CHAPTER III
ORDER STRUCTURE OF POSET $DP(X)$

Since last several years topologists have been studying the order structure of an associated family of extensions of a space and have obtained results like Theorem 1.3. obtained by Magill in 1968. The situation in which the topology of a space is determined by the order structure of an associated family of mappings is illustrated by Theorem 1.17.

In case of the poset $DP(X)$, the order structure of $DP(X)$ is always determined by the topology of the space $X$, i.e. if topological spaces $X$ and $Y$ are homeomorphic then $DP(X)$ and $DP(Y)$ are order isomorphic. In this chapter we deal with the converse problem, that is, if $DP(X)$ and $DP(Y)$ are order isomorphic then can we say that $X$ is homeomorphic to $Y$? In section 1 of this chapter we define and study primary and dual member in $DP(X)$. In section 2, we introduce the notion of cln-bijection and using cln-bijections we prove in the last section that if $X$ and $Y$ are countably compact $T_3$ spaces without isolated points then $DP(X)$ and $DP(Y)$ are order isomorphic implies $X$ is homeomorphic to $Y$.

Some results of this chapter are being published in the Bulletin of the Australian Mathematical Society, 72 (2005).
1. Primary and dual members in $DP(X)$.

In this section we define primary and dual members in $DP(X)$ and characterize them. We also derive the formula for the greatest lower bound of two primary members in $DP(X)$.

**Definition 3.1.1.** Let $X$ be a topological space. Then an $f$ in $DP(X)$ is said to be

(i) **primary** if the corresponding dp-partition $\varphi(f)$ generated by $f$ has at most one non-singleton member.

(ii) **dual** if $f$ is primary and the corresponding dp-partition $\varphi(f)$ generated by $f$ contains exactly one doubleton.

**Notation.** If $f \in DP(X)$ is such that $\varphi(f)$ contains $n$ non-singleton members, say $K_1, K_2, ..., K_n$, then $f$ is denoted by $(f; K_1, K_2, ..., K_n)$. In particular, if $K$ is a non-singleton closed nowhere dense set in $X$, then $(f; K)$ denotes the natural density preserving map defined on $X$ obtained by collapsing $K$ to a point.

**Examples 3.1.2.(a)** Consider the usual space $\mathbb{R}$ of real numbers and take closed nowhere dense set as the set $\mathbb{Z}$ of all integers. The natural quotient map $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ obtained by identifying $\mathbb{Z}$, is a density preserving map and the only non-singleton member in $\varphi(f)$ is $\mathbb{Z}$. Therefore $(q; \mathbb{Z})$ is a primary member in $DP(\mathbb{R})$ but is not a dual member.
3.1.2(b) The natural quotient map $f$ on the usual space $\mathbb{R}$ of real numbers obtained by identifying two distinct points $x$ and $y$ in $\mathbb{R}$ is a density preserving map. Observe that the only non-singleton member of $\wp(f)$ is the set $\{x,y\}$. Hence this primary member $f$ in $DP(\mathbb{R})$ is also a dual.

**Theorem 3.1.3.** Let $X$ be a topological space. Then an $f$ in $DP(X)$, $f \neq I_X$, is primary if and only if there do not exist duals $g, h \in DP(X)$ such that $f \wedge g = f \wedge h \neq f$ and the only dual points greater than $g \wedge h$ are $g$ and $h$.

**Proof.** If $f$ in $DP(X)$ represents the family of constant members then for any two duals $g, h$ in $DP(X)$, we get $f \wedge g = f \wedge h = f$. Therefore $f$ is primary. Suppose $f$ is a non constant member in $DP(X)$ such that $f \neq I_X$ and there do not exist duals $g$ and $h$ in $DP(X)$ such that $f \wedge g = f \wedge h \neq f$ and the only duals greater than $g \wedge h$ are $g$ and $h$. Then we need to show that $f$ is primary or equivalently dp-partition $\wp(f)$ generated by $f$ contains exactly one non-singleton member. Suppose $\wp(f)$ contains more than one non-singleton member say $K$ and $H$. Then choose distinct points $a, b \in H$ and $c, d \in K$ and consider the natural dual members $(g;\{a,c\})$ and $(h;\{b,d\})$ in $DP(X)$. Since $H \cap K = \emptyset$, the dp-partition $\wp(f \wedge (g;\{a,c\}))$ generated by $f \wedge (g;\{a,c\})$ consists of $[\wp(f) - \{K, H\}] \cup \{K \cup H\}$. Observing similarly for $f \wedge (h;\{b,d\})$, we obtain $f \wedge (g;\{a,c\}) = f \wedge (h;\{b,d\}) \neq f$. 

33
Also, besides $g$ and $h$, there are other duals in $DP(X)$ greater than $g \wedge h$. In fact $g \wedge h$ is the natural quotient map $(g \wedge h; \{a,b,c,d\})$ and $(k;\{a,b\})$ is a dual member in $DP(X)$ different from $g$ and $h$ satisfying $(g \wedge h) \leq k$. This contradicts our hypothesis. Hence such an $f$ is primary.

Conversely, suppose $f \neq I_X$ is primary. In case $f$ is constant we are through. Let $f$ be non-constant and let $K$ be the non-singleton member of $\varphi(f)$. Let $(g;\{a,b\})$ and $(h;\{c,d\})$ be two distinct dual members in $DP(X)$.

Then $g \neq h$ implies

$$\{a,b\} \neq \{c,d\}$$

$$\Rightarrow \{a,b\} \cap \{c,d\} = \emptyset \text{ or } \{a,b\} \cap \{c,d\} \text{ is a singleton.}$$

Now suppose $\{a,b\} \cap \{c,d\} = \emptyset$. In case $\{a, b\} \cap K = \emptyset$ and $\{c, d\} \cap K = \emptyset$, then

$$f \wedge g \approx (f \wedge g; K, \{a, b\})$$

(1)

and

$$f \wedge h \approx (f \wedge h; K, \{c, d\}).$$

(2)

On the other hand consider the case when $\{a, b\} \cap K \neq \emptyset$ and $\{c, d\} \cap K \neq \emptyset$.

Let $b \in K$ and $d \in K$. Then

$$f \wedge g \approx (f \wedge g; K \cup \{a\})$$

(3)

and

$$f \wedge h \approx (f \wedge h; K \cup \{c\}).$$

(4)

Relations (1) to (4) imply that if $\{a,b\} \cap \{c,d\} = \emptyset$, then $f \wedge g \neq f \wedge h$ and we are through.
Next, suppose \( \{a, b\} \cap \{c, d\} \) is a singleton. In case \( f \wedge g \neq f \wedge h \), we are through. Suppose \( f \wedge g \approx f \wedge h \neq f \). In this case we obtain a dual member \( \eta \) different from \( g \) and \( h \) which is greater than \( h \wedge g \). In fact, if \( \{a, b\} \cap \{c, d\} = \{a\} \), where \( a=c \), then

\[
(h; \{a, b\}) \wedge (g; \{c, d\}) \approx (h \wedge g; \{a, b, d\}).
\]

and the dual member \( (\eta; \{b, d\}) \) is different from \( g \) and \( h \) and is greater than \( h \wedge g \).

Therefore if \( \varphi(f) \) contains exactly one non-singleton member then there do not exist dual points \( g \) and \( h \) such that \( f \wedge g \approx f \wedge h \neq f \) and the only duals greater than \( h \wedge g \) are \( g \) and \( h \).

**Theorem 3.1.4.** Let \( X \) be a topological space. Then an \( f \) in \( DP(X) \) is a dual if and only if there does not exist \( g \) in \( DP(X) \) such that \( f < g < I_x \).

**Proof.** Suppose \( f \in DP(X) \) is a dual. Then there exists precisely one non-singleton member in \( \varphi(f) \) which is a doubleton. If possible, suppose there exists \( g \in DP(X) \) such that \( f < g < I_x \). Then

\[
f < g \Rightarrow \varphi(g) \subseteq \varphi(f)
\]

\[
\Rightarrow g^{-1}(z) \text{ is a singleton for each } z \text{ in } Rg
\]

\[
\Rightarrow g \approx I_x - \text{a contradiction.}
\]

Conversely, if possible, suppose \( f \) is not a dual. Then the dp-partition \( \varphi(f) \) contains one non-singleton member, say \( K \), containing more than two elements. Choose distinct points \( a, b, c \) in \( K \). Then \( (h; \{a, b\}) \) is a dual member in \( DP(X) \) satisfying \( f < h < I_x \).
**Theorem 3.1.5.** Let $X$ be a topological space. Then for any two closed nowhere dense subsets $K_1$ and $K_2$ of $X$,

$$
(f; K_1) \land (g; K_2) = \begin{cases} (h; K_1, K_2), & \text{if } K_1 \cap K_2 = \emptyset \\ (h; K_1 \cup K_2), & \text{if } K_1 \cap K_2 \neq \emptyset. \end{cases}
$$

**Proof.** Suppose $K_1 \cap K_2 = \emptyset$. Then by Lemma 2.3.3,

$$\varphi(f) \subseteq \varphi(h) \text{ and } \varphi(g) \subseteq \varphi(h) \Rightarrow (f; K_1) \geq (h; K_1, K_2) \text{ and } (g; K_2) \geq (h; K_1, K_2).$$

If $\sigma \in DP(X)$ is another member such that $f \geq \sigma$ and $g \geq \sigma$, then by Lemma 2.3.3,

$$f \geq \sigma \text{ and } g \geq \sigma \Rightarrow \varphi(f) \subseteq \varphi(\sigma) \text{ and } \varphi(g) \subseteq \varphi(\sigma),$$

which implies $K_1$ is a subset of some member in $\varphi(\sigma)$ and $K_2$ is also a subset of some member in $\varphi(\sigma)$. But this gives $\sigma \leq (h; K_1, K_2)$ which proves $(f; K_1) \land (g; K_2) = (h; K_1, K_2)$.

Next, suppose $K_1 \cap K_2 \neq \emptyset$, then $(f; K_1) \geq (h; K_1 \cup K_2)$ and $(g; K_2) \geq (h; K_1 \cup K_2)$. Suppose $\sigma \in DP(X)$ is such that $f \geq \sigma$ and $g \geq \sigma$. Then by Lemma 2.3.3,

$$f \geq \sigma \text{ and } g \geq \sigma \Rightarrow \varphi(f) \subseteq \varphi(\sigma) \text{ and } \varphi(g) \subseteq \varphi(\sigma),$$

which implies that $K_1$ is a subset of some member, say $H_1$ in $\varphi(\sigma)$ and $K_2$ is a subset of some member, say $H_2$ in $\varphi(\sigma)$. Since $K_1 \cap K_2 \neq \emptyset$, we have $H_1 = H_2$. Thus $(h; K_1 \cup K_2) \geq \sigma$. This proves that $(f; K_1) \land (g; K_2) = (h; K_1 \cup K_2)$, if $K_1 \cap K_2 \neq \emptyset$. 

36
Theorem 3.1.6. Let $X$ and $Y$ be topological spaces. Then an order isomorphism $\varphi : DP(X) \to DP(Y)$ maps duals to duals.

Proof. Suppose $f \in DP(X)$ is a dual. If $\varphi(f)$ is not a dual member then by Theorem 3.1.4, there exists $g \in DP(Y)$ such that $\varphi(f) < g < I_X$. Since $\varphi$ is an order isomorphism, we have $f < \varphi^{-1}(g) < I_X$ which contradicts that $f \in DP(X)$ is a dual. Hence an order isomorphism $\varphi : DP(X) \to DP(Y)$ maps duals to duals.

2. cln-bijection.

In this section we define cln-bijection and prove that for Hausdorff spaces $X$ and $Y$ without isolated points, an order isomorphism $\varphi : DP(X) \to DP(Y)$ induces a cln-bijection from $X$ to $Y$.

Definition 3.2.1. A bijection $f : X \to Y$ from a topological space $X$ to a topological space $Y$ is called a cln-bijection if the family $\{f(A) \mid A \text{ is a closed nowhere dense subset of } X\}$ is precisely the family of all closed nowhere dense subsets of $Y$.

We recall that a point $p$ in the Stone-Čech compactification $\beta X$ of a Tychonoff space $X$ is called a remote point of $X$ if $p \in \beta X - X$ but $p \notin Cl_{\beta X} A$, for every nowhere dense subset $A$ of $X$.

Example 3.2.2. Consider the usual space $Q$ of rational numbers. Suppose $p$ and $q$ are remote points of $Q$ such that Stone extension of none of the
self-homeomorphism of $Q$ maps $p$ to $q$. Consider the subspaces $Q \cup \{p\}$ and $Q \cup \{q\}$ of the Stone-Čech compactification $\beta Q$ and the map $f: Q \cup \{p\} \to Q \cup \{q\}$ defined by $f(x) = x$ if $x \in Q$ and $f(p) = q$. Then $f$ is a cln-bijection.

Lemma 3.2.3. Let $X$ and $Y$ be Hausdorff spaces without isolated points and let $\varphi: DP(X) \to DP(Y)$ be an order isomorphism. Then there exists a cln-bijection $F: X \to Y$ such that for each $f \in DP(X)$ we have $\varphi(\varphi(f)) = \{F(A) \mid A \in \varphi(f)\}$.

Proof. Let $p \in X$. Choose distinct points $q, r \in X - \{p\}$. By Theorem 3.1.6, $\varphi(f,\{p, q\}), \varphi(g,\{p, r\})$ are dual points of $DP(Y)$ say $(\bar{f},\{a, b\})$ and $(\bar{g},\{c, d\})$ respectively. Clearly

$$(\bar{f};\{a, b\}) \wedge (\bar{g};\{c, d\}) = \varphi(f \wedge g;\{p, q, r\}).$$

If $\{a, b\} \cap \{c, d\} = \emptyset$, then

$$(\bar{f};\{a, b\}) \wedge (\bar{g};\{c, d\}) = (\bar{f} \wedge \bar{g};\{a, b\},\{c, d\}).$$

Thus in this case $(\bar{f};\{p, q\}), (\bar{g};\{p, r\}), (\bar{h};\{q, r\})$ are three dual points in $DP(X)$ greater than $(\bar{f} \wedge \bar{g};\{p, q, r\})$ whereas $(\bar{f};\{a, b\}), (\bar{g};\{c, d\})$ are the only two dual points in $DP(Y)$ greater than $(\bar{f} \wedge \bar{g};\{a, b\},\{c, d\})$ which is not possible. Therefore $\{a, b\} \cap \{c, d\} \neq \emptyset$, in fact it is a singleton, say $\{a\}$. Define $F: X \to Y$ by $F(p) = a$. We now show that the choice of $a$ does not depend upon the choices of $r$ and $q$. Let $s \in X - \{p, q, r\}$. Then there exist points $y$ and $z$ in $Y$ such that $\varphi(k;\{p, s\}) = (\bar{k};\{y, z\})$. We have
\( \varphi(f;\{p, q\}) = (f;\{a, b\}) \). Assume \( \varphi(g;\{p, r\}) = (g;\{a, c\}) \). Using similar arguments we conclude that \( \{y, z\} \) intersects both \( \{a, b\} \) and \( \{a, c\} \) in exactly one point. If \( a \notin \{y, z\} \) then \( \{y, z\} = \{b, c\} \). Therefore by Theorem 3.1.5,

\[
\varphi(f \land g \land k; \{p, q, r, s\}) = (f;\{a, b\}) \land (g;\{a, c\}) \land (k;\{b, c\}) = (f \land g \land k; \{a, b, c\}).
\]

This implies there are six dual points greater than \( (f \land g \land k;\{p, q, r, s\}) \) while there are only three duals greater than \( (f \land g \land k;\{a, b, c\}) \), which is impossible as \( \varphi \) is an order isomorphism. Thus our assumption that \( a \notin \{y, z\} \) is not possible. This proves that for any \( s \in X - \{p\} \), if \( \varphi(k;\{p, s\}) = (k;\{y, z\}) \) then \( a \in \{y, z\} \). Also if \( s' \) is any other point in \( X - \{p, q\} \) and if \( \varphi(\sigma;\{p, s'\}) = (\sigma;\{y', z'\}) \), then \( \{y', z'\} \cap \{y, z\} = \{a\} \). Thus we have defined map \( F \).

We now show that \( F \) maps closed nowhere dense sets to closed nowhere dense sets. Let \( H \) be a closed nowhere dense set in \( X \). Consider \( f \in DP(X) \) of the form \( (f;H) \). If \( \varphi(f;H) = \bar{f} \) then \( \bar{f} = (\bar{f};K) \) for some closed nowhere dense subset \( K \) of \( Y \). Further, if \( p, q \in H \), \( p \neq q \), then by Lemma 2.3.3 \( (g;\{p, q\}) \geq (f;H) \) which implies \( (g;\{a, b\}) \geq (\bar{f};K) \). Hence \( \{a, b\} \subseteq K \). This proves that \( F(\{p, q\}) = \{a, b\} \subseteq K \). Since \( p, q \in H \) are arbitrary, it follows that \( F(H) \subseteq K \).

Similarly, using \( \varphi^{-1} \), we can define \( \bar{F}:Y \to X \) as follows: Let \( a \in Y \). Choose distinct points \( b \) and \( c \) in \( Y - \{a\} \). Then, \( \varphi^{-1}(\bar{f};\{a, b\}) \) and
are dual points of $DP(X)$ say $(f; \{p, q\})$ and $(g; \{r, s\})$ respectively. Then using similar arguments as above we can show that $\{p, q\} \cap \{r, s\}$ is a singleton say $p$ and the choice of $p$ does not depend upon the choice of $b$ and $c$. Define $\overline{F}: Y \to X$ by $\overline{F}(a) = p$. Arguing as above, one can show that $\overline{F}(K) \subseteq H$.

We now prove that $\overline{F} \circ F$ is identity on $X$. Let $p \in X$ and $q \in X - \{p\}$. Clearly $\varphi(f; \{p, q\})$ is a dual, say $(\overline{f}; \{a, b\})$. Then $F(p) \in \{a, b\}$. Assume $F(p) = a$. Suppose $\overline{F}(a) \neq p$. Then $\overline{F}(a) = q$. Choose $r \in X - \{p, q\}$. Then there exists $c \in Y$ such that $\varphi(g; \{p, r\})$ is a dual point say $(\overline{g}; \{a, c\})$. Since $\overline{F}(a) \in \{p, r\}$ and $\overline{F}(a) \neq p$, therefore $\overline{F}(a) = r$, a contradiction as $\overline{F}(a) = q \neq r$. Therefore we conclude that $\overline{F}(a) = p$. This proves $\overline{F} \circ F$ is identity on $X$. Similarly, we can prove that $F \circ \overline{F}$ is identity on $Y$. Hence $F: X \to Y$ is a bijective map which preserves closed nowhere dense sets. Also, by the definition of the map $F$, it follows that if $\varphi(f; K) = (\overline{f}; K)$, then $F(H) = K$ and hence $\varphi(\varphi(f)) = \{F(A) | A \in \varphi(f)\}$.

3. Topology of $X$ and the order structure of $DP(X)$.

We shall see an easy proof of the fact that if topological spaces $X$ and $Y$ are homeomorphic topological spaces then $DP(X)$ and $DP(Y)$ are order isomorphic. To find when the converse is true, we shall use the following known fact:
Note. Let $X$ be a countably compact $T_3$ space without isolated points. Then $A \subseteq X$ is closed if and only if whenever $B \subseteq A$ and $\text{Cl}_X B$ is nowhere dense in $X$ then $\text{Cl}_X B \subseteq A$ [22].

In fact, we shall use the given order isomorphism between $DP(X)$ and $DP(Y)$ to construct a cln-bijection $F$ between $X$ and $Y$ and then use the fact stated in the above note to prove that both $F$ and $F^{-1}$ are closed maps.

**Theorem 3.3.1.** Let $X$ and $Y$ be countably compact $T_3$ spaces without isolated points. Then $DP(X)$ and $DP(Y)$ are order isomorphic if and only if $X$ and $Y$ are homeomorphic.

**Proof.** Suppose $X$ and $Y$ are homeomorphic and let $h : X \rightarrow Y$ be a homeomorphism. Define $\varphi : DP(X) \rightarrow DP(Y)$ by $\varphi(f) = f \circ h$. The map $\varphi$ thus defined is a bijective order preserving map from $DP(X)$ to $DP(Y)$. Hence $DP(X)$ and $DP(Y)$ are order isomorphic.

Conversely, suppose $DP(X)$ and $DP(Y)$ are order isomorphic. Then by Lemma 3.2.3 there is a cln-bijection $F : X \rightarrow Y$. We shall show that $F$ is a closed map. By symmetry, it will follow that $F^{-1}$ is also a closed map and hence $F$ will be a homeomorphism. Let $M \subseteq X$ be such that $\text{Cl}_X M$ is nowhere dense. First observe that $F(\text{Cl}_X M) = \text{Cl}_Y F(M)$. Clearly, $M \subseteq \text{Cl}_X M$ and hence $F(M) \subseteq F(\text{Cl}_X M)$. Since $\text{Cl}_X M$ is a closed nowhere dense set and $F$ is a cln-bijection, $\text{Cl}_Y F(M)$ is a closed nowhere dense set satisfying $\text{Cl}_Y F(M) \subseteq F(\text{Cl}_X M)$. For the reverse containment, observe that
$F^{-1}(Cl_y F(M))$ is a closed nowhere dense set containing $M$ which implies that $F(Cl_x M) \subseteq Cl_y F(M)$. This proves $F(Cl_x M) = Cl_y F(M)$, whenever $Cl_x M$ is a nowhere dense set.

We complete the proof by showing that $F$ is a closed map. Let $C$ be a closed subset of $X$. Suppose $B \subseteq F(C)$ and $Cl_y B$ is a nowhere dense set. Then $Y$ being countably compact $T_3$ space without isolated points, to establish that the map $F$ is closed, it is sufficient to show that $Cl_y B \subseteq F(C)$.

Now,

$$B \subseteq F(C) \Rightarrow F^{-1}(B) \subseteq C.$$ 

Also $C$ is closed implies $Cl_x F^{-1}(B) \subseteq C$ and hence

$$F^{-1}(Cl_y B) = Cl_x F^{-1}(B) \subseteq C.$$

This proves $Cl_y B \subseteq F(C)$ and hence the required result.

**Note.** A cln-bijection between non-countably compact spaces without isolated points need not be a homeomorphism is justified by Example 3.2.2. In fact, the map $f : Q \cup \{p\} \to Q \cup \{q\}$ defined in that example is a cln-bijection but we know that the spaces $Q \cup \{p\}$ and $Q \cup \{q\}$ considered there are not homeomorphic.