CHAPTER - II

ABSOlUTE CONVERGENCE FACTORS

2.1. Definitions and Notations.

Let $\Sigma a_n$ be a given infinite series with $s_n$ for the $n$th partial sum. Let $t^p_n$ and $t^q_n$ denote, respectively, the sequence of $(N, p_n)$ and $(\bar{N}, q_n)$ means of the sequence $\{s_n\}$. Other relevant definitions and notations will be the same as in Chapter I.

2.2. A series $\Sigma a_n$ is said to be summable $|(N, p_n)(\bar{N}, q_n)|$, if the sequence $t^p_n q \in BV$, where $t^p_n q$ denotes the $(N, p_n)$ mean of the sequence $\{t^p_n\}$. Summability $|(\bar{N}, q_n)(N, p_n)|$ is analogously defined.

The necessary and sufficient conditions for the regularity of the Norlund method $(N, p_n)$ are known to be (1.5.7) and (1.5.8).

We write

\[ (2.2.1) \quad p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad P(x) = \sum_{n=0}^{\infty} P_n x^n. \]

Both the power series in (2.2.1) are convergent for $|x| < 1$ whenever (1.5.7) holds. The sequence $\{c_n\}$ is defined by means of the identity

\[ (2.2.2) \quad \sum_{n=0}^{\infty} c_n x^n = \left( \sum_{n=0}^{\infty} p_n x^{p_n} \right)^{-1}. \]

For convenience of analysis, we use $c_m$ for $m = 1, 2, \ldots,$ in the sense that $c_{-m} = 0$ for each such $m.$
From (2.2.2), we have

(i) \( c_0 = p_0 = 1; \)

(ii) \( \sum_{j=0}^{\infty} c_{n-j} p_j = 0 \) \( (n \geq 1); \)

(iii) \( \sum_{j=0}^{n} c_{n-j} p_j = 1 \) \( (n = 0, 1, 2, \ldots) \)

The familiar inversion formula for expressing \( s_n \) in terms of its Norlund mean \( t_n \) is the following:

\[
(2.2.4) \quad s_n = \sum_{j=0}^{n} c_{n-j} p_j t_j \quad (n = 0, 1, 2, \ldots)
\]

We shall denote by \( M \), the class of sequences \( \{p_0\} \) for which the following hold.

\[
(2.2.5) \quad p_n > 0, \quad p_{n+1} \leq p_{n+2} \leq p_{n+1} \leq 1 \quad (n = 0, 1, \ldots)
\]

2.3. Concerning \( \overline{p}, q_n \) and \( \overline{N}, p_n \) sumability factors of infinite series Peyerimhoff and Das, respectively, established the following:

**Theorem 2.1.** Let the sequence \( q_n \) be non-negative and such that \( q_n \rightarrow \infty \),

\[
q_{n+1} |Q_{n+1}| = O(q_n |Q_n|).
\]

Then

\[
\sum_{n=0}^{\infty} |a_n q_n| Q_n < \infty,
\]

whenever \( \sum a_n \) is summable \( \overline{N}, q_n \).

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1) This is explicitly stated for the first time in Das [17], and is referred to by him as a special case of a more general formula contained in Mc Fadden [36]

2) Peyerimhoff [48], Theorem 17.
Theorem 2.2. Let the sequence \( \{p_n\} \) satisfy the following:

(2.3.1) \( \sum_{n=0}^{\infty} |c_n| < \infty \);

(2.3.2) \( \sum_{n=0}^{\infty} |p_n| = O(|p|) \).

Then a necessary and sufficient condition for \( \sum_{n=0}^{\infty} a_n \) to be absolutely convergent whenever \( \sum_{n=0}^{\infty} a_n \) is sumable \( |N, p_n| \) is that

\[ p_n \xi_n = O(1). \]

Recently, generalizing these results, Mohapatra established the following theorems.

Theorem 2.3. Let the sequence \( \{p_n\} \) satisfy (2.3.1) and (2.3.2). Then necessary and sufficient conditions that \( \sum_{n=0}^{\infty} a_n \) is absolutely convergent whenever \( \sum_{n=0}^{\infty} a_n \) is \( |(N, p_n) (\bar{N}, q_n)| \) are:

(2.3.3) \( \xi_n p_n q_n / q_n = O(1) \)

(2.3.4) \( \xi_n p_n-1 q_n-2 \mid q_n = O(1) \)

Theorem 2.4. Let the sequence \( \{q_n\} \) satisfy (2.3.1) and (2.3.2), and let the sequence \( \{q_n\} \) satisfy the following:

(2.3.5) \( q_n \mid Q_n = O(1) \)

\[ q_n \mid Q_n = O(\mid q_n \mid / \mid q_n \mid) \]

1) Das [16]

2) Mohapatra [41], Chapter II.
for all $n > m$. Then a necessary and sufficient condition for $\Sigma a_n c_n$ to be absolutely convergent whenever $\Sigma a_n$ is summable $|(\bar{N}, q_n) (N, \bar{p}_n)|$ is:

$$(2.3.7) \quad c_n p_n q_n / q_n = O(1).$$

Here we observe that if we put $q_n = c^n (c > 1)$ in theorem 2.3. and theorem 2.4., then we get theorem 2.2; for in this case $(\bar{N}, q) \sim (c, c)$. Again by putting $p_q = 1, p_n = 0 (n > 1)$ in theorems 1 and 2 he obtains the following:

**Theorem 2.5.** $\Sigma a_n c_n$ is absolutely convergent whenever $\Sigma a_n$ is summable $|\bar{N}, q_n|$, if and only if

$$c_n q_n / q_n = O(1),$$

and

$$c_n q_{n-2} / q_{n-1} = O(1).$$

If the sequence $\{q_n\}$ satisfy requirements of Theorem 2.1, Theorem 2.5. reduces to Theorem 2.1, for in this case

$$q_{n-2} c_n / q_{n-1} \leq K(q_{n-1} / q_{n-1})(q_n / q_n) + K(q_n) \leq K.$$  

But, if we compare the conditions necessary for the truth of Theorem 2.4 an A, we observe that if $\{q_n\}$ is positive then (2.3.5) is automatically satisfied, which (2.3.6) implies

$$q_{n+1} c_n / q_{n+1} = O(q_n / q_n),$$

for non negative $q_n$.

1) *Hardy and Reisz [25].*
Mahapatra further observed that:

on a look at the proof of theorem 2.4, we find that (2.3.7) and (2.3.5) imply the necessity of \( P_n \xi_n = O(1) \). On the other hand when this condition alone is satisfied, the second term of \( S_n, \gamma \) is bounded, and thus in effect the necessary and sufficient condition in Theorem 2.4 reduce to:

\[
\left| \frac{Q_{\gamma+1}}{Q_{\gamma+1}} P_{\gamma+1} \right| \left\{ \sum_{n=\gamma+1}^{\infty} |\xi_n| + |\Delta C_n| \right\} = K.
\]

which can be written as

\[
(2.3.6) \quad \left| \frac{Q_{\gamma+1}}{Q_{\gamma+1}} P_{\gamma+1} \right| \sum_{n=0}^{\infty} |\xi_{\gamma+n} + 1| |\Delta C_n| \leq K.
\]

Thus he also obtained the following:

**Theorem 2.7.** The necessary and sufficient conditions that

\[
\sum_{n=1}^{\infty} |a_n \xi_n| \text{ is convergent when ever } \sum_{n=1}^{\infty} |a_n| \text{ is summable}
\]

\[
|\{N_n, q_n\} (N_p) \text{ when } \{p_n\} \text{ and } \{q_n\} \text{ satisfy}
\]

(2.3.1), (2.3.2) and (2.3.5) are:

\[
P_n \xi_n = O(1),
\]

and

\[
\left| \frac{Q_{\gamma+1}}{Q_{\gamma+1}} P_{\gamma+1} \right| \sum_{n=0}^{\infty} |\xi_{n+\gamma+1}| |\Delta C_n| \leq K.
\]

He deduced following useful results as corallaries from Theorem 2.3 and 2.4.
Theorem 2.8. Let \( \{p_n\} \in M \) and \( \{q_n\} \) satisfies (2.3.5) and (2.3.6). Then \( \sum |a_n \in \sigma_n| < \infty \) whenever \( \sigma_n \) is summable \( |(N, p_n)(\overline{N}, q_n)| \) \( |(\overline{N}, q_n)(N, p_n)| \), if and only if

\[
Q_n p_n \in n / q_n = o(1)
\]

and

\[
Q_{n-2} p_{n-1} \in n | q_{n-1} = o(1)
\]

\[
[l(\sum \in n) \in q_n = o(1)]
\]

Theorem 2.9. \( \|a_n \sigma_n| < \infty \) whenever \( \sigma_n \) a is summable \( |(c,1)(N, 1/\pi+1)| \) \( |(1(N, 1/(n+1))(c,1)| \) if and only if

\[
\in n = o(1/\pi \log n).
\]

Theorem 2.10. \( \|a_n \sigma_n| < \infty \) whenever \( \sigma_n \) a is summable \( |(c,\alpha)(R, \log n, 1)| \) \( |(R, \log n, 1)(c,\alpha)| \) (\( \alpha > 0 \)), if and only if

\[
\in n = o\left(\frac{1}{\alpha+1}\right).
\]

Theorem 2.11. \( \|a_n \sigma_n| < \infty \) whenever \( \sigma_n \) a is summable \( |(N, 1/\pi+1)(R, \log n, 1)| \) \( |(R, \log n, 1)(N, 1/(n+1)| \), if and only if

\[
\in n = o\left(\frac{1}{n(\log n)^2}\right).
\]

Since Theorem 2.3. and 2.4 of general nature we proceed to produce their proofs in Sections 2.5 and 2.6, while in Section 2.4. we shall state the lemmas needed to prove these theorems.
2.4. We shall use the following lemmas for the proof of the
Theorems 2.3 and 2.4.

Lemma 1. If \( \{p_n\} \) satisfies (2.3.2), then, uniformly for
\( m \geq n \),
\[
 p_n - p_m = O(1)
\]

Lemma 2. If
\[
 t_{p,q} = (p_n)^{-1} \sum_{\nu=0}^{n} p_{n-\nu} t_{\nu}
\]
then for \( n \geq 2 \)
\[
 a_n = \sum_{\nu=0}^{n} c_{p,q} \Delta \chi_{\nu}^\prime \left( \frac{Q_n}{q_n} \Delta \chi < n - \nu \right) - \frac{Q_{n-2}}{q_{n-1}} \Delta (c_{n-1}) \cdot
\]

Lemma 3. If
\[
 t_{p,q} = (Q_n)^{-1} \sum_{\nu=0}^{n} q_{\nu} t_{\nu},
\]
then, for \( n \geq 1 \),
\[
 a_n = \sum_{\nu=0}^{n} c_{p,q} \left\{ \sum_{\mu=0}^{\nu+1} p_{\mu} \Delta \mu c_{n-\mu} - \frac{p_{\nu+1}}{q_{\nu+1}} Q_{\nu+1} \Delta \chi c_{n-\nu-1} \right\}
\]

Lemma 4. If
\[
 U_n = \sum_{\nu=1}^{\infty} S_{n,\nu} u_{\nu} \quad (n = 1, 2, \ldots)
\]
where \( \{S_{n,\nu}\} \) is a double sequence, then a necessary and
sufficient condition that the series \( \sum_{n=1}^{\infty} U_n \) be convergent
whenever

1) Knoop and Lorentz [30].
\[ \sum_{n=1}^{\infty} |u_n| < \infty, \]
\[ \sum_{n=1}^{\infty} |s_{n,y}| < K, \]

where \( K \) is a positive constant independent of \( \lambda \).

Lemma 5. Let the sequence \( \{p_n\} \) satisfy (2.3.1), (2.3.2), and sequence \( \{c_n\} \) be such that (2.3.3) holds. Then the necessary and sufficient conditions for \( \sum a_n c_n \) to be absolutely convergent whenever \( \sum a_n \) is summable is

\[ \Psi(\lambda) = \sum_{n=r}^{\infty} \left| \frac{q_{n-r-2}}{q_{n-r}} c_n \right| \sum_{r=0}^{\infty} p_r \Delta c_{n-r-1} \leq K. \]

2.5. Proof of Theorem 2.3.

The necessity of 2.3.3

writing

\[ a_n c_n = \sum_{\gamma=0}^{\infty} s_{n;\gamma} \Delta_t^p q \quad (n \geq 2), \]

we have from Lemma 2,

\[ s_{n;\gamma} = \begin{cases} 0 & (\gamma > n), \\ s_n \gamma & (\gamma \leq n). \end{cases} \]

(2.5.1)

\[ s_{n;\gamma} = \left\{ \begin{array}{ll} 0 & \gamma > n, \\ \varepsilon_n \sum_{\gamma=0}^{\infty} \frac{q_{n-r-2}}{q_{n-r}} \Delta_q c_{n-r-1} & \gamma \leq n. \end{array} \right. \]

\[ \times p_\gamma (\gamma \leq n) \]

1) Mahapatra [41], Lemma 6, Chapter II.
Now, from (2.5.1) and Lemma 4, a necessary and sufficient condition that \( \sum_{n=1}^{\infty} |a_n| \) is convergent whenever 
\( n\to\infty \), \( p, q \), 
\( \sum_{n=1}^{\infty} \Delta t_n \) is convergent is 
\( \sum_{n=1}^{\infty} |S_{n, v}| \leq K. \)

This necessitates 
\( S_{v+1, v} = O(1). \)

Since by (2.5.1)

\[
S_{v+1, v} = \frac{q_{v+1}}{q_{v+1}} \left( \sum_{r=0}^{v+1} \frac{P \Delta r \gamma_{v+1-r \gamma}^1 S_{v+1}}{q_{v+1}} - \frac{q_{v+1}}{q_{v+1}} \left( \sum_{r=0}^{v+1} P \Delta r \gamma_{v+1-r \gamma}^1 S_{v+1, v} \right) \right)
\]

\( = - \frac{q_{v+1}}{q_{v+1}} \frac{P \Delta v_{v+1} \gamma_{v+1}^1}{q_{v+1}} \)

(Since \( c_o = 1 \))

by

\[
(2.5.2) \sum_{\lambda=0}^{n} P_{\lambda} \Delta \lambda \gamma_{n-\lambda}^1 C_{n-\lambda} = \begin{cases} 0 & (n \geq 1) \\ 1 & (n = 0) \end{cases}
\]

Thus (2.3.3) is necessary.

The necessity of (2.3.4). Now that (2.3.3) holds, we have from Lemma 5, a necessary and sufficient condition for 
\( \sum_{n=0}^{\infty} |a_n| < \infty \) whenever 
\( n\to\infty \)

\( \sum a_n \) is summable \( |(N, n_{p, n})(N, n_{n})| \)
\( \Psi(\mathcal{V}) \leq K, \)

which implies the boundedness of each individual term of 
\( \Psi(\mathcal{V}) \), since each term is non-negative. The first term of 
\( \Psi(\mathcal{V}) \) that does not vanish is for \( n = \mathcal{V} + 2 \) and the bounded-
ness of this alone shows that

\[
|c_{n+2} p_{n+1} q_{\mathcal{V}_1} / q_{\mathcal{V}_1 + 1}| \leq K,
\]

is necessary.

**Sufficiency:** In view of Lemma 6, we need only to show 
that (2.3.4) implies \( \Psi(\mathcal{V}) \leq K \), when (2.3.1) and (2.3.2) 
hold

\[
\text{By (2.3.4) and since}
\]

\[
(2.5.3) \sum_{r=0}^{\mathcal{V}} p_{\mathcal{V}} \Delta^{C_{n-r}} - \sum_{r=0}^{\mathcal{V}} p_{\mathcal{V}} C_{n-r} - C_{n-r-1},
\]

we have,

\[
\Psi(\mathcal{V}) \leq K \sum_{n=\mathcal{V}+1}^{\infty} (|p_{n-1}|)^{-\mathcal{V}} \sum_{r=0}^n p_{\mathcal{V}} \Delta^{C_{n-r-1}}
\]

\[
\leq K \sum_{n=\mathcal{V}+1}^{\infty} (|p_{n-1}|)^{-\mathcal{V}} \sum_{r=0}^n p_{\mathcal{V}} \Delta^{C_{n-r-1}} +
\]

\[
+ K |p_{\mathcal{V}}| \sum_{n=\mathcal{V}+1}^{\infty} (|p_{n-1}|)^{-\mathcal{V}} C_{n-\mathcal{V}-2}
\]

\[
\leq \frac{K}{|p_{\mathcal{V}}|} \sum_{r=0}^{\mathcal{V}} p_{r} \sum_{n=r+1}^{\infty} C_{n-r-1} + K \sum_{n=\mathcal{V}+2}^{\infty} C_{n-\mathcal{V}-2}
\]

\[
\leq K
\]

by Lemma 1.
Theorem 2.3 is thus established.

2.6. Proof of Theorem 2.4.

Necessity. From Lemma 3, writing

\[ a_n \epsilon_n = \sum_{j=0}^{\infty} a_n \Delta \gamma (n \geq 1), \]

we have

\[ (2.6.1) S_n, \gamma = \begin{cases} 0 & (\forall >n) \\ \epsilon_n \left\{ \sum_{n=0}^{\infty} \frac{P_{n} \Delta \gamma C_{n+1}}{\gamma+1} - \frac{Q_{n+1}}{\gamma+1} P_{\gamma+1} C_{n+1} \right\} (\forall \leq n) \end{cases} \]

As in Theorem 2.3, a necessary condition for

\[ \sum_{n=1}^{\infty} \epsilon_n < \infty, \text{ whenever } \sum_{n=1}^{\infty} -P_{n} \gamma_q < \infty, \]

is:

\[ S_{\gamma+1, \gamma} = 0(1). \]

Now,

\[ S_{\gamma+1, \gamma} = \epsilon \left\{ \sum_{n=0}^{\infty} -P_{n} \Delta \gamma C_{\gamma+1} - \frac{Q_{\gamma+1}}{\gamma+1} P_{\gamma+1} C_{\gamma+1} \right\} \]

\[ = -\frac{Q_{\gamma+1}}{\gamma+1} P_{\gamma+1} \epsilon, \]

by (2.5.2). Thus (2.3.7) is necessary.

Sufficiency:- By Lemma 4 and (2.6.1), it will be enough to show that

\[ \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} |S_{n, \gamma}| \leq K. \]

Now, by (2.3.7), Lemma 1, (2.3.6), (2.5.3) and (2.3.5),
\[ \sum_{n=1}^{\infty} |s_{n,\gamma}| \leq \left| \frac{p_{\gamma+1}}{q_{\gamma+1}} \right| \sum_{n=\gamma+1}^{\infty} |e_{n}| |\Delta_{\gamma}c_{n-\gamma-1}| + \sum_{n=\gamma+1}^{\infty} |e_{n}| \sum_{\mu} |p_{\mu}| \Delta_{\mu} c_{n-\mu} \]

\[ \leq K \sum_{n=\gamma+1}^{\infty} |c_{n-\gamma-1}| + K \sum_{n=\gamma+1}^{\infty} |c_{n-\gamma-2}| + \sum_{\mu} |p_{\mu}| \sum_{n=\gamma+1}^{\infty} |e_{n}| |c_{n-\mu}| + |p_{\gamma+1}| \sum_{n=\gamma+1}^{\infty} |e_{n}| |c_{n-\gamma-2}| \]

\[ \leq K + K (|p_{\gamma+1}|) \sum_{\mu} |p_{\mu}| + K \sum_{n=\gamma+2}^{\infty} |c_{n-\gamma-2}| \]

\[ \leq K. \]

by (2.3.1) and (2.3.2)

This completes the proof of Theorem 2.4.