CHAPTER - 5

FUZZY ABSOLUTE AS A SET OF F-OPEN ULTRA FILTERS®

5.0. Introduction

If X is a fuzzy topological space, then τ(X) - the set of all fuzzy open subsets of X forms a pseudo Boolean algebra with pseudo complement defined as, for μ ∈ τ(X), μᵀ = 1 - f-cl(μ) = f-cl(1 - μ). In the third chapter we have already introduced fuzzy open filters (f-open filters for short), f-open ultrafilters and fixed f-open ultrafilters.

It is known that in crisp topology, absolutes can be constructed using open ultrafilters on X. So in this chapter we are constructing fuzzy absolutes by taking τ(X), instead of the fuzzy regularly closed subsets of X. The absolute so constructed is denoted by f-E'X. Then the underlying set of f-E'X is the fixed f-open ultra filters on X. The second section of this chapter gives some properties of f-E'X. Though they are similar to that of f-EX in chapter 4, we are explicitly proving this to note the differences in the two cases.

5.1. Fuzzy absolute using f-open ultrafilters

5.1.1. Definition: Let X be a fuzzy topological space. Then the set of all convergent f-open ultra filters on X will be denoted by f-E'X. If x_p is a fuzzy point in X with support x then N(x_p) = {μ ∈ τ(X): p ≤ μ(x)}.  

® Some results of this chapter were communicated to Ganitha Sandesh, Rajasthan Ganitha Parishad.
5.1.2 Lemma:

Let $X$ be a fuzzy topological space and $x_p$ be a fuzzy singleton in $X$. Let $u \in f\text{-}E'X$. Then,

i) if $x_p \in a(u)$, $N(X_p) \subseteq u$.

ii) there is exactly one $x \in X$ such that $a(u)(x) = 1$.

Proof:

(i) Since $u \in f\text{-}E'X$ it is a fixed $f$-open ultrafilter. Therefore the result follows from 3.1.8.

ii) By the result 3.1.11, if $x_p \in a(u)$, then $x_p$ is a cluster point. But by lemma 3.1.12 $C(u)$, the set of all cluster points of $u$ contains exactly one point. Therefore, there exists exactly one $x \in X$ such that $x_p \in a(u)$, i.e. there exists exactly one $x \in X$ such that $a(u)(x) = 1$.

5.1.3 Remark: From the example 3.1.10 we can see that the converse of 5.1.2(i) need not be true in the case of non crisp sets. In the case of crisp sets corresponding to every $x \in X$, there exists a $u(x) \in E$ $X$ such that $a(u(x)) = \{x\}$. But because of the example 3.1.10 this is not true in the case of non crisp sets.

5.1.4 Definition: Let $X$ be a fuzzy topological space and let $f\text{-}\theta'X$ denote the set of all $f$-open ultrafilters on $X$. If $\mu \in \tau(X)$, let $O(\mu) = \{ u \in f\text{-}\theta'X : \mu \in u \}$. Then $f\text{-}E'X = \{ u \in f\text{-}\theta'X : a(u) \neq 0 \}$. 
5.1.5. Lemma: Let $\mu, \gamma \in \tau(X)$. Then,

(i) $O(\mu) = \phi$ if and only if $\mu = 0$.

(ii) $O(\mu \land \gamma) = O(\mu) \cap O(\gamma)$

(iii) $(O(\mu))^c \subset O(\mu^c)$.

(iv) $O(\mu) = f\cdot \theta'X$ if $\mu$ is dense in $X$ and in particular $O(1) = f\cdot \theta'X$

(v) \{O(\mu) : \mu \in \tau(X)\} is a base for a Hausdorff topology on $f\cdot \theta'X$

Proof:

i) Let $\mu = 0$. Since there is no ultrafilter contains zero, $O(\mu) = \phi$.

Conversely suppose $\mu \neq 0$. Then there exists $x \in X$ such that $\mu(x) \neq 0$. Let $F = \{\gamma \in \tau(X) : \gamma(x) > 0\}$. Then $F$ is an $f$-open ultrafilter containing $\mu$.

i.e. $F \in O(\mu) \therefore O(\mu) \neq \phi$.

i.e. $O(\mu) = \phi$ if and only if $\mu = 0$.

ii) Let $U \in O(\mu \land \gamma)$

$U \in O(\mu \land \gamma) \iff \mu \land \gamma \in U$

$\iff \mu \in U$ and $\gamma \in U$ by definition of $f$-open ultrafilter.

$\iff U \in O(\mu)$ and $U \in O(\gamma)$

$\iff U \in O(\mu) \cap O(\gamma)$

$\therefore O(\mu \land \gamma) = O(\mu) \cap O(\gamma)$

iii) Let $U \in (O(\mu))^c$

Then $U \notin O(\mu)$. i.e. $\mu \notin U$.

Therefore by note 3.1.5 $\mu^c \in U$ and so $U \in O(\mu^c)$

$\therefore (O(\mu))^c \subset O(\mu^c)$.
The reverse inclusion does not hold here. The example 1.2.7 will prove this since the set of all constant functions from $X$ to $[0,1]$ forms a fuzzy topology on $X$.

iv) If $\mu \in \tau(X)$, then $1 - \bar{\mu} = 0$ if and only if $\mu$ is dense in $X$.

Therefore, $O(\mu^c) = \phi$ if and only if $\mu$ is dense in $X$.

But $(O(\mu))^c \subset O(\mu^c)$. Therefore, $O(\mu)^c = \phi$ if $\mu$ is dense in $X$.

That is $O(\mu) = f- \emptyset^X$ if $\mu$ is dense in $X$.

So in particular $O(1) = f- \emptyset^X$

v) By i), ii) and iv), the set $\{O(\mu) : \mu \in \tau(X)\}$ forms a basis for $f- \emptyset^X$.

Let $U$ and $V$ be two distinct $f$-open ultrafilters. Then $U \not\subset V$.

Therefore there exists $\mu \in U$ such that $\mu \not\in V$.

Since $\mu \not\in V$, by theorem 3.1.4 there exists $\gamma \in V$, such that $\mu \wedge \gamma = 0$.

Therefore $U \in O(\mu)$ and $V \in O(\gamma)$ such that $O(\mu) \cap O(\gamma) = \phi$.

$\therefore$ $f- \emptyset^X$ is Hausdorff.

Note: By the above lemma, $f- \emptyset^X$ can be regarded as a topological space with $\{O(\mu) : \mu \in \tau(X)\}$ is a basis for open sets.

5.1.6. Definition:

Let $X$ be a fuzzy topological space.
The set $f\theta'X$ is topologised by giving the subspace topology inherited from the space $f\theta'X$. That is $\{O(\mu) \cap f\theta'X : \mu \in \tau(X)\}$ is an open base for the topology of $f\theta'X$. This topological space $f\theta'X$ is called the fuzzy absolute of $X$.

**Note:** To each $u \in f\theta'X$, there is a unique point $x$ in $X$ such that $a(u)(x) = 1$. This defines a function from $f\theta'X$ into $X$, which will be denoted by $K_{f\theta'X}$, i.e. $K_{f\theta'X}(u) = x$.

**5.1.7 Result:** For $\mu \in \tau(X)$, $O(\mu)$ is clopen in $f\theta'X$

**Proof:**

$O(\mu) = \{ u \in f\theta'X : \mu \in u \}$

$(O(\mu))^c = \{ v \in f\theta'X : \mu \notin v \}$

When $\mu \notin v$ by theorem 3.1.4 there exists a $\gamma \in v$ such that $\mu \land \gamma = 0$. For every $\mu \notin v$ there is some $\gamma$ satisfying this condition.

Therefore $(O(\mu))^c = \cup_i (O(\gamma_i))$, where $\mu \land \gamma_i = 0$.

$O(\gamma_i)$ is open for every $i$ and so is $\cup O(\gamma_i)$.

$\therefore O(\mu)$ is closed.

i.e. $O(\mu)$ is clopen in $f\theta'X$.

Therefore $\{O(\mu) : \mu \in \tau(X)\} \subset B(f\theta'X)$ where $B(f\theta'X)$ is the set of all clopen sets in $f\theta'X$. 
5.1.8. Theorem: Let $X$ be a fuzzy topological space. Then $f$-$\theta'X$ is a B-extremally disconnected, zero dimensional, compact space.

Proof: Let $U$ be any open set in $f$-$\theta'X$. Then there exists a finite subset $D$ of $\tau(X)$ such that $U = \cup \{O(\mu_i) : \mu_i \in D\}$

Since $\tau(X)$ is complete, $\forall \mu_i$ exist.

Let $\forall \mu_i = \gamma$. Then $\gamma \in D$

$\mu_i \leq \gamma$

$\therefore O(\mu_i) \subseteq O(\gamma)$

Therefore $\cup O(\mu_i) \subseteq O(\gamma)$

$O(\gamma)$ is clopen by 5.1.7.

$\therefore \text{cl} (\cup O(\mu_i)) \subseteq O(\gamma)$

That is $\text{cl} (U) \subseteq O(\gamma)$. (1)

Therefore $f$-$\theta'X$ is B-extremally disconnected.

By 5.1.7, we have $O(\mu)$ is clopen in $f$-$\theta'X$. Also $\{O(\mu) : \mu \in \tau(X)\}$ forms an open base for $f$-$\theta'X$. Therefore by definition 1.3.1, $f$-$\theta'X$ is zero dimensional.

To show that $f$-$\theta'X$ is compact.

Let $\mathcal{F}$ be a filter of closed sets in $f$-$\theta'X$ and

$G = \{\mu \in \tau(X) : O(\mu) \subseteq F \text{ for some } F \in \mathcal{F}\}$. 
\{O(\mu) : \mu \in \tau(X)\} \text{ is a base for closed sets in } f\text{-}\theta'X.

Therefore \( \cap F = \cap \{O(\mu) : \mu \in G\} \).

Then \( G \) is an \( f \)-open filter. So there exists an \( f \)-open ultrafilter say \( u \) such that \( G \subseteq u \).

Therefore \( \mu \in u \) for every \( \mu \in G \).

That is, \( u \in O(\mu) \) for all \( \mu \in G \).

Therefore, \( u \in \cap \{O(\mu) : \mu \in G\} \).

\( \therefore u \in \cap F \). Hence \( \cap F \neq \emptyset \).

\( \therefore f\text{-}\theta'X \) is compact.

5.1.9. Note: As in the case of \( f\text{-}\text{EX} \), the reverse inclusion of (1) in the above theorem does not hold. As in example 1.4.4 we can prove this.

Example: Let \( X \) be any non empty set.

Define \( \mu_1 : X \to [0,1] \) as \( \mu_1(0) = \frac{1}{3} \) and \( \mu_1(x) = 0 \), \( \forall x \neq 0 \).

Consider the fuzzy topology \( \tau \) on \( X \) generated by the set of all constant functions and \( \mu_1 \).

Let \( f\text{-}\theta'X \) be the collection of all \( f \)-open ultrafilters on \( X \). Then the base for open sets in \( f\text{-}\theta'X \) is \( \{O(\gamma) : \gamma \in \tau(X)\} \).

Consider the open set \( U = O(\mu_1) = \{u \in f\text{-}\theta'X : \mu_1 \in u\} \).
Then the $f$-open ultrafilter $U[0]=\{\mu \in \tau(X) : \mu(0)>0\}$ will contain $\mu_1$.

Let $\mu_2 : X \to [0,1]$ be defined as $\mu_2(x) = \frac{1}{3}, \quad \forall x \in \mathbb{R}$

Then $\mu_1 \leq \mu_2$.

Therefore $O(\mu_1) \subseteq O(\mu_2)$

That is $U \subseteq O(\mu_2)$

i.e. $\text{cl}(U) \subseteq O(\mu_2)$

Let $V[x] = \{\mu \in \tau(X) : \mu(x)>0, x \neq 0\}$.

Then $V[x]$ is an $f$-open ultrafilter containing $\mu_2$ but not $\mu_1$.

i.e. $V[x] \in O(\mu_2)$ and $V[x] \not\in O(\mu_1)$.

\[ \therefore O(\mu_2) \not\subseteq \text{cl}(U). \]

5.1.10. Result:

$B(f^-\tau'X) = \{O(\mu) : \mu \in \tau(X)\}$

**Proof:** By the result 5.1.7, we have $\{O(\mu) : \mu \in \tau(X)\} \subseteq B(f^-\tau'X)$ \hspace{1cm} (1)

Now let $C \in B(f^-\tau'X)$. Then $C$ is both open and closed. Since $C$ is open there exists a subset $D$ of $\tau(X)$ such that $C = \cup\{O(\mu) : \mu \in D\}$. $C$ is also closed. Therefore $C$ is compact. Therefore there exists a finite subset $H$ of $D$ such that

$C = \cup\{O(\mu) : \mu \in H\}$

$\subseteq \{O(\vee\mu_i) : \mu \in H \}$

$\subseteq \{O(\mu_i) : \mu \in \tau(X)\}$. 


Therefore, \( C \in \{ O(\mu_i); \mu_i \in \tau(X) \} \).

\[
B(f-\theta'(X)) \subset \{ O(\mu); \mu \in \tau(X) \}
\]

Hence \( B(f-\theta'X) = \{ O(\mu); \mu \in \tau(X) \} \).

**5.2. Properties of the pair \((f-E'X, K_{f-x'})\)**

**5.2.1 Definition:** A fuzzy singleton in \( X \) which is also \( f \)-open is called an \( f \)-open point in \( X \).

**5.2.2 Theorem:** \( f-E'X \) is a dense \( B \)-extremally disconnected zero dimensional subspace of \( f-\theta'X \).

**Proof:** Being a subspace of \( f-\theta'X \) which is zero dimensional \( f-E'X \) is also zero dimensional.

To show that \( f-E'X \) is dense in \( f-\theta'X \).

Let \( \mu \in \tau(X) \) such that \( \mu \neq 0 \). Choose an \( f \)-open point \( p \) in \( X \) such that \( p(x) = 1 \).

Let \( F = \{ \gamma \in \tau(X) : \gamma(x) = 1 \} \).

Then \( F \) is an \( f \)-open filter and so there is an \( f \)-open ultrafilter say \( U \) such that \( F \subset U \).

\( \mu \in F \), therefore \( \mu \in U \). i.e. \( U \in O(\mu) \).

If \( a(U) = 0 \), then there is \( \eta \in U \) such that \( \bar{\eta}(x) = 0 \). i.e. \( \eta(x) = 0 \)

\[ :. p \land \eta = 0, \text{ which is not true since } p \in U. \]

\[ :. a(U) \neq 0. \text{ i.e. } U \in f-E'X. \]
Therefore, \( U \in O(\mu) \cap f-E'X \).

i.e. for \( \mu \in \tau(X) \), \( f-E'X \cap O(\mu) \neq \emptyset \). Therefore \( f-E'X \) is dense in \( f-\theta'X \).

By result 1.4.2, any dense subset of a B-extremally disconnected space is B-extremally disconnected. \( f-E'X \) is a dense subspace of \( f-\theta'X \) which is B-extremally disconnected. Therefore \( f-E'X \) is B-extremally disconnected.

**5.2.3 Theorem:** For \( \mu \in \tau(X) \), \( K_{f,x}(O(\mu) \cap f-E'X) = \{p_i\} \) where \( \{p_i\} \) is subordinate to \( \bar{\mu} \).

**Proof:** Let \( U \in (O(\mu) \cap f-E'X) \) and \( K_{f,x}(U) = p \) where \( p(x_0) = 1 \) and \( p(x) = 0 \) for every \( x \neq x_0 \).

Then, \( U \in O(\mu) \) and \( U \in f-E'X \)
i.e. \( \mu \in U \) and \( a(U) \neq 0 \).

Since \( K_{f,x}(U) = p \), \( a(U)(x_0) = 1 \)
i.e. \( \gamma(x_0) = 1 \) for every \( \gamma \in U \).

\[ \therefore \bar{\mu}(x_0) = 1 \], since \( \mu \in U \).
i.e. \( p \leq \bar{\mu} \).

Therefore, \( K_{f,x}(O(\mu) \cap f-E'X) = \{p_i\} \) where \( p_i \leq \bar{\mu} \) for every \( i \).

Therefore by definition 3.3.1 \( \{p_i\} \) is subordinate to \( \bar{\mu} \).
5.2.4 Theorem: Let \( p \) be an f-open point in \( X \) with support \( x \) and \( \mu \in \tau(X) \). If \( p \in f\text{-int } (f\text{-cl}(\mu)) \), then \( K_{f,x}(p) \subseteq O(\mu) \).

Proof: Let \( p \in f\text{-int } (\mu) \). i.e. \( f\text{-int } \mu(x) = 1 \). Therefore, \( \mu(x) = 1 \).

Let \( U \in K_{f,x}(p) \). Then \( K_{f,x}(U) = p \)

Therefore, \( a(U)(x) = 1 \).

i.e. \( \gamma(x) = 1 \) for every \( \gamma \in U \).

\( \mu(x) = 1 \). Therefore, \( 1 - \mu(x) \neq 1 \).

i.e. \( 1 - \mu \notin U \). That is \( \mu \notin U \). Therefore, \( \mu \in U \).

That is \( U \in O(\mu) \).

\( \therefore K_{f,x}(p) \subseteq O(\mu) \).

Note: The converse of this theorem is not true in the case of non crisp sets. By an example similar to that of 4.3.4, we can prove this.

5.2.5 Theorem: The function \( K_{f,x} \) from \( f\text{-E}'X \) into \( X \) is s-continuous and compact.

Proof: Similar to that of 4.3.7.

5.2.6 Note: The mapping \( K_{f,x} \) is not a closed map. An example similar to that of 4.3.8 will serve the purpose.

5.2.7 Definition: An f-open ultrafilter \( U \) on \( X \) is said to be strong fixed (s-fixed) if \( a(U) = 0 \) and \( \bigcap \{ \sigma_0(\mu) : \mu \in U \} \neq \emptyset \).
5.2.8 Remark:

As in the case of s-fixed ultrafilters, if \( \cap \{ \sigma_0(\mu) : \mu \in U \} \neq \emptyset \), then it reduces to be a singleton. Therefore the only s-fixed f-open ultrafilters are of the form \( F [x] = \{ \mu \in \tau(X) : 0 \leq \mu(x) \} \). They are called principal f-open ultrafilters.

So as in section 4.4 we can construct the absolutes using the principal f-open ultrafilters. The results proved in section 4.4 hold in this case also.

In such construction corresponding to every \( x \in X \), we can have \( U [x] \) belonging to f-E'X such that \( \cap \{ \sigma_0(\mu) : \mu \in U \} = \{x\} \). Therefore the mapping \( K_{f,x} \) becomes a surjection from f-E'X onto X.