Appendix F

Excess parameter $\gamma_2(m, f_m)$ in terms of $SU(\Omega)$ Racah coefficients

The formula for $\gamma_2(m, f_m)$, given by Eq. (4.4.7), involves $\langle H^4 \rangle_{m, f_m}$. As the Hamiltonian in Eq. (4.3.1) is a direct sum of matrices in $f_2 = \{2\}$ and $\{1^2\}$ spaces, we have

$$\langle H^4 \rangle_{m, f_m} = \langle (H_{\{2\}} + H_{\{1^2\}})^4 \rangle_{m, f_m}.$$  \hspace{1cm} (F1)

Expanding the RHS of Eq. (F1) using the cyclic invariance of the averages and applying the property that terms with odd powers of $H_{\{2\}}$ and $H_{\{1^2\}}$ will vanish [see Eq. (4.3.6)], we have

$$\langle H^4 \rangle_{m, f_m} = \langle (H_{\{2\}})^4 \rangle_{m, f_m} + \langle (H_{\{1^2\}})^4 \rangle_{m, f_m} + 4\langle (H_{\{2\}})^2(H_{\{1^2\}})^2 \rangle_{m, f_m}$$
$$+ 2\langle H_{\{2\}}H_{\{1^2\}}H_{\{2\}}H_{\{1^2\}} \rangle_{m, f_m}.$$  \hspace{1cm} (F2)

Writing $H$ in terms of the unit tensors $B$'s using Eq. (4.4.3), the first two terms in Eq. (F2) will give

$$\langle H^4 \rangle_{f_2} = \frac{1}{d_2(f_m)_{\nu_1, \nu_2, \nu_3, \nu_4, \omega_{\nu_1}, \omega_{\nu_2}, \omega_{\nu_3}, \omega_{\nu_4}}} \sum_{\nu_1, \nu_2, \nu_3, \nu_4} \langle f_m v_1 | B(f_2 F_{\nu_1} \omega_{\nu_1}) | f_m v_2 \rangle$$

277
Using Eq. (4.4.5), it is easy to see that the term \( \langle H^4_{f_1} \rangle_{m_1} \) will have non-zero contribution in three cases, (i) \( \delta_{F_1,F_2} = 1, \delta_{w_1,w_2} = 1, \delta_{w_3,w_4} = 1 \); (ii) \( \delta_{F_1,F_4} = 1, \delta_{w_1,w_4} = 1, \delta_{w_2,w_3} = 1 \); and (iii) \( \delta_{F_2,F_3} = 1, \delta_{w_1,w_3} = 1, \delta_{w_2,w_4} = 1 \). The first two cases are equivalent due to cyclic invariance of the traces and they can be called direct terms whereas the third case involves cross-correlations and thus is called the exchange term. For (i) and (ii), applying the Wigner-Eckart theorem and carrying out simplifications using the properties of the Wigner coefficients (see Appendix E), the direct terms reduce to \( \langle H^4_{f_2} \rangle_{m_1} \).

Similarly, for the exchange term, reordering of the Wigner coefficients \([\text{see Eq. (E7)}]\) yields an expression in terms of a new Racah coefficient. With these, we have

\[
\langle H^4_{f_2} \rangle_{m_1} = \left[ \langle H^4_{f_2} \rangle_{m_1} \right]^2 + \lambda^4_{f_2} [d_4(F_2)]^2 d_0(f_m)
\]

\[
\times \sum_{F_1,F_2,F_3,F_4} \frac{1}{d_0(F_1)d_0(F_2)} U(f_m \bar{f}_m f_m \bar{f}_m; (F_1)_{\rho_1 \rho_2} (F_2)_{\rho_3 \rho_4})
\times \langle f_m || B(f_2 F_1) || f_m \rangle_{\rho_1} \langle f_m || B(f_2 F_2) || f_m \rangle_{\rho_2}
\]


\[
\times \langle f_m || B(f_2 F_1) || f_m \rangle_{\rho_3} \langle f_m || B(f_2 F_2) || f_m \rangle_{\rho_4}.
\]

In Eq. (F4), the multiplicity labels appearing in the new \(U\)-coefficient \([\text{this is quite different from the} U\)-coefficient appearing in Eq. (4.4.10)]\) can be easily understood from the corresponding labels in the reduced matrix elements. Similarly, we have

\[
\langle H^2_{f_2} H^2_{[2]} \rangle_{m_1} = \left\{ \langle H^2_{f_2} \rangle_{m_1} \right\} \left\{ \langle H^2_{[2]} \rangle_{m_1} \right\},
\]

278
\[
\langle H_{[2]} H_{[2]} H_{[2]} \rangle_{m}^{f_{m}} = \lambda_{m}^{2} \lambda_{m}^{2} d_{4}(\{2\}) d_{4}(\{1\}) d_{4}(f_{m})
\]

\[
\times \sum_{F_{v1}, F_{v2}, \rho_{1}, \rho_{2}, \rho_{3}, \rho_{4}} \frac{1}{d_{\Omega}(F_{v1}) d_{\Omega}(F_{v2})} U(f_{m}, f_{m}^{\rho_{1}}; F_{v1}) \rho_{1} \rho_{3} (F_{v2}) \rho_{2} \rho_{4}
\]

\[
\times \langle f_{m} \mid B(\{2\} F_{v1}) \mid f_{m} \rangle \rho_{1} \langle f_{m} \mid B(\{1\} F_{v2}) \mid f_{m} \rangle \rho_{2}
\]

\[
\times \langle f_{m} \mid B(\{2\} F_{v1}) \mid f_{m} \rangle \rho_{3} \langle f_{m} \mid B(\{1\} F_{v2}) \mid f_{m} \rangle \rho_{4}
\]

Substituting the results in Eqs. (F4), (F5a) and (F5b) in Eq. (F2) gives \( \langle H^{4} \rangle_{m}^{f_{m}} \). Using this and Eqs. (4.5.5) and (4.4.7), we have the analytical result for the excess parameter \( \gamma_{2}(m, f_{m}) \). This involves \( SU(\Omega) \) Racah coefficients with multiplicity labels and evaluation of these is in general complicated [Gl-05, Kl-09]. Similarly, evaluation of the reduced matrix elements in Eq. (F4) is also complicated. The only simple situation is, when the multiplicity labels are all unity. We denote the \( U(\Omega) \) irreps that satisfy this as \( f_{m}^{(g)} \) and we have verified that one of these irreps is \( \{4r\} \) where \( m = 4r \). For these irreps, the expression for \( \gamma_{2} \) is,

\[
\left[ \gamma_{2}(m, f_{m}^{(g)}) + 1 \right] = \left[ \langle H^{4} \rangle_{m}^{f_{m}^{(g)}} \right]^{-2}
\]

\[
\times \left\{ \sum_{f_{a}, f_{b} = \{2\}, \{1\}} \frac{\lambda_{fa}^{2} \lambda_{fb}^{2}}{d_{\Omega}(fa) d_{\Omega}(fb)} \sum_{F_{v1}, F_{v2}} \frac{d_{\Omega}(f_{m}^{(g)})}{\sqrt{d_{\Omega}(F_{v1}) d_{\Omega}(F_{v2})}}
\right\}
\]

\[
\times U(f_{m}^{(g)} f_{m}^{(g)} f_{m}^{(g)}, F_{v1}, F_{v2}) \mathcal{O}_{v1}(fa:m, f_{m}^{(g)}) \mathcal{O}_{v2}(fb:m, f_{m}^{(g)})
\]

The \( \mathcal{O}_{v}(f_{2}: f_{m}) \) in Eq. (F6) are defined by Eq. (4.5.6). They can be calculated using \( X_{UU} \) given in Table 4.4. Therefore the only unknown in Eq. (F6) is the \( SU(\Omega) \) Racah coefficient \( U(f_{m}^{(g)} f_{m}^{(g)} f_{m}^{(g)}, F_{v1}, F_{v2}) \). There are many attempts in the past to derive analytical formulation and also to develop numerical methods for evaluating general \( SU(N) \) Racah coefficients [Bi-68, Lo-70a, Lo-70, Bi-87, Bi-82, Se-88, Vi-95]. There are also attempts to derive analytical formulas for some simple class of Racah coefficients; see [Vi-95, Li-90] and references therein. In addition, there is a recent effort
to develop a new numerical method for evaluating $SU(N)$ Racah coefficients with multiplicities [Gl-05, Kl-09]. From all the attempts we made in trying to use these results, we conclude that further group theoretical work on $SU(N)$ Racah coefficients is needed to be able to derive analytical formulas for, or for evaluating numerically, the Racah coefficients appearing in Eq. (F6).