_p(f, \frac{2^n}{2^{n-1}}) \right)^{\theta/p'} 2^{r(1-\frac{\theta}{p'})} \]
\[ = O \left( \frac{2^{r(1-\frac{\theta}{p'})}}{2^{r \delta \beta/p}} \right) \left( \frac{\log(2^n/\pi)^{\omega_p(f, \frac{2^n}{2^{n-1}})} \log \log(2^n/\pi)^{1+\varepsilon}}{\log(2^n/\pi)^{\omega_p(f, \frac{2^n}{2^{n-1}})} \log \log(2^n/\pi)^{1+\varepsilon}} \right) \]

Therefore
\[ \sum_{2^{n-1}}^{2^n} |c_n|^\theta \log^T n = O \left( \frac{2^{r(1-\frac{\theta}{p'})}}{2^{r \delta \beta/p}} \log^T 2^{r^2} \right) \]
\[ = O \left( \frac{2^{r(1-\frac{\theta}{p'})}}{2^{r \delta \beta/p}} \log^T 2^{r^2} \right) \]
\[ = O \left( \frac{1^{1+\varepsilon}}{r \log r} \right) \text{ for } \beta = \frac{(1+\varepsilon)p}{(1+\varepsilon)\beta} \]
\[ \text{and } \delta = 1+ \frac{p(1-\beta)}{\beta}. \]
Thus we conclude that
\[ \sum_{n=2}^{\infty} |c_n|^{\beta} \log n = O \left( \sum_{n=2}^{\infty} \frac{1}{r \log n} \right) \]
\[ < \infty. \]

This proves theorem 12.

**Remark.** Theorem 11 remains true even if
\[ \omega^{(m)}(f,h) = O \left( h^{\frac{1}{(\log_2 h)^{\alpha}} - 1} \cdots \left( \log_k h \right)^{-1} \left( \log_{k+1} h \right)^{-1 - \alpha(h)} \right), \]
where \( \log_1 h = \log h \) and \( \log_n h = \log \log_{n-1} h \).

Theorem 12 remains true if
\[ \omega^{(m)}(f,h) = O \left( h^{\frac{1}{(\log_2 h)^{\alpha}} - 1} \cdots \left( \log_k h \right)^{-1} \left( \log_{k+1} h \right)^{-1 - \alpha(h)} \right), \]

More general results than that of Min-Teh Cheng have been established by different authors. In particular O. Szász proved the following generalization of Theorem 1.13:

**Theorem D.** If \( f \in \text{Lip}(\alpha, p) \), \( 0 < \alpha < 1 \), \( 1 \leq p \leq 2 \),

\[ \text{then} \]
\[ \sum_{n=1}^{\infty} n^{\beta} \left( |a_n| + |b_n| \right) \]

converges for every \( \beta < \alpha - p^{-1} \).

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1) Szász [35]
The series (5) may not converge for \( \beta = \kappa - p^{-1} \).

We prove below a more general theorem than theorem of 0. Sand.

**Theorem 13.** If \( f \in L^p, 1 \leq p \leq 2 \), and

\[
\omega_p(n) = O \left( \frac{\frac{h^{\alpha_1}}{[l_1(h) l_2(h) \ldots l_k(h)]^{2+\varepsilon}}} {h^{2+\varepsilon} [l_1(h) \ldots l_k(h)]^{\frac{p-1}{2} p^p \kappa}} \right) \quad \text{as } h \to 0,
\]

where \( \varepsilon > 0 \), then the series (5) converges for

\[
\beta = \kappa - p^{-1}.
\]

**Theorem 13** can be extended as follows:

**Theorem 14.** Let \( 1 \leq p \leq 2 \), \( 0 < \rho_2 \leq p \leq p_2 \),

\[
l_1, n_2, n_3, p_1 + n_2 p_2 = p, \quad p_1 + p_2 = 1.
\]

If \( \omega_p(n) = O \left( \frac{h^{\alpha_1}}{[l_1(h) \ldots l_k(h)]^{\frac{p-1}{2} p^p \kappa}} \right) \)

and \( \omega_p(n) = O \left( \frac{h^{\alpha_2}}{[l_1(h) \ldots l_k(h)]^{\frac{p-1}{2} p^p \kappa}} \right) \)

then the series (5) is convergent for

\[
\beta = \frac{l_1 p_1 + l_2 p_2}{p} + \frac{l_3}{p}.
\]

**Proof of Theorem 13.** From (4) and using the hypothesis
of the theorem we get
\[
\sum_{n=1}^{\infty} |c_n| \leq B \frac{x(1-\frac{1}{r})}{l_1(n/2^m) l_2(n/2^n)} - x
\]
\[
= C \frac{2^x(1-\frac{1}{p})}{1+\varepsilon} \frac{(r-2)}{(r-2) \log (r-2)}
\]

Hence
\[
\sum_{n=1}^{\infty} n^\beta |c_n| \leq C \frac{2^x \beta (1-\frac{1}{p})}{1+\varepsilon} \frac{(r-2)}{(r-2) \log (r-2)}
\]

Therefore
\[
\sum_{n=1}^{\infty} n^\beta |c_n| = \sum_{n=1}^{\infty} \sum_{n=1}^{2^x} n^\beta |c_n| \leq C \frac{1}{(r-2) \log (r-2)}
\]

This proves Theorem 18 if we take into account that
\[
c_n = (1/2)(c_n - y_0) \text{ for } n > 0.
\]

Remark. For the convergence of the series (\textit{5}) it is enough that
\[ \int_0^1 h^{-y} \omega_p^{(m)}(h) \, dh = (1), \text{ where } Y = \beta + \gamma - 1. \]

**Proof.** It is a well-known fact that \( \omega_p^{(m)}(h) \) is a non-decreasing function of \( h \), and hence the above remark can easily be proved by following an analysis similar to the proof of Remark 1 in Chapter II.

**Proof of Theorem 14.** Using a result proved by Sanez in [36] and following an analysis similar to the proof of Theorem 13 we get by virtue of condition of the theorem

\[
\sum_{n=1}^{\infty} n^\beta |c_n| \leq C \sum_{n=1}^{\infty} \left( \frac{\Theta(\beta+\gamma-1)}{2 n^{\alpha_1,\alpha_2,\alpha_3}} \right)^{\beta} \left( \frac{1}{(r-2) \log^{1+\varepsilon}(r-2)} \right)
\]

\[
\leq C \sum_{n=1}^{\infty} \left( \frac{1}{2^{\alpha_1,\alpha_2,\alpha_3}} \right)^{\beta} \left( \frac{1}{(r-2) \log^{1+\varepsilon}(r-2)} \right)
\]

\[
\leq \infty, \text{ for } \beta = \frac{1}{p} \left( \int h \, \alpha_1 + \int h \, \alpha_2 \right) - \frac{1}{p}.
\]

If in the hypotheses of above theorem 13 and 14, we replace the expression

\[ L_1(h) L_2(h) \ldots L_r(h) \]

by \( L \), we get the following weaker forms of Theorem 13 and Theorem 14 respectively.
Theorem 15. If \( \omega_p^{(m)}(n) = O(n^{\alpha}) \), then the series (5) is convergent for \( \beta < \alpha - p^{-1} \).

Theorem 16. In Theorem 14 if \( l_1(n)l_2(n)\ldots l_{k+\varepsilon}(n) \) is replaced by 1, then the series (5) is convergent for \( \beta < \frac{1}{p} (p_1 \alpha_1 + p_2 \alpha_2) - \frac{1}{p} \).

It may be remarked that Theorem 15 and 16 are more general than Theorem 5 above.