3.1 Definitions and Notations:

Let $\sum a_n$ be a given infinite series and $s_n$ its $n$th partial sum. Let $t^p_n$ and $t^q_n$ denote the $(N, p_n)$ and $(N, q_n)$ means of sequence $\{s_n\}$ respectively. Other definitions and notations will be the same as in Chapter I.

Throughout the chapter, we shall write:

(i) $\frac{P_n}{P_{n-1}} = (n, n)'$

(ii) $\overline{l}_n$ and $s'_n$ to denote the $(N, \frac{1}{n+1})$ and $(c, l)$ mean of the sequence $\{s_n\}$ respectively.

(iii) $\{\lambda_n\}$ to denote a positive non-decreasing sequence of $n$.

It is to be noted that summability $|\overline{N}, q_n|$ $(q_n > 0, q_n \to \infty)$ is absolutely regular.

3.2 For absolute summability of infinite series the following inclusion relations are known:
Since the converse relations do not necessarily hold, the natural question arises as to what suitable sequences of factors may be multiplied with the term of a series to ensure the converse relations.

This question has been answered in the affirmative by number of authors.

Singh\(^1\) established the following:

**Theorem 3.1** If \( \sum a_n \) is summable \(|c,1|\), then
\[
\sum n^{-1} \log (n+1) a_n \mbox{ is summable } |N,-\frac{1}{n+1}| .
\]

In a different context Kishore\(^2\) established the following result which also generalizes Theorem 3.1.

**Theorem 3.2** If a series \( \sum a_n \) is summable \(|c,1|\) and if \( \{p_n\} \) be a non-decreasing sequence of real and non-negative numbers, then the series \( \sum a_n p_n/\sqrt{n} \) is summable \(|N,p_n|\).

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(1) Singh [56] 
(2) Kishore [27]
Das, Srivastava and Mohapatra \(^1\) established the following results:

Theorem 3.3: Let \( \sum a_n \) be summable \(|c, \alpha|, 0 < \alpha < 1 \)
then \( \sum a_n \epsilon_n \) is summable \(|N, \frac{1}{n+1}\), if 

(i) \( \epsilon_n = O(\log n/n^\alpha) \)

(ii) \( n \Delta \epsilon_n = O(1) \)

Theorem 3.4: The necessary and sufficient conditions that \( \sum a_n \epsilon_n \) should be summable \(|N, \frac{1}{(n+1)}\) whenever \( \sum a_n \) is summable \(|c, 1|\), are the following:

(i) \( \epsilon_n = O(\log n/n) \)

(ii) \( n \epsilon_n = O(1) \)

Theorem 3.5: If \( \sum a_n \) is summable \(|c, \alpha|, \alpha > 0\), then 
\( \sum (\log n)/n^\alpha \) is summable \(|N, \frac{1}{n+1}\).

Recently, L.S. Bosanquet and G. Das, in a joint note, have proved the following theorem:

Theorem 3.6: The necessary and sufficient conditions that \( \sum \epsilon_n a_n \) should be summable \(|N, \frac{1}{n+1}\) whenever \( \sum a_n \) is summable \(|c, \alpha|\) \((\alpha > 0)\) are:

\(^{1}\) Mohapatra \([9]\), Theorem 1, 2, 3.
\( (i) \quad \varepsilon_n = 0 \left( \log n / n^\alpha \right) \)

\( (ii) \quad n^\alpha \varepsilon_n = O(1) \)

Instead of restricting to two of the methods of inclusion chain at a time, Mohapatra recently considered two fairly general methods which contain all the methods referred to have (in the inclusion chain) as special cases. He has proved the following:

**Theorem 3.7** 2) Let the sequences \( \{p_n\} \) and \( \{q_n\} \) be such that the following hold:

\[ (3.2.1) \quad p_n > 0 \text{ and } \{p_n\} \text{ is monotonic non-decreasing} \]

\[ (3.2.2) \quad (P_n/p_n) = O(f_n), \text{ where } \{f_n\} \text{ is a positive non-diminishing sequence.} \]

\[ (3.2.3) \quad n f_n p_n/p_n = O(1) ; \]

\[ (3.2.4) \quad q_0 > 0, \quad q_n \geq 0 (n \geq 1), \quad \mathcal{Q}_n \to \infty, \quad \text{and} \quad q_n p_n/q_n = O(1) . \]

Then the necessary and sufficient conditions that \( \Sigma a_n \varepsilon_n \) should be summable \( |N,p_n| \) whenever \( \Sigma a_n \) is summable \( |N,q_n| \), are

1. Mohapatra [41]
2. Mohapatra [41], chapter II, theorem 1.
(3.2.5) $\varepsilon_n = o(q_n p_n / q_n)$,
(3.2.6) $\Delta \varepsilon_n = o(q_n / q_n)$.

Theorem 3.8 Let the sequences $\{p_n\}$ and $\{q_n\}$ satisfy (3.2.2), (3.2.3) and the following:

(3.2.7) $p_n > 0$ and $\{p_n\}$ is monotonic non-diminishing
(3.2.8) $q_0 > 0$, $q_n \geq 0$ ($n \geq 1$) and $n q_n | q_n = o(1)$.

Then $\sum a_n \xi_n$ is summable $|N, p_n|$ whenever $\sum a_n$ is summable $|N, q_n|$ if

(3.2.9) $\varepsilon_n = o(n q_n | q_n)$,

(3.2.10) $\Delta \varepsilon_n = o(qn | q_n)$.

Now we observe that the summability $(F, \log n, 1)$ of a given series does not in general imply its absolute cesaro summability of any positive order and a fortiori absolute Harmonic summability of the same, for there exist the series

$$\sum_{n=1}^{\infty} \frac{1}{n^c}$$

($c > 0$)
which is summable \((R, \log n, 1)\) but not even summable 
\((A)^1\), and therefore not summable \(|A|\).

In view of this, Mahapatra\(^2\) obtained a suitable 
sequence of factors so that the series with factors is 
summable \(\{N, p_n\}\) under an assumption of some what 
weaker nature than the summability \((\bar{N}, q_n)\) of the given 
series. He proved the following

**Theorem 3.8.** Let the sequences \(\{p_n\}\) and \(\{q_n\}\) satisfy 
\((3.2.1), (3.2.2), (3.2.3)\) and \((3.2.11)\)

\((3.2.11)\) \(q_0 > 0, q_n > 0 (n \geq 1)\) and \(q_{n+1}/q_{n+1} = O(q_n)\)

Then the necessary and sufficient conditions

that \(\sum q_n e_n\) is summable \(|N, p_n|\) whenever \(t^\n = O(\lambda_n)\)

are

\((3.2.12)\) \(\sum_{n=1}^{\infty} \{ |e_n| q_n \lambda_n \left| p_n q_n \right| < \infty \} \).

\((3.2.13)\) \(\sum_{n=1}^{\infty} q_n \lambda_n \left| \Delta(e_n, q_n) \right| < \infty \).

\(\sum_{n=1}^{\infty} \left\{ |e_n| q_n \lambda_n \left| p_n q_n \right| < \infty \} \).

\(\sum_{n=1}^{\infty} q_n \lambda_n \left| \Delta(e_n, q_n) \right| < \infty \).

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1. Hardy [22] p. 163
2. Mohapatra [41] Chapter II
Theorem 3.10 1) : Let the sequences \( \{p_n\} \) and \( \{q_n\} \) satisfy (3.2.2), (3.2.3), (3.2.7) and (3.2.11). Then 
\[
\sum_{n=1}^{\infty} |\epsilon_n| \lambda_n / (n+1)q_n < \infty;
\]
(3.2.14) \[
\sum_{n=1}^{\infty} \eta_n \lambda_n / (n+1) \eta_n < \infty,
\]
(3.2.15) \[
\sum_{n=1}^{\infty} \eta_n \lambda_n |\Delta(\Delta \epsilon_n/q_n)| < \infty.
\]

3.3 It will not be out of place to mention the following results which Mohapatra 2) deduced from his theorems 3.7 - 3.11.

Theorem 3.11 2) : The series \( \sum a_n \epsilon_n \) is summable 
\[
|c, \alpha| (0 \leq \alpha \leq 1) \text{ whenever } \sum a_n \text{ is summable } |c, 1|, \text{ if and only if }
\]
\[
\epsilon_n = O(n^{\alpha-1}),
\]
\[
n \Delta \epsilon_n = O(1).
\]

Theorem 3.12 3) : The series \( \sum a_n \epsilon_n \) is summable 
\[
|c, \alpha| (0 \leq \alpha \leq 1) \text{ whenever } \sum a_n \text{ is summable } |R, \log n, 1|, \text{ if and only if }
\]
\[
\log n \epsilon_n = O(n^{\alpha-1})
\]
\[
\log n \epsilon_n = O(n^{\alpha-1}).
\]

1. Mohapatra [41], Chapter II, Theorem 4.
2. Mohapatra [41], Chapter II.
3. This result is contained in Chow [14] and Peyerimhoff.
4. Mohapatra [41], Chapter II.
Theorem 3.13: The series $\sum a_n \varepsilon_n$ is summable

$| N, \frac{1}{(n+1)} |$, whenever $\sum a_n$ is summable $| R, \log n, 1 |$, if and only if

$$\varepsilon_n = O(n^{-1})$$

$$\Delta \varepsilon_n = O(\frac{1}{n \log n})$$

Theorem 3.14: The series $\sum a_n \varepsilon_n$ is summable

$| c, \alpha |, (\alpha > 1)$, whenever $\sum a_n$ is summable $| R, \log n, 1 |$

if,

$$\varepsilon_n = O(\frac{1}{\log n})$$

$$\Delta \varepsilon_n = O(\frac{1}{n \log n})$$

Theorem 3.15: The series $\sum a_n \varepsilon_n$ is summable $| c, \alpha |$

$(0 \leq \alpha \leq 1)$ whenever $\bar{L}_n = O(\lambda_n)$ if and only if

$$\sum_{n=1}^{\infty} n^{1-\alpha} |\varepsilon_n| \log n \lambda_n < \infty,$$

$$\sum_{n=1}^{\infty} \log n \lambda_n |\Delta((n+1)\Delta \varepsilon_n)| < \infty.$$

Theorem 3.16: The series $\sum a_n \varepsilon_n$ is summable $| N, \frac{1}{(n+1)} |$

whenever $\bar{L}_n = O(\lambda_n)$ if and only if

$$\sum_{n=1}^{\infty} n |\varepsilon_n| \lambda_n < \infty;$$

$$\sum_{n=1}^{\infty} \lambda_n \log (n+1) |\Delta((n+1)\Delta \varepsilon_n)| < \infty.$$

Mohapatra also obtains theorem 3.4 as a corollary.

1. Mohapatra [41], Chapter II, Corollary 1.