0.1 Extensions and Absolutes

An extension of a topological space $X$ is a space that contains $X$ as a dense subspace. The different kinds of extensions especially compactification formed a major area of study in topology. As per one method of construction, the points of the extensions are ultrafilters on lattices such as lattices of open sets, lattices of zero sets, and lattices of clopen sets. From the introduction of Stone Čech compactification in the thirties this area caught the attention of researchers and many papers appeared there after in this area.

Related dual concepts of Absolutes due to Iliadis and Banaschewski also formed a major area of activity in Topology. Associated with each Hausdorff space always there exists an extremally disconnected zero dimensional Hausdorff space $EX$ called the Iliadis absolute of $X$ and a perfect irreducible $Θ$ continuous surjection from $EX$ on to $X$. In most of the constructions of absolutes the points of $EX$ are ultrafilters on lattices associated with $X$. Thus extensions and absolutes although conceptually dual in nature are constructed using similar tools.

Perfect continuous mappings always preserve certain topological properties. Therefore, whenever dealing with a new topological property $P$ and if $X$ has $P$ we have to look for a space which is a perfect continuous image of $X$ or that can be mapped on to $X$ by a perfect continuous surjection. If it is
possible to find such a space we can study the properties of that space and from that we can infer the properties of $X$.

If $X$ is regular, always there exists such a space called Iliadis absolute of $X$. It can be mapped on to $X$ by a perfect continuous surjection $K_X$ which is irreducible, i.e. if $A$ is a proper closed subset of $EX$ then $K_X(A)$ is a proper closed subset of $X$. The space $EX$ is zero dimensional and also extremally disconnected. That is every point of $EX$ has a clopen basis and closure of its open sets are always open.

Even though $X$ is not regular, we can have an extremally disconnected space. This is called Banaschewski absolute of $X$ denoted as $PX$. Here also there is a perfect irreducible surjection $\pi_X: PX \to X$. Since $X$ is not regular $PX$ is not regular and so not zero dimensional. When $X$ is regular $PX$ and $EX$ coincide and so does $K_X$ and $\pi_X$.

Since perfect continuous mapping preserves some topological properties, $PX$ and $X$ have certain properties in common. More over there are certain properties which are preserved by the irreducibility of $\pi_X$. Examples of such properties are cellularity, $\pi$ weight, density character and feeble compatness $\{P;W\}$.

The absolute $EX$ of $X$ arises in the following situation. Suppose with each space $X$ we can associate some algebraic object $A(X)$. Let $A(\hat{X})$ be the
algebraic completion of $A(X)$. If $X$ has certain nice properties, then $A(\hat{X})$ is isomorphic to $A(EX)$.

The original motivation behind the study of absolutes was the problem of characterizing the projective objects in the category of compact spaces and continuous functions. In 1958 Gleason [GL] solved this problem. He showed that the projective objects of this category are precisely the compact extremally disconnected spaces. He constructed $EX$ (when $X$ is compact) as a part of the solution of this problem.

0.2 Fuzzy set theory

The concept of Fuzzy sets introduced by the American Cyberneticist L.A. Zadeh started a revolution in every branch of knowledge and in particular in every branch of mathematics. Zadeh introduced the fuzzy set theory [ZA] in 1965 inorder to study the control problem of complicated systems and dealings with fuzzy information. This theory described Fuzziness mathematically for the first time. Fuzziness is a kind of uncertainty and uncertainty of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol belongs.

Since the 16th century probability theory has been studying a kind of uncertainty – randomness – i.e. the uncertainty of the occurrence of an event. But in this case the event itself is completely certain, the only uncertain thing is whether the event will occur or not. Fuzziness is another kind of
theory of fuzzy topology. Using fuzzy sets introduced by Zadeh, C.L. Chang [CH] defined fuzzy topological space in 1968 for the first time. In 1976 Lowen [LO] suggested a variant of this definition. Since then an extensive work on fuzzy topological space has been carried out by many researchers.

Many mathematicians while developing fuzzy topology have used different lattices for the membership sets like (1) completely distributive lattice with 0 and 1 by T.E. Gantner, R.C. Steinlage and R.H. Warren [G;S;W] (2) complete and completely distributive lattice equipped with order reversing involution by Bruce Hutton and Ivan Reilly [H;R] (3) complete and completely distributive non atomic Boolean algebra by Mira Sarkar [SA] (4) complete chain by Robert Bernard [BE] and F. Conard [CO] (5) complete Brouwerian lattice with its dual also Brouwerian by Ulrich Hohle [HO] (6) Complete and distributive lattice by S.E. Rodabaugh [RO] (7) complete Boolean algebra by Ulrich Hohle [HO].

We take the definition of fuzzy topology in the line of Chang with membership set as the closed unit interval [0,1].

0.4 About this thesis

The theory of extensions has got a rich parallel theory in fuzzy topological spaces. Mathematicians like Cerutti, U [CE], Liu Ying Ming and Mao-Kang Luo [Y,M] have done a detailed study in this area. But not much work has been done regarding the theory of absolutes. In our work we are
investigating this-fuzzy absolutes. In the development of a parallel theory based on fuzzy sets, here we can specifically notice many differences between the two theories.

As a prelude to construction of fuzzy absolutes we have done a detailed study of the proper analogue in the fuzzy context for the concepts like Stone spaces, regularly closed filters and open filters. The concepts of fuzzy filters introduced by A.K. Katsaras [KA], and fuzzy regular closed sets by K.K. Azad [AZ] have been used for this purpose.

0.5 Summary of the thesis

The thesis is divided into six chapters including this chapter 0.

The general preliminary results which are used in the succeeding chapters are given in the next section of this chapter. Due references are given wherever necessary. Some of the preliminary results which are related to each chapter are given at the beginning of the corresponding chapter itself.

The Stone duality theory was developed by Marshal H. Stone [ST] in 1937. Using the notion of ultrafilters he introduced the Stone space of a Boolean algebra and proved the Stone representation theorem.

In the first chapter we are doing the fuzzy analogue of this concept. Here we define a function $\Lambda: I^* \to P(\Omega(X))$ where $\Omega(X)$ is the set of all fuzzy ultrafilters on $X$ and prove some of its properties. Using this we have
introduced the Stone space of fuzzy sets denoted as f-S(X) and have proved the "Fuzzy Stone representation theorem". With suitable examples we point out the differences with the crisp situation.

In Chapter 2 we have introduced the concept of fuzzy regularly closed filters (FRC-filter) similar to the notion of fuzzy filters introduced by A.K. Katsaras [KA]. Then FRC ultrafilters are studied and an equivalent formulation for such filters have been given. M.A. De Prade Vicente in her paper [P;A] proved that every fuzzy ultrafilter is free. With suitable examples here we prove the existence of fixed FRC ultrafilters. Here we have also defined the s-fixed FRC ultrafilters in the same line as that by De Prade [P;A].

It is known that in the crisp situation the absolutes can be constructed using open ultrafilters also. So in the third chapter we introduce fuzzy open filters; and prove some of its properties. Also we define the fuzzy Hausdorff-closed spaces (f-H closed space) analogues to the concept of H-closed spaces. A characterization for f-H closed space has been given. Then we introduce an s-continuous mapping from a topological space to a fuzzy topological space and prove that the image of an H-closed space under an s-continuous mapping is f-H closed. Here we have also proved that the arbitrary product $\prod f_i$ and the sum $\Theta f_i$ of the s-continuous maps $f_i$ are also s-continuous.

Using the concepts in chapter 1 and chapter 2 in the fourth chapter we have introduced the fuzzy absolutes of a fuzzy topological space as a
subspace of fuzzy Gleason space which is the Stone space of \( FRC(X) \). Thus associated with each fuzzy Hausdorff space there is a pair \((f-EX, K_{f,X})\) consisting of a B- extremally disconnected zero dimensional space \( f-EX \) and an s-continuous mapping \( K_{f,X} \) which is compact (but not closed and hence not irreducible) from \( f-EX \) into \( X \). Significant properties of the pair \((f-EX, K_{f,X})\) are given and have proved the uniqueness of the fuzzy absolutes. Fuzzy absolutes of sums and products of fuzzy topological spaces are also studied.

Fuzzy absolutes also can be constructed using the strong fixed (s-fixed) \( FRC \) ultrafilters. Properties of fuzzy absolutes in this situation is studied and identifies the differences occurring in the two cases.

In the fifth chapter we have given another construction for the fuzzy absolutes. Instead of using the fuzzy regularly closed subsets of \( X \) we can construct fuzzy absolutes by taking the pseudo-complemented lattice \( \tau(X) \) of all fuzzy open subsets of \( X \). It is denoted as \( f-E'X \). Then the underlying set of \( f-E'X \) is the fixed f-open ultrafilters on \( X \). Properties which are proved in chapter 4 are explicitly proved here for \( f-E'X \). Suitable examples are given wherever necessary.

### 0.6 Basic Definitions:

The following definitions are adapted from \([G;S],[CH],[LO]_1\), \([WO]_1\), \([G;K;M]\), \([MA]\), \([P,W]\) \([Y;M]\).
0.6.1 Definition: - Let $\mu$ and $\gamma$ be fuzzy subsets of a non empty set $X$. Then

\[ \mu = \gamma \iff \mu(x) = \gamma(x) \ \forall \ x \in X. \]
\[ \mu \leq \gamma \iff \mu(x) \leq \gamma(x) \ \forall \ x \in X. \]
\[ \mu \lor \gamma = \eta \iff \eta(x) = \max(\mu(x), \gamma(x)) \ \forall \ x \in X. \]
\[ \mu \land \gamma = \delta \iff \delta(x) = \min(\mu(x), \gamma(x)) \ \forall \ x \in X. \]
\[ \mu' = \lambda \iff \lambda(x) = 1 - \mu(x) \ \forall \ x \in X. \]

More generally, $\lor \mu_i$ and $(\land \mu_i)$ are defined as $(\lor \mu_i)(x) = \lor(\mu_i(x))$ and $(\land \mu_i)(x) = \land(\mu_i(x)), \ \forall \ x \in X.$

The symbol $0$ is used to denote the empty fuzzy subset defined by $\mu(x) = 0$, $\forall x \in X$ and $1$ is used to denote the whole set $X$ defined by $\mu(x) = 1 \ \forall x \in X.$

0.6.2 Definition: - If $\mu$ is a fuzzy subset of $X$, then $\{ x \in X : \mu(x) > 0 \}$ is called the support of $\mu$ and is denoted as $\text{supp } \mu$ or $\sigma_0(\mu)$. 

0.6.3 Definition:

A fuzzy point $x_p$ in $X$ is a fuzzy set in $X$ defined by

\[ x_p(y) = p \quad (p \in (0, 1]) \text{ for } y = x \]
\[ = 0 \quad \text{for } y \neq x. \]

$x$ and $p$ are the support and value of $x_p$. A fuzzy point $x_p$ belongs to a fuzzy set $\mu$ in $X$ if and only if $p \leq \mu(x)$. In this case we use the notation $x_p \in \mu$. When $p = 1$, $x_p$ is said to be fuzzy singleton.
0.6.4 Definition [CH]: A fuzzy topology on X is a subset $\delta \subset 1^X$ such that

i) $0, 1 \in \delta$.

ii) If $\mu, \gamma \in \delta$ then $\mu \land \gamma \in \delta$.

iii) If $\mu_i \in \delta$ for each $i$, then $\lor_i \mu_i \in \delta$.

$\delta$ is called a fuzzy topology on X and the pair $(X, \delta)$ is a fuzzy topological space or fts for short. Every member of $\delta$ is called an open fuzzy set. A fuzzy set is closed if and only if its complement is open.

0.6.5 Definition [LO]: $\delta \subset 1^X$ is a fuzzy topology on X if and only if

i) for all constants $\alpha, \alpha \in \delta$.

ii) for all $\mu, \gamma \in \delta$, $\mu \land \gamma \in \delta$.

iii) If $\mu_i \in \delta$ for each $i$, then $\lor_i \mu_i \in \delta$.

0.6.6 Definition: Let $(X, \tau)$ be a fuzzy topological space. For $Y \subset X$ and $Y \neq \emptyset, \tau / Y = \{\mu / Y, \mu \in \tau\}$ is a fuzzy topology on Y. Then $(Y, \tau / Y)$ is called a subspace of $(X, \tau)$.

0.6.7 Definition: Let $\{X_\alpha\} \alpha \in I$, be a family of fuzzy topological spaces with fuzzy topology $\tau_\alpha$. Let $X = \pi X_\alpha$ be the usual product space and $P_\alpha$ be the projection from $X$ onto $X_\alpha$. Then for $B \in \tau_\alpha$, $P_\alpha^{-1}(B)$ is a fuzzy set in X. Let $S = \{P_\alpha^{-1}(B) / B \in \tau_\alpha, \alpha \in I\}$. Let $\emptyset$ be the family of all finite intersections of members of $\delta$ and $\tau$ be the family of all unions of members of $\emptyset$. Then $\tau$ is a
fuzzy topology for $X$ with $\emptyset$ as a base and $S$ as a sub-base. Then $(X, \tau)$ is called the product fuzzy topological space.

**0.6.8 Definition:** Let $(X_i, \tau_i)_{i \in I}$ be a family of pair wise disjoint fuzzy topological spaces. Consider the set $X = \cup_i X_i$. Define the sum topology of \{${\tau_i, \alpha \in I}$} on $I^X$ i.e. $\Theta \tau$ as follows. For every $\mu \in I^X$, $\mu \in \Theta \tau$ if and only if $\mu / X_\alpha \in \tau_\alpha$. Then $(X, \tau_\alpha)$ is called sum fuzzy topological space or sum fts.

**0.6.9 Definition:** Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, \tau)$. Then the largest open fuzzy set contained in $\mu$ is called the interior of $\mu$ and is denoted as “f-int $\mu$” or $\mu^\circ$.

\[ \mu^\circ = \vee \{ \lambda : \lambda \in \tau, \lambda \leq \mu \}. \]

The smallest closed fuzzy set containing $\mu$ is called the closure of $\mu$ denoted as ‘f-cl $\mu$’ or $\mu$.

\[ \mu = \wedge \{ \lambda' : \lambda \in \tau, \lambda' \geq \mu \}. \]

**0.6.10 Definition:** Let $(X, F)$ be a fuzzy topological space. A fuzzy subset $\mu$ on $X$ is dense in $(X, F)$ provided that $\text{cl}_F(\mu) = 1$. If $(Y, H)$ is a fuzzy subspace of $(X,F)$ then $(Y,H)$ is dense in $(X,F)$ provided $\mu_Y$ is dense in $(X,F)$.

**0.6.11 Definition:** A fuzzy topological space $(X, \delta)$ is said to be fuzzy Hausdorff or fT$_2$ if for each distinct pair of points $x$ and $y$ in $X$, there exist open fuzzy sets $\mu$ and $\gamma$ such that $\mu(x) = \gamma(y) = 1$ and $\mu \wedge \gamma = 0$. 
0.6.12 Definition: Let \((X, \tau)\) be a fuzzy topological space. A family \(\mathcal{A}\) of fuzzy sets is a cover of a fuzzy set \(\mu\) if and only if \(\mu \leq \vee \{ \lambda : \lambda \in \mathcal{A} \}\). It is an open cover if and only if each member of \(\mathcal{A}\) is an open fuzzy set. A sub cover of \(\mathcal{A}\) is a subfamily which is also a cover.

0.6.13 Definition: Let \((X, \delta)\) be a fuzzy topological space. Then the family \([\delta]\) of supports of all crisp subsets in \(\delta\) is an ordinary topology on \(X\). Then the topological space \((X, [\delta])\) is called the background space of \((X, \delta)\).

0.6.14 Definition: A function \(f:X \to I\) is lower semi continuous if and only if for each \(\alpha \in f^{-1}(\alpha, 1]\) is open in \(X\). The characteristic function \(\chi_A:X \to [0,1]\) is lower semi continuous if and only if \(A\) is open in \(X\).

0.6.15 Definition: A space \(X\) is zero dimensional if and only if each point of \(X\) has a neighbourhood base consisting of clopen sets.

For the elementary definitions and results in topology references may be made to [WI], [JO]. For the theory of Stone space and absolutes to [P;W],[WA].