CHAPTER VI
NEARLY HAUSDORFF COMPACTIFICATIONS

Mathematicians including Alexandroff, Urysohn, Čech, Cartan, Wallman, Tychonoff and Lubben laid the foundation of the modern theory of Hausdorff extensions. Once the term "COMPACTNESS" was defined, it was a natural problem to try and extend a non-compact space to a compact space. The first general method in this direction was the one-point compactification in 1924 due to Alexandroff. In 1937 Čech developed a compactification having the maximal extension property by extending Tychonoff's idea of embedding a completely regular Hausdorff space $X$ in a cube. Stone developed a similar compactification independently. This compactification is termed as Stone-Čech compactification and is denoted by $\beta X$ for a space $X$. In fact, $\beta X$ is maximal compactification of a Tychonoff space $X$. In 1938, Wallman gave a general method for constructing a $T_i$ compactification coinciding with $\beta X$.

A family $B$ of subset of a space $X$ is called a ring of sets if it is closed under finite intersections and finite unions. A subfamily $\alpha$ of non-empty members of a ring $B$ is called a $B$-filter if $\alpha$ is closed under finite intersections and super sets. A $B$-filter $\alpha$ is a $B$-ultrafilter if it is not properly contained in any other $B$-filter. The filter concept was introduced in order to study convergence. Besides for describing convergence, collections of $B$-
ultrafilters have been used to construct topological spaces. Let $\sigma(B)$ denote the collection of all $B$-ultrafilters on $X$. For $z \in B$, let $z^\sigma$ denote the members of $\sigma(B)$ which contain $z$. Taking $\{z^\sigma | z \in B\}$ as a base for closed sets we get topology on $\sigma(B)$ which is useful in the formation of compactifications. In 1938, Wallman considered the case in which $B$ is the family of all closed sets of a $T_1$ space $X$. Wallman showed that under these conditions $\sigma(B)$ is not necessarily Hausdorff. We recall the definition of a Wallman base. A Wallman base $L$ on a space $X$ is a ring of subsets of $X$ satisfying:

(i) $\varnothing, X \in L$,

(ii) $L$ is a closed base for $X$,

(iii) if $A \in L$ and $x \in X - A$ then there is a $B \in L$ such that $x \in B$, $A \cap B = \varnothing$ and

(iv) if $A, B \in L$ such that $A \subseteq X - B$ then there are $C, D \in L$ such that $A \subseteq X - C \subseteq D \subseteq X - B$.

We have observed that for a $T_1$ topological space $X$ having more than one point, the family $R(X)$ of all regular closed subsets of $X$, is not a ring in general but if we consider the family $R_f(X)$ of all finite intersections of members of $R(X)$ then the family $R_f(X)$ forms a ring. In this chapter, we attempt to construct a compactification $rX$ for a non-Tychonoff space $X$ by using the family $R_f(X)$. We observe that the resulting compactification $rX$ is a non-Hausdorff $T_1$ space. A separation axiom stronger than $T_1$ but weaker
than $T_2$ naturally exists on $rX$ which we term as nearly Hausdorffness. In the section 1, of this chapter we define and study this separation axiom. In the section 2, we discuss the construction of the space $rX$ and in the last section we discuss the natural question under what conditions $rX = \beta X$?

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1. Nearly Hausdorff Spaces.

In this section, we define and study "Nearly Hausdorffness" a separation axiom stronger than $T_1$ but weaker than $T_2$. We introduce a topological property $\pi$ and note that a space with property $\pi$ is a nearly Hausdorff space if and only if it is Urysohn. A flow diagram showing various implications about separation axioms supported by necessary counter examples is included in this section.

**Definition 6.1.1.** Distinct points $x$ and $y$ in a topological space $X$ are said to be separated by subsets $A$ and $B$ of $X$ if $x \in A - B$ and $y \in B - A$.

**Definition 6.1.2.** A topological space $X$ is called a nearly Hausdorff space if for every pair of distinct points of $X$ there exists a pair of regular closed sets in $X$ separating them.
We observe that the notion of nearly Hausdorff spaces coincides with the notion of weakly Hausdorff spaces defined by Soundararajan in [26]. A topological space $X$ is called weakly Hausdorff if each of its points is an intersection of regular closed sets. That a nearly Hausdorff space is weakly Hausdorff follows because for each pair of distinct points, there exist regular closed sets separating them and hence each point is an intersection of regular closed sets. Conversely, suppose $X$ is a weakly Hausdorff space then each point of $X$ is an intersection of regular closed sets. Hence for every pair of distinct points, there exists a pair of regular closed sets separating them. Thus a weakly Hausdorff space is a nearly Hausdorff space.

We introduce here a topological property $\pi$ for a topological space $X$.

**Definition 6.1.3.** A topological space $X$ is said to have property $\pi$ if for every $F \in R(X)$ and $x \notin F$ there exists an $H \in R(X)$ such that $x \in \text{Int}H$ and $H \cap F = \emptyset$.

We denote a topological space $X$ with property $\pi$ by $X(\pi)$.

We recall that a space $X$ is called a Urysohn space if for every pair of distinct points $x, y$ in $X$ there exist open sets $G$ and $H$ containing $x$ and $y$ respectively such $ClG \cap ClH = \emptyset$ [33]. Following flow diagram expresses the relationship of nearly Hausdorffness with other separation axioms.
Examples given below [27, 33] justify that unidirectional implications in the above flow diagram need not be reversible. In addition, example 6.1.4.(b) shows that nearly Hausdorffness is not a closed hereditary property.

**Examples 6.1.4.(a)** A $T_i$ space need not be nearly Hausdorff for example an infinite cofinite space is a $T_i$ space but not a nearly Hausdorff space.

6.1.4.(b) The following example justifies that a nearly Hausdorff space need not be a Hausdorff space: Consider $\mathbb{N}$, the set of natural numbers with cofinite topology and $I = [0, 1]$ with the usual topology. Let $X = \mathbb{N} \times I$ and define a topology on $X$ as follows:

(i) neighborhoods of the points of the form $(n, y), y \neq 0$ are usual neighborhoods $\{(n, z) \in X \mid y - \varepsilon < z < y + \varepsilon\}$ in $I = \{n\} \times I$ for small positive $\varepsilon$;

(ii) neighborhoods of the points of the form $(n, 0)$ are of the form $\{(m, z) \in X \mid m \in U, 0 \leq z < \varepsilon_m\}$, where $U$ is a neighborhood of $n$ in $\mathbb{N}$ and $\varepsilon_m$ is a small positive number for each $m \in U$.

The resulting space $X$ is a non-Hausdorff space as the distinct points in $X$ of the form $(n, 0)$ and $(m, 0)$ cannot be separated by disjoint open sets. We now observe that it is a nearly Hausdorff space. Let $(n, x)$ and $(m, y)$ be two distinct points in $X$. We consider the following cases:
Case (i) Let \( m = n \). Then choose \( \varepsilon < \frac{1}{2}|x-y| \). The regular closed sets 
\( \{(n,z) \in X \mid x-\varepsilon \leq z \leq x+\varepsilon \} \) and 
\( \{(m,z) \in X \mid y-\varepsilon \leq z \leq y+\varepsilon \} \) separates 
\((n,x)\) and \((m,y)\).

Case (ii) Let \( m \neq n \). Then the regular closed sets 
\( \{(n,z) \in X \mid 0 \leq z < x \} \) and 
\( \{(m,z) \in X \mid 0 \leq z < y \} \), where \( \varepsilon \) and \( \delta \) are small positive numbers, separate 
\((n,x)\) and \((m,y)\).

Note. (1) The set \( \{(n,0) \in X \mid n \in \mathbb{N} \} \) in the previous example is a closed 
subspace of \( X \) but not a nearly Hausdorff space. Thus a closed subspace of 
an \( \text{nh} \)-space need not be an \( \text{nh} \)-space.

(2) Also the space \( X \) does not possess property \( \pi \) because the set 
\( F = \{(1,z) \in X \mid 0 \leq z \leq \frac{1}{2} \} \) is a regular closed set and \((2,0) \notin F\) but there does 
not exist a regular closed set \( H \) in \( X \) such that \((2,0) \in \text{Int}H\) and \( H \cap F = \emptyset \).

Therefore a nearly Hausdorff space need not always have the property \( \pi \).

6.1.4.(c) In the previous example we saw that a nearly Hausdorff space need 
not have property \( \pi \). Now, we give an example showing that even a 
Hausdorff space need not have the property \( \pi \).

Let \( A \) be the linearly ordered set \( \{1, 2, 3, \ldots, \omega, \ldots, -3, -2, -1\} \) with the 
interval topology and let \( \mathbb{N} \) be the set of natural numbers with the discrete 
topology. Define \( X \) to be \( A \times \mathbb{N} \) together with two distinct points say \( a \) and 
\(-a\), which are not in \( A \times \mathbb{N} \). The topology \( \mathcal{I} \) on \( X \) is determined by the 
product topology on \( A \times \mathbb{N} \) together with basic neighborhoods 
\( M_n^+(a) = \{a\} \cup \{(i,j) \mid i < \omega, j > n\} \) and 
\( M_n^-(a) = \{-a\} \cup \{(i,j) \mid i > \omega, j > n\} \) about \( a \) and
Resulting space $X$ is a non Urysohn Hausdorff space without property $\pi$. The space $X$ is not Urysohn because there do not exist disjoint open sets $U$ and $V$ containing $a$ and $-a$ respectively such that $\overline{U} \cap \overline{V} = \varnothing$. That $X$ does not have property $\pi$ follows from the fact that $a \notin M_n(-a)$ and there does not exist a regular closed set $F$ containing $a$ such that $a \in \text{Int}F$ and $F \cap M_n(-a) = \varnothing$. Thus a Hausdorff space need not possess the property $\pi$.

6.1.4.(d) The following example justifies that a Urysohn space need not possess property $\pi$. Let $S$ be the set of rational lattice points in the interior of the unit square except those whose $x$-coordinate is $\frac{1}{2}$. Define $X$ to be $S \cup \{(0, 0)\} \cup \{(1, 0)\} \cup \{(\frac{1}{2}, r\sqrt{2})| r \in \mathbb{Q}, 0 < r\sqrt{2} < 1\}$. Topologize $X$ as follows:

Local base for points in $S \subseteq X$ are same as those inherited from the Euclidean topology and for other points following local bases are taken:

$$U_n(0,0) = \{(x,y) \in X | 0 < x < \frac{1}{4}, 0 < y < \frac{1}{n}\} \cup \{(0, 0)\},$$

$$U_n(1,0) = \{(x,y) \in X | \frac{3}{4} < x < 1, 0 < y < \frac{1}{n}\} \cup \{(1, 0)\},$$

$$U_n(\frac{1}{2}, r\sqrt{2}) = \{(x,y) \in X | \frac{1}{4} < x < \frac{1}{2} \text{ and } |y - r\sqrt{2}| < \frac{1}{n}\} \cup \{(0,0)\}.$$ 

The resulting space $X$ is a Urysohn space without property $\pi$. That space $X$ does not have property $\pi$ follows since $(0,0) \notin H = \{(x,y) \in S | \frac{1}{4} \leq x \leq \frac{3}{4}\}$ and there does not exist regular closed set $F$ such that $(0,0) \in \text{Int}F$ and $\text{Int}F \cap H = \varnothing$.

6.1.4.(e) The following example justifies that a Urysohn space with property $\pi$ need not be a regular space: Let $X$ be the set of real numbers with neighborhoods of any non-zero point as in the usual topology while...
neighborhoods of 0 will have the form $U - A$, where $U$ is a neighborhood of 0 in the usual topology and $A = \{ \frac{1}{n} | n \in \mathbb{N} \}$.

The resulting space $X$ is a non-regular Urysohn space with property $\pi$. That the space $X$ is not regular follows because $0 \notin A$, $A$ is closed in $X$ but there do not exist disjoint open sets $U$ and $V$ such that $0 \in U$ and $A \subseteq V$.

The space $X$ has property $\pi$ follows from the fact that the topology on $X$ is finer than the usual topology on the set of real numbers.

The space $X$ is a Urysohn space since for every pair of distinct points $x$ and $y$ in $X$, there exist disjoint open sets $(x - \eta, x + \eta)$ and $(y - \eta, y + \eta)$ having disjoint closures, where $\eta < \frac{1}{2} |x - y|$.

In an approach to unify the separation axioms between $T_0$ and completely Hausdorff, F. G. Arenas, J. Dontchev and M. L. Puertas in [1], have studied the relation of the separation axiom weakly Hausdorffness with $kd$-space, $ko$-space, $us$-space, $hT_1^R$ space. In [1], authors have observed that a space $X$ is weakly Hausdorff if its semiregularization is $T_1$, i.e., if each singleton is $\delta$-closed. We recall the following terms. A point $x$ in a topological space $X$ is called a $\delta$-cluster point of a subset $A$ of $X$ if $A \cap U \neq \varnothing$ for every regular open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $Cl_\delta(A)$. If $A = Cl_\delta(A)$ then $A$ is called $\delta$-closed. A topological space $X$ is called semiregular if regular open sets form a base for the topology of $X$. 75
Further, for a topological space \((X, \tau)\), the family of all regular open sets forms a base for a new topology \(\tau_s\), coarser than \(\tau\), which is called the \textit{semiregularization} of \(\tau\). Thus a space \((X, \tau)\) is semiregular if and only if \(\tau_s = \tau\). We observe the following result:

**Lemma 6.1.5.** A semiregular \(T_1\) space is a nearly Hausdorff space.

**Proof.** Let \(X\) be a semiregular \(T_1\) space and let \(x, y \in X\), \(x \neq y\). Since \(X\) is a \(T_1\) space, there exist open sets \(U\) and \(V\) separating \(x\) and \(y\) respectively. Further \(X\) is a semiregular space which implies there exist regular open sets \(G_x\) and \(G_y\) containing \(x\) and \(y\) respectively such that \(G_x \subset U\) and \(G_y \subset V\). The result now follows by observing that \(X - G_y\) and \(X - G_x\) are regular closed sets separating \(x\) and \(y\).

**Theorem 6.1.6.** A non-empty product of a nearly Hausdorff space is a nearly Hausdorff space if and only if each factor is a nearly Hausdorff space.

**Proof.** Let \(X = \prod_{\lambda} X_{\lambda}\), where \(\{X_{\lambda}\}_{\lambda \in \Lambda}\) is a family of nearly Hausdorff spaces, \(\lambda \neq \emptyset\). Consider two distinct points \(x, y\) in \(X\). Then \(x \neq y\)

\[
\Rightarrow \quad x_\alpha \neq y_\alpha \quad \text{for some} \quad \alpha \in \Lambda.
\]

Since each \(X_\alpha\) is a nearly Hausdorff space, for \(x_\alpha \neq y_\alpha\) in \(X_\alpha\) there exist regular closed sets \(F\) and \(H\) in \(X_\alpha\) separating \(x_\alpha\) and \(y_\alpha\). Define \(U = \prod_{\lambda \in \Lambda} U_\gamma\) and \(V = \prod_{\lambda \in \Lambda} V_\gamma\), where \(U_\gamma = V_\gamma = X_\gamma\) for \(\gamma \neq \alpha\) and \(U_\alpha = \text{Int}F\).
Then the regular closed sets $CIU$ and $CIV$ separate $x$ and $y$ respectively.

Conversely, suppose $X = \prod_{\gamma \in \Lambda} X_{\gamma}$ is a nearly Hausdorff space. Let $x_\alpha$, $y_\alpha$ be two distinct points in $X_\alpha$. Choose points $x$, $y$ in $X$ such that they differ only in $\alpha$th co-ordinate and their $\alpha$th co-ordinates are $x_\alpha$ and $y_\alpha$ respectively. Since $X$ is a nearly Hausdorff space, there exist regular closed sets $F$ and $H$ in $X$ separating $x$ and $y$ respectively. Since $\text{Int} F$ and $\text{Int} H$ are open sets in $X$ therefore $\text{Int} F = \prod_{\gamma \in \Lambda} U_{\gamma}$ and $\text{Int} H = \prod_{\gamma \in \Lambda} V_{\gamma}$, where $U_{\gamma}$ and $V_{\gamma}$ are open sets in $X_{\gamma}$ for each $\gamma$ and $U_{\gamma} = X_{\gamma}$, $V_{\gamma} = X_{\gamma}$ except for finitely many values of $\gamma$. The regular closed sets $CIU_\alpha$ and $CIV_\alpha$ separate $x_\alpha$ and $y_\alpha$ respectively. This proves that for each $\alpha \in \Lambda$, $X_\alpha$ is a nearly Hausdorff space.

The following result was proved in [8] for a regular Hausdorff space. We now observe that it is true for nearly Hausdorff space also.

**Theorem 6.1.7.** Let $X$ be a nearly Hausdorff space and let $f : X \to Y$ be a density preserving epimorphism. Then

(A) for a regular closed set $H$ of $Y$, we have $Clf(Cl^{-1}(\text{Int} H)) = H$ and hence $R(Y) = \{Clf(F) | F \in R(X)\}$.

(B) $Clf(F) \in R(Y)$ whenever $F \in R(X)$.
Proof. (A) Clearly, \( \text{Clf}(\text{Clf}^{-1}(\text{Int}H)) \subseteq H \). For the reverse containment, let \( x \in H \). Then we consider the following cases:

Case (i) Let \( x \in \text{Int}H \). Then we have \( x \in \text{Clf}(\text{Clf}^{-1}(\text{Int}H)) \) which implies \( \text{Int}H \subseteq \text{Clf}(\text{Clf}^{-1}(\text{Int}H)) \) and therefore \( H \subseteq \text{Clf}(\text{Clf}^{-1}(\text{Int}H)) \).

Case (ii) Let \( x \) be a limit point of \( H \). Then every open set \( U_x \) containing \( x \) has a non-empty intersection with \( \text{Int}H \). But this implies

\[
\begin{align*}
  f^{-1}(U_x) \cap \text{Clf}^{-1}(\text{Int}H) &\neq \varnothing \\
  \Rightarrow \quad f\left(f^{-1}(U_x) \cap \text{Clf}^{-1}(\text{Int}H)\right) &\neq \varnothing \\
  \Rightarrow \quad U_x \cap f(\text{Clf}^{-1}(\text{Int}H)) &\neq \varnothing .
\end{align*}
\]

Therefore \( x \in \text{Clf}(\text{Clf}^{-1}(\text{Int}H)) \) and hence \( H \subseteq \text{Clf}(\text{Clf}^{-1}(\text{Int}H)) \).

(B) If \( F = \varnothing \) then the result follows trivially. Let \( F \in R(X) - \{\varnothing\} \). Then

\[ \text{ClInt}(\text{Clf}(F)) \subseteq \text{Clf}(F). \]  \hspace{1cm} (1)

For the reverse containment, we shall show that \( \text{Clf}(F) \cap (Y - \text{ClInt}\text{Clf}(F)) = \varnothing \). Let \( G = Y - \text{ClInt}\text{Clf}(F) \). Suppose \( G \cap \text{Clf}(F) \neq \varnothing \). Then \( G \) being open and \( f \) being a density preserving epimorphism, we have

\[
\begin{align*}
  G \cap f(F) &\neq \varnothing \\
  \Rightarrow \quad f^{-1}(G) \cap F &\neq \varnothing \\
  \Rightarrow \quad f^{-1}(G) \cap \text{Int}F &\neq \varnothing 
\end{align*}
\]

Let \( H = \text{Cl}(f^{-1}(G) \cap \text{Int}F) \). Then

\[
\varnothing \neq \text{IntClf}(H) = \text{IntClf}(\text{Clf}^{-1}(G) \cap \text{Int}F) \subseteq G \cap \text{IntClf}(F) = \varnothing ,
\]

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which is a contradiction. Therefore our assumption that $G \cap f(F) \neq \emptyset$ is wrong. Hence

$$\text{Clf}(F) \subseteq \text{ClInt}(\text{Clf}(F)). \quad (2)$$

From (1) and (2), we have $\text{ClInt}(\text{Clf}(F)) = \text{Clf}(F)$, whenever $F \in R(X)$.

**Note.** (1) Observe that the first projection of the space $N\times I$ in Example 6.1.4 (b) shows that a continuous image of a nearly Hausdorff need not be a nearly Hausdorff space.

(2) In the same example if we consider the second projection of $N\times I$ onto $[0, 1]$ with the cofinite topology then we get that even a continuous density preserving image of a nearly Hausdorff space need not be a nearly Hausdorff space.

2. **The space $rX$.**

   In this section we obtain "$\beta X$ like" compactification for a nearly Hausdorff space $X$ with property $\pi$. Consider the family $R_f(X)$ of finite intersections of members of $R(X)$, where $R(X)$ is the family of all regular closed subsets of $X$. For a topological space $X$, an $\alpha \subseteq R_f(X) - \{\emptyset\}$ is called an $r-$filter if it is closed under finite intersections and superset. A maximal $r-$filter is called an $r-$ultrafilter. A filter $\alpha$ is said to be fixed (free) depending upon whether $\cap \alpha$ is non-empty (empty).

**Lemma 6.2.1.** Let $X$ be a nearly Hausdorff space. Then,
(i) for each \( x \in X \), there exists a unique \( r \)-ultrafilter \( \alpha_x \) such that 
\[ \cap \alpha_x = \{x\}, \text{ where } \alpha_x = \{F \in \mathcal{R}(X) | x \in F\}. \]

(ii) \( X \) is a compact space if and only if each \( r \)-ultrafilter in \( X \) is fixed.

Proof. (i) Follows from the fact that for each pair of distinct points of \( X \) there exist regular closed sets separating them.

(ii) If \( X \) is a compact space, then each \( r \)-ultrafilter being a family of closed sets with finite intersection property, has arbitrary intersection non-empty.

Converse follows from the fact that for each \( r \)-ultrafilter \( \alpha \), the family of open sets \( C_\alpha = \{X - F | F \in \alpha\} \) is such that if no finite sub collection of \( C_\alpha \) covers \( X \) then \( C_\alpha \) does not cover \( X \).

For a nearly Hausdorff space \( X \) with property \( \pi \), denote by \( rX \), the set of all \( r \)-ultrafilters in \( X \). Further for \( F \in \mathcal{R}(X) \) define 
\[ \overline{F} = \{\alpha \in rX | F \in \alpha\}. \] Topologize the set \( rX \) by taking \( B = \{\overline{F} | F \in \mathcal{R}(X)\} \) as a base for closed sets in \( rX \). We use the following result from [3] to show that \( B \) is a base for closed sets in \( rX \).

A collection \( B \) of subsets of a set \( X \) is a closed base for a topological space \( X \) if and only if the following conditions are satisfied:

(i) The intersection of members of \( B \) is empty.

(ii) For each \( F_1 \) and \( F_2 \) in \( B \) and \( x \notin F_1 \cup F_2 \), there exists an \( F \) in \( B \) such that \( x \notin F \supseteq F_1 \cup F_2 \).
Lemma 6.2.2. Let $X$ be a nearly Hausdorff space with property $\pi$. Then the set $B = \{ \overline{F} \mid F \in R(X) \}$ forms a base for closed sets in $rX$.

Proof. Observe that for $F_1, F_2 \in B$, $F_1 \cup F_2 = \overline{F_1} \cup \overline{F_2}$. Let $\alpha \in \overline{F_1} \cup \overline{F_2}$. Then $F_1 \cup F_2 \in \alpha$ and $\alpha$ is an $r$-ultrafilter implies that either $F_1 \in \alpha$ or $F_2 \in \alpha$, i.e. either $\alpha \in \overline{F_1}$ or $\alpha \in \overline{F_2}$. Hence $\alpha \in \overline{F_1} \cup \overline{F_2}$. This proves that $\overline{F_1} \cup \overline{F_2} \subseteq \overline{F_1} \cup \overline{F_2}$. The reverse containment can be proved similarly. Since $X$ is a nearly Hausdorff space it follows that intersection of members of $B$ is empty. Hence $B$ is a base for closed sets in $rX$.

Note. Let $X$ be a nearly Hausdorff space with property $\pi$. Then for each $F \in R(X)$, $\text{Cl}_X \text{Int}_X \overline{F} = \overline{F}$. Clearly $\text{Cl}_X \text{Int}_X \overline{F} \subseteq \overline{F}$. To observe the reverse containment, let $\alpha \in \overline{F}$. If possible suppose $\alpha \notin \text{Cl}_X \text{Int}_X \overline{F}$. Then there exists an open set $U$ in $rX$ containing $\alpha$ such that

$$U \cap (\text{Int}_X \overline{F}) = \emptyset$$

$$\Rightarrow U \cap \text{Int}_X F = \emptyset$$

$$\Rightarrow U \cap F = \emptyset,$$

a contradiction since $U$ is an open set containing $\alpha$ and $F \in \alpha$.

Theorem 6.2.3. Let $X$ be a nearly Hausdorff space with property $\pi$. Then the space $rX$ of all $r$-ultrafilters in $X$ is a compact $nh$-space which contains $X$ as a dense subspace.
Proof. Define $r : X \rightarrow rX$ by $r(x) = \alpha_x$ where $\alpha_x = \{ F \in Rf(X) \mid x \in F \}$. We first prove that $\alpha_x$ is an $r$-ultrafilter. Clearly, $\varnothing \notin \alpha_x$ and $X \in \alpha_x$. Also $\alpha_x$ is closed under finite intersections and supersets. We now prove that $\alpha_x$ is a maximal subfamily of $Rf(X)$ with finite intersection property.

Let $A \in Rf(X)$ be such that $A \cap F \neq \varnothing$, for all $F \in \alpha_x$. Let $A = \bigcap_{i=1}^{n} A_i$ where $A_i \in R(X)$ for each $i \in \{1, 2, \ldots, n\}$. Now,

$$A \cap F \neq \varnothing \quad \text{for all } F \in \alpha_x$$

$$\Rightarrow \quad A_i \cap F \neq \varnothing \quad \text{for all } F \in \alpha_x.$$  

It is sufficient to prove that $A_i \in \alpha_x$ for each $i \in \{1, 2, \ldots, n\}$. If possible, suppose $A_i \notin \alpha_x$ for some $i$. This implies $x \notin A_i$. Since $X$ has property $\pi$, there exists $H$ in $R(X)$ such that $x \in \text{Int}H$ and $H \cap A_i = \varnothing$. Since $x \in H$ we have $H \in \alpha_x$. But this contradicts $A_i \cap F \neq \varnothing$ for each $F \in \alpha_x$. Therefore our assumption that $A_i \notin \alpha_x$ for some $i$, is wrong. This prove $\alpha_x$ is an $r$-ultrafilter.

That the map $r$ is well defined and is one-one follows by Lemma 6.2.1.(i). We now prove that $r(F) = \overline{F} \cap r(X)$, where $F \in R(X)$. Note that

$$\alpha_x \in \overline{F} \cap r(X) \Leftrightarrow F \in \alpha_x, \alpha_x \in r(F) \Leftrightarrow x \in F, \alpha_x \in r(X) \Leftrightarrow \alpha_x \in r(F).$$

The identity $r(F) = \overline{F} \cap r(X)$ implies $r^{-1}(\overline{F}) = F$, i.e. inverse image of every basic closed set in $rX$ is closed in $X$. This proves $r$ is continuous. That the map $r$ is a closed map onto its image follows from the fact that
$r(F) = \overline{F} \cap r(X)$ and the fact that the family $Rf(X)$ form a base for closed sets in $X$.

We now prove that $Cl_{rX}r(F) = \overline{F}$, where $F \in R(X)$. The identity $r(F) = \overline{F} \cap r(X)$, $F \in R(X)$, implies $Cl_{rX}r(F) \subseteq \overline{F}$. For the reverse containment let $\overline{K}$ be a basic closed set in $rX$ containing $r(F)$. Then $\overline{K} \supseteq \overline{F}$ since

$$r(F) \subseteq \overline{K}$$

$$\Rightarrow \{\alpha \in rX \mid x \in F\} \subseteq \overline{K}$$

$$\Rightarrow K \in \alpha$$ for each $x \in F$

$$\Rightarrow x \in K$$ for each $x \in F$

$$\Rightarrow F \subseteq K$$

$$\Rightarrow \overline{F} \subseteq \overline{K}.$$ Therefore every basic closed set $\overline{K}$ containing $r(F)$ contains $\overline{F}$. Since $Cl_{rX}r(F)$ is the intersection of all closed sets in $rX$ containing $r(F)$, it follows that $Cl_{rX}r(F) = \overline{F}$.

We now establish that the space $rX$ is compact. Let $\{\overline{F}\}_{\lambda}$ be a family of basic closed sets in $rX$ with finite intersection property, where $\lambda$ is a subfamily of $Rf(X)$. Observe that the family $\lambda$ also has the finite intersection property. For if $\bigcap_{i=1}^{n} F_i = \varnothing$, $F_i \in \lambda$ for each $i \in \{1, 2, \ldots, n\}$ then

$$\bigcap_{i=1}^{n} \overline{F_i} = \{\alpha \in rX \mid \bigcap_{i=1}^{n} F_i \in \alpha\}$$
which is a contradiction. An \( r \)-ultrafilter is a maximal subfamily of \( Rf(X) \) with finite intersection property. Hence \( \lambda \) is contained in some \( r \)-ultrafilter say \( \alpha \). Now

\[ \alpha \in \bigcap_{F \in \alpha} F \subseteq \bigcap_{K \in \lambda} K \]

proves that \( \bigcap_{K \in \lambda} K \neq \emptyset \). Hence \( rX \) is compact.

We note that the space \( rX \) is an nh-space. Let \( \alpha \) and \( \zeta \) in \( rX \) be two distinct \( r \)-ultrafilters. Then \( \alpha \neq \zeta \) implies there exists \( F \in \alpha \) such that \( F \notin \zeta \). Now \( F \notin \zeta \) implies that there exists \( H \in \zeta \) such that \( F \cap H = \varnothing \). The regular closed sets \( \overline{F} \) and \( \overline{H} \) separate \( \alpha \) and \( \zeta \) respectively.

**Note.** If a space \( X \) with property \( \pi \) is a non-Hausdorff, nearly Hausdorff space then \( rX \) cannot be Hausdorff. For, if \( rX \) is Hausdorff then \( X \) being subspace of a compact Hausdorff space must be a completely regular Hausdorff space, which is a contradiction.

**Example 6.2.4.** Following example justifies that a one point compactification of a non-Urysohn Hausdorff space without property \( \pi \) can be a nearly Hausdorff. Consider the subspace \( Y = \{(\frac{1}{n}, \frac{1}{m}) | n \in \mathbb{N}, m \in \mathbb{N}\} \cup \{\frac{1}{n}, 0) | n \in \mathbb{N}\} \) of the usual Euclidean space \( \mathbb{R}^2 \). Set \( X = Y \cup \{p, q\} \), where \( p, q \notin Y \) and topologize \( X \) by taking sets open in \( Y \) as open in \( X \) and a set \( U \) containing \( p \) (respectively \( q \)) is open in \( X \) if for some \( r \in \mathbb{N} \),
\{(\frac{1}{n}, \frac{1}{m})| n \geq r, m \in \mathbb{N}\} \subseteq U \quad \text{(respectively) \quad \{(\frac{1}{n}, \frac{1}{m})| n \geq r, -m \in \mathbb{N}\} \subseteq U\).  The resulting space \(X\) is a non-Urysohn Hausdorff space without property \(\pi\) and its one point compactification is a nearly Hausdorff space.

The space \(X\) is not a Urysohn space follows from the fact that distinct points \(p\) and \(q\) cannot be separated by open sets such that their closures are disjoint.

Let \(Z = X \cup \{(0,0)\}\). Topologize \(Z\) by declaring sets open in \(X\) as open in \(Z\) and the open sets about \((0,0)\) are those inherited from the subspace of Euclidean space \(\mathbb{R}^2\). Resulting space \(Z\) is a compact nh-space.

Let \(\mu\) be an open cover of \(Z\). Then choose open sets \(U, V, W\) containing \(p, q\) and \((0,0)\) respectively. Let \(n\) be the largest natural number such that \((\frac{1}{n}, 0) \in U \cup V \cup W\). Choose basic open sets \(U_r\) about each \((\frac{1}{r}, 0)\), \(1 \leq r \leq n\). Then the open set \(U \cup V \cup W \cup \left(\bigcup_{r=1}^{n} U_r\right)\) covers all but finitely many points of \(Z\). The remaining finitely many points of \(Z\) are isolated points. This proves that \(Z\) is compact.

We now show that \(Z\) is a nearly Hausdorff space. Let \(x, y \in Z, x \neq y\).

Then we consider the following cases:

Case (i) Let \(x, y \in Y \cup \{(0,0)\}\). Then \(x\) and \(y\) be separated by regular closed sets as \(Y \cup \{(0,0)\}\) inherits the usual Euclidean space \(\mathbb{R}^2\).

Case (ii) Let \(x = p\) and \(y = (0,0)\). Then the regular closed sets

\[F = \{(\frac{1}{n}, \frac{1}{m})| 1 \leq m \leq r, n \in \mathbb{N}\} \cup \{p\}\]

and
\[ H = \{(\frac{1}{n}, 0) | n \in \mathbb{N}\} \cup \{(\frac{1}{n}, \frac{1}{m}) | n \in \mathbb{N}, -m \in \mathbb{N}\} \cup \{q\} \]

separate \( x \) and \( y \).

Case (iii) Let \( x = q \) and \( y = (0,0) \). Then the regular closed sets

\[ F = \{(\frac{1}{n}, \frac{1}{m}) | -r \leq -m \leq -1, n \in \mathbb{N}\} \cup \{q\} \]

and

\[ H = \{(\frac{1}{n}, 0) | n \in \mathbb{N}\} \cup \{(\frac{1}{n}, \frac{1}{m}) | n \in \mathbb{N}, m \in \mathbb{N}\} \cup \{p\} \]

separate \( x \) and \( y \).

Case (iv) Let \( x = p \) and \( y = q \). Then the regular closed sets

\[ F = \{(\frac{1}{n}, \frac{1}{m}) | 1 \leq m \leq r, n \in \mathbb{N}\} \cup \{p\} \]

and

\[ H = \{(\frac{1}{n}, \frac{1}{m}) | -r \leq -m \leq -1, n \in \mathbb{N}\} \cup \{q\} \]

separate \( x \) and \( y \).

**Theorem 6.2.5.** Let the spaces \( X \) and \( rX \) be as in Theorem 6.2.3. Then \( X \) is \( C^* \)-embedded in \( rX \).

**Proof.** Let \( f \in C^*(X) \). Suppose image of \( f \subseteq [0,1] = I \). For \( \alpha \) in \( rX \), define

\[ f^\#(\alpha) = \{ H_1 \cup H_2 \in R \mid Cl_X f^{-1}(Int H_1 \cup Int H_2) \in \alpha \} \].

Observe that \( f^\#(\alpha) \) satisfies finite intersection property. In fact for \( F, H \) in \( f^\#(\alpha) \), \( Cl_X f^{-1}(Int H) \),\( Cl_X f^{-1}(Int F) \) \( \in \alpha \) and therefore

\[ Cl_X f^{-1}(Int H) \cap Cl_X f^{-1}(Int F) \neq \emptyset \]

\[ \Rightarrow \emptyset \neq \emptyset \left( (Cl_X f^{-1}(Int H)) \cap (Cl_X f^{-1}(Int F)) \right) \]

\[ \subseteq f((Cl_X f^{-1}(Int H)) \cap f((Cl_X f^{-1}(Int F))) \]
Let $\alpha \in rX$. Then choose an open set $G$ of $I$ such that $rf(\alpha) \in G$. If $rf(\alpha) = t$ then using regularity of $I$ successively we obtain open sets $G_1, G_2$ satisfying
\[ t \in G_1 \subseteq \overline{G_1} \subseteq G_2 \subseteq \overline{G_2} \subseteq G. \]

Set \( F_i = \text{Cl}_{1-G_2} \) and \( H_i = \text{Cl}_{1-(1-\text{Cl}_{1-G_i})} \). Since \( \text{Int}_1 F_i \cup \text{Int}_1 H_i = 1 \), we have \( F_i \cup H_i \in f^*(\alpha) \) and as \( t \notin H_i \), \( F_i \in f^*(\alpha) \) and \( H_i \notin f^*(\alpha) \). If \( K_i = \text{Cl}_x f^{-1}(\text{Int}_1 F_i) \) and \( L_i = \text{Cl}_x f^{-1}(\text{Int}_1 H_i) \) then \( \alpha \notin \overline{L_i} \) and the open set \( rX - \overline{L_i} \) contains \( \{\alpha\} \). Finally the containment \( rf(rX - \overline{L_i}) \subseteq G \) establishes the continuity of \( rf \). For the assertion, one may use the above technique to note that \( \{F \in R(I) | t \in \text{Int}_1 F\} \subseteq f^*(\alpha) \).

**Theorem 6.2.6.** Let \( X \) be a nearly Hausdorff space with property \( \pi \). Then there exists a compact nearly Hausdorff space \( rX \) in which \( X \) is densely C*-embedded.

**Proof.** Follows from Theorem 6.2.3 and Theorem 6.2.5.

**Corollary 6.2.7.** If \( X \) is a regular Hausdorff space, then it is densely C*-embedded in \( rX \).

**3. When \( rX = \beta X \)?**

In this section we answer the natural question when \( rX = \beta X \)? We observe that if \( Rf(X) \) forms a Wallman base for a nearly Hausdorff space \( X \) then \( rX = \beta X \). As a consequence we have that if \( X \) is normal or zero-dimensional then \( rX = \beta X \).
Lemma 6.3.1. Let $X$ be a normal space and let $R_f(X)$ be the collection of all finite intersections of members of $R(X)$. Then $R_f(X)$ is a Wallman base.

Proof. Clearly $R_f(X)$ is closed under finite intersections and finite unions. We observe the following:

(i) $\varnothing, X \in R_f(X)$.

(ii) Note that $R_f(X)$ forms a closed base for $X$: Since $X$ is a normal space, $\cap R_f(X) = \varnothing$. Further for each $F, H \in R_f(X)$ such that $x \notin F \cup H$ implies that there exist disjoint open sets $U$ and $V$ such that $x \in U$, $F \cup H \subseteq V$ and $U \cap V = \varnothing$. Clearly, $V \in R_f(X)$ and $x \notin V \supseteq F \cup H$.

(iii) Let $A \in R_f(X)$ and $x \in X - A$. Then $X$ being a normal space, there exists an open set $V$ such that $x \in \overline{V} \subset X - A$.

(iv) Let $A, B \in R_f(X)$ be such that $A \subseteq X - B$. Since $X$ is a normal space, the closed set $A \subseteq X - B$ implies that there exists an open set $U$ such that

$$A \subseteq U \subseteq \overline{U} \subseteq X - B \quad (1)$$

Further,

$$A \subseteq U \Rightarrow X - U \subseteq X - A.$$  

Since $X$ is a normal space there exists open set $W$ such that $X - U \subseteq W \subseteq \overline{W} \subseteq X - A$ which implies $X - U \subseteq \overline{W} \subseteq X - A$ which in turn gives

$$A \subseteq X - \overline{W} \subseteq U \subseteq \overline{U} \quad (2)$$

From (1) and (2), it follows that $A \subseteq X - \overline{W} \subseteq \overline{U} \subseteq X - B$. Therefore for $A \subseteq X - B$, there exist $\overline{U}, \overline{W} \in R_f(X)$ such that $A \subseteq X - \overline{W} \subseteq U \subseteq \overline{U}$. 89
Lemma 6.3.2. Let $X$ be a nearly Hausdorff space such that $Rf(X)$ is a Wallman base. Then $X$ is a regular space.

Proof. Let $x \in X$ and let $F$ be a closed subset of $X$ such that $x \notin F$. Since $Rf(X)$ forms a base for closed sets in $X$, $F = \bigcap_{H \in \beta} H$, where $\beta$ is some subfamily of $Rf(X)$. Now, $x \notin F$ implies that $x \notin H$ for some $H \in \beta$. Since $Rf(X)$ forms a Wallman base,

$$x \notin H, \quad H \in Rf(X)$$

$$\Rightarrow \quad \text{there exists } K \in Rf(X) \text{ such that } x \in K \text{ and } K \cap H = \emptyset$$

Further, $K \cap H = \emptyset$ implies that $H \subset X - K$. Again using the fact that $Rf(X)$ is a Wallman base, there exist $C, D \in Rf(X)$ such that

$$H \subset X - C \subset D \subset X - K.$$ 

The open sets $X - C$ and $X - D$ are disjoint and contain $F$ and $x$, respectively.

Theorem 6.3.3. Let $X$ be a nearly Hausdorff space such that $Rf(X)$ is a Wallman base. Then $rX = \beta X$.

Proof. It is sufficient to show that $rX$ is Hausdorff. Let $x, y \in rX, x \neq y$. Then there exist $r$-ultrafilters $\alpha_x$ and $\alpha_y$ in $rX$ with limit points $x$ and $y$, respectively. Since $\alpha_x$ and $\alpha_y$ are distinct $r$-ultrafilters, there exist $F \in \alpha_x$ such that $F \cap K = \emptyset$, for some $K \in \alpha_y$. Therefore $F \subset X - K$. Since $Rf(X)$ is a Wallman base therefore there exist $E$ and $H$ in $Rf(X)$ such that

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Then \( rX - \overline{H} \) and \( rX - \overline{E} \) are disjoint open sets containing \( x \) and \( y \), respectively. Hence \( rX \) is a Hausdorff space.

**Corollary 6.3.4.** A nearly Hausdorff space \( X \) for which \( Rf(X) \) forms a Wallman base is a Tychonoff space.

*Proof.* Follows from Theorem 6.3.3 since \( X \) is a subspace of \( \beta X \).

**Corollary 6.3.5.** If \( X \) is a normal space or zero-dimensional space then \( rX = \beta X \).

*Proof.* Follow from Lemma 6.3.1 and Theorem 6.3.3.