CHAPTER 1
PRELIMINARIES

This chapter contains a concise account of preliminary definitions and results on Lattice Theory, Fuzzy Set Theory, Evidence Theory and Possibility Theory.

1.1 Lattice Theory

The definitions and results are from [4, 10]

1.1.1 Definition

A partially ordered set (poset for short) \((P; \leq)\) is a nonempty set \(P\) with a partial order relation \(\leq\) defined on it. If there exists an element 0 in \(P\) such that \(0 \leq x\) for all \(x \in P\), then 0 is called the \textit{least element} of \(P\) and if there exists an element 1 in \(P\) such that \(x \leq 1\) for all \(x \in P\), then 1 is called the \textit{greatest element} of \(P\). The elements 0 and 1 are called \textit{universal bounds} of \(P\). Two elements \(x, y \in P\) are said to be \textit{comparable} if either \(x \leq y\) or \(y \leq x\). A poset \((P; \leq)\) is a \textit{chain} if every two elements in \(P\) are comparable.

1.1.2 Definition

Let \((P, \leq)\) be a poset and let \(S \subseteq P\). \(x \in P\) is called an \textit{upper bound} of \(S\) iff \(s \leq x\) for all \(s \in S\). An upper bound \(x\) of \(S\) is called \textit{least upper bound} (l.u.b) of \(S\) (or \textit{supremum} of \(S\)) iff \(x \leq y\) for any upperbound \(y\) of \(S\) and is denoted by \(\sup S\) or \(\vee S\). The notions of \textit{lower bound} of \(S\) and \textit{greatest lower bound} of \(S\) (or \textit{infimum} of \(S\)) are
defined dually. The greatest lower bound (g.l.b) of S is denoted by \( \inf S \) or \( \land S \). When S consists of only two elements say a and b, \( \lor \{a, b\} \) is denoted as \( a \lor b \) (a join b) and \( \land \{a, b\} \) is denoted as \( a \land b \) (a meet b).

1.1.3 Definition

A poset \((L; \leq)\) is called a lattice if \( x \lor y \) and \( x \land y \) exist in \( L \) for all \( x, y \in L \).

A lattice \( L \) is complete when \( \lor S \) and \( \land S \) exist in \( L \) for every nonempty subset \( S \) of \( L \). A lattice \( L \) is said to be conditionally complete if \( \lor S \) and \( \land S \) exist in \( L \) for every nonempty bounded subset \( S \) of \( L \).

1.1.4 Definition

A subset \( S \) of \( L \) is a sublattice of the lattice \((L; \land, \lor)\) if \( S \) is closed for \( \land \) and \( \lor \). A subset \( S \) of \( L \) is called convex, if \( a, b \in S \), \( c \in L \) and \( a \leq c \leq b \) imply that \( c \in S \). For \( a, b \in L \), \( a \leq b \), the lattice interval \([a, b]\) is defined as \([a, b] = \{x \in L : a \leq x \leq b\}\).

1.1.5 Lattice algebra

In any lattice \( L \), the operations of meet \((\land)\) and join \((\lor)\) satisfy the following laws.
\[ L_1 \quad x \land x = x, \quad x \lor x = x \quad \text{(idempotence)} \]

\[ L_2 \quad x \land y = y \land x, \quad x \lor y = y \lor x \quad \text{(commutativity)} \]

\[ L_3 \quad x \land (y \land z) = (x \land y) \land z \quad \text{and } \quad x \lor (y \lor z) = (x \lor y) \lor z \quad \text{(associativity)} \]

\[ L_4 \quad x \land (x \lor y) = x \lor (x \land y) = x \quad \text{(absorption)} \]

Moreover, \( x \leq y \) is equivalent to each of the following conditions:

\[ L_5 \quad x \land y = x \quad \text{and } \quad x \lor y = y \quad \text{(consistency)} \]

If \( L \) is a lattice with universal bounds 0 and 1, then

\[ L_6 \quad 0 \land x = 0 \quad \text{and } \quad 0 \lor x = x \quad \text{for all } x \in L \]

\[ 1 \land x = x \quad \text{and } \quad 1 \lor x = 1 \quad \text{for all } x \in L \]

\( L \) is a **distributive lattice** iff it satisfies either of the following equivalent identities

\[ L_7 \quad x \land (y \lor z) = (x \land y) \lor (x \land z) \]

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \text{for all } x, y, z \in L \]

1.1.6 Remark

Any chain is a distributive lattice.

1.1.7 Definition

In a bounded lattice \( L \) with bounds 0 and 1, \( y \in L \) is a **complement** of \( x \in L \) if

\[ x \land y = 0 \quad \text{and } \quad x \lor y = 1 \]
L is complemented if every element in L has complement.

1.1.8 Definition

A complemented distributive lattice is called *Boolean lattice*.

Complements are unique in any distributive lattice.

1.1.9 Result

In any Boolean lattice, each element x has one and only one complement \( x^c \).

More over,

\[
L_8 \quad (x \land y)^c = x^c \lor y^c \\
(x \lor y)^c = x^c \land y^c
\]  
(De Morgan's laws)

\[
L_9 \quad (x^c)^c = x \quad \text{(involution)}
\]

\[
L_{10} \quad \begin{cases} 
  x \land x^c = 0 \\
  x \lor x^c = 1
\end{cases}
\]  
(complementarity)

1.1.10 Definition

Boolean lattices can be considered as algebras with two binary operations \((\land, \lor)\) and one unary operation \(^c\). When so considered they are called Boolean algebras. Hence a *Boolean algebra* is an algebra with operations \(\land, \lor, ^c\) satisfying \(L_1\) to \(L_{10}\).
A De Morgan algebra (De Morgan lattice) is an algebra with operations \( \land, \lor, \complement \) satisfying $L_1$ to $L_9$.

1.1.11 Definition

An element 'a' in a lattice L is an atom if 'a' covers 0 (a > 0) i.e., if a > 0 and for no x, a > x > 0.

A complete lattice L with least element 0 is atomic if $L \setminus \{0\}$ has a nonempty subset of minimal elements known as atoms such that every element of L is the supremum of all the atoms below it.

1.1.12 Definition

Let $f : L \to M$ be a function from a lattice L to a lattice M. Then f is

(i) monotone when

\[ x \leq y \text{ implies } f(x) \leq f(y) \text{ for all } x, y \in L \]

(ii) a homomorphism when

\[ f(x \lor y) = f(x) \lor f(y) \]

and \[ f(x \land y) = f(x) \land f(y) \text{ for all } x, y \in L \]

A homomorphism is called (a) an isomorphism if it is a bijection (b) an automorphism if it is an isomorphism and $L = M$

1.1.13 Definition

A valuation on a lattice L is a real valued function $V$ on L which satisfies
\[ V(x \lor y) + V(x \land y) = V(x) + V(y) \text{ for all } x, y \in L \]

\[ V \text{ is monotone iff } x \leq y \text{ implies } V(x) \leq V(y) \text{ and positive iff } x < y \text{ implies } V(x) < V(y). \]

1.1.14 Definition

In any lattice \( L \) with a positive valuation \( V \), the distance function

\[ d(x, y) = V(x \lor y) - V(x \land y) \quad ; \quad x, y \in L \]

is a metric and \( L \) is called \textit{metric lattice}.

1.1.15 Definition

Let \( \{x_\alpha\} \) be any net in a complete lattice \( L \), where \( \alpha \) varies over some directed index set.

Define

\[ \liminf \{x_\alpha\} = \bigvee_{\beta \geq \alpha} x_\alpha \]

\[ \limsup \{x_\alpha\} = \bigwedge_{\beta \geq \alpha} x_\alpha \]

Then, \( \{x_\alpha\} \) is said to order converge to \( \alpha \) in \( L \) when

\[ \liminf \{x_\alpha\} = \limsup \{x_\alpha\} = \alpha \]
A subset $A$ of $L$ is said to be \textit{closed} if every order convergent net of elements of $A$ order converges to an element of $A$. The closed sets thus established define the \textit{order topology} on $L$.

\textbf{1.1.16 Definition}

A \textit{topological lattice} is a lattice with a specified convergence topology in which

\[ x_\alpha \to x \text{ and } y_\beta \to y \text{ imply } x_\alpha \wedge y_\beta \to x \wedge y \]

\[ x_\alpha \to x \text{ and } y_\beta \to y \text{ imply } x_\alpha \vee y_\beta \to x \vee y \]

where \{\{x_\alpha\}\} and \{\{y_\beta\}\} are nets in the lattice.

\textbf{1.1.17 Result}[4]

Any metric lattice is a topological lattice in its metric topology (the topology induced by the metric).

\textbf{1.1.18 Result [4]}

A complete distributive lattice $L$ is a topological lattice under order convergence iff it satisfies the \textit{infinite distributivity laws}:

\[ a \wedge (\bigvee x_\alpha) = \bigvee (a \wedge x_\alpha) \]

\[ a \vee (\bigwedge x_\alpha) = \bigwedge (a \vee x_\alpha) \]

for any \{\{x_\alpha\}\} and any $a$ in $L$ where $\alpha$ belongs to any arbitrary index set.
1.1.19 Definition

A lattice $L$ is Brouwerian if when $a, b \in L$, the set $\{x \in L: a \land x \leq b\}$ has a greatest element $b : a$ (called the relative pseudo complement of $a$ in $b$).

A lattice $L$ is dual Brouwerian if when $a, b \in L$, the set $\{x \in L: a \lor x \geq b\}$ has a least element $a : b$.

1.1.20 Result [4]

A complete lattice which satisfies the infinite distributive laws is both Brouwerian and dual Brouwerian.

1.1.21 Result [34]

The following laws are valid for complete lattices.

\[
\bigvee_{i,j} a_{ij} = \bigvee_{j} \bigvee_{i} a_{ij}
\]

\[
\bigwedge_{i,j} a_{ij} = \bigwedge_{j} \bigwedge_{i} a_{ij}
\]

for arbitrary indices $i$ and $j$.

1.1.22 Definition [60]

A lattice $L$ which is a semigroup under $*$ and also satisfies the distributive laws

\[
x * (y \lor z) = (x * y) \lor (x * z)
\]
\[(x \lor y) \ast z = (x \ast z) \lor (y \ast z)\]

is called a lattice ordered semi group and is denoted \( (L; \land, \lor, \ast) \) where \( \land \) and \( \lor \) are the lattice operations.

Moreover, \( L \) is said to be a lattice ordered semigroup with unity 1 and zero 0 if the following are satisfied for any \( x \) in \( L \).

\[
x \lor 0 = x \quad x \ast 0 = 0 \ast x = 0
\]

\[
x \lor 1 = 1 \quad x \ast 1 = 1 \ast x = x
\]

1.1.23 Definition [60]

A semi ring \( (R; +, \times) \) is a set \( R \) with two operations addition (+) and multiplication (\( \times \)) such that + is associative and commutative and \( \times \) is associative and distributive over +.

A semi ring is unitary if \( \times \) has a unit 1 and is commutative if \( \times \) is commutative and is a semi ring with zero if + has an identity 0 such that

\[
0 \times a = a \times 0 = 0
\]
1.2 Fuzzy Set Theory

The following results and definitions are from [54].

1.2.1 Definition

Let $X$ be a nonempty set and $I = [0, 1]$ with usual ordering. Then a fuzzy set $A$ in $X$ (i.e., a fuzzy subset $A$ of $X$) is characterised by a member of $I^X$, the family of all functions from $X$ to $I$. For each $x \in X$, $A(x)$ is called the membership value of $x$ in the fuzzy set $A$.

Obviously, a fuzzy set is a generalised subset of a crisp set.

*The collection of all fuzzy subsets of $X$ is denoted by $\mathcal{F}(X)$*

1.2.2 Algebra of fuzzy sets

Let $A_1$ and $A_2$ be two fuzzy sets in $X$. Then

(i) $A_1 \subseteq A_2$ iff $A_1(x) \leq A_2(x)$ for all $x \in X$. (inclusion)

(ii) $A_1 = A_2$ iff $A_1(x) = A_2(x)$ for all $x \in X$. (equality)

If $A_n$, $n$ belongs to any arbitrary index set are fuzzy sets, then

(iii) $\bigvee A_n$ is the fuzzy set $A$ where $A(x) = \bigvee n \{A_n(x)\} \forall x \in X$ and
(iv) \( \bigcap A_n \) is the fuzzy set \( A \) where \( A(x) = \bigwedge_n \{A_n(x)\} \); \( \forall x \in X \)

and if \( A \) is a fuzzy set, then

(v) the complement of \( A \) is the fuzzy set \( A^c \)

where \( A^c(x) = 1 - A(x) \), \( \forall x \in X \)

1.2.3 Note

Union \( (\cup) \), intersection \( (\cap) \), and complementation \( (^c) \) defined as above are called *standard fuzzy set theoretic operations*.

1.2.4 Note

\( \mathcal{F}(X) \) is a complete De Morgan lattice under the standard fuzzy set theoretic operations. But the complement operation on \( \mathcal{F}(X) \) is not a complement operation in the lattice sense and hence \( \mathcal{F}(X) \) is not a complete Boolean lattice under the standard fuzzy set operations.

1.2.5 Generalisation of fuzzy sets

A fuzzy set can be generalised in several ways.

If \( A(x) \) is allowed to take values as sub intervals of \([0, 1]\), then \( A \) is called an *interval valued fuzzy set*. i.e., \( A : X \rightarrow \mathcal{I}([0, 1]) \) where \( \mathcal{I}([0, 1]) \) is the family of all...
closed sub intervals of \([0, 1]\). If \(A\) is allowed to take values as fuzzy sets in the unit interval \([0, 1]\), it is called a fuzzy set of type 2.

i.e., \(A: X \rightarrow \mathcal{F}([0, 1])\)

An ordinary fuzzy set is a fuzzy set of type 1. If the value set \([0, 1]\) in the ordinary fuzzy set is replaced by a lattice \(L\), then it is called an \(L\)-fuzzy set.

i.e., \(A: X \rightarrow L\)

1.2.6 Definition

Given a fuzzy set \(A\) in \(X\) and any real number \(\alpha \in [0, 1]\),

(i) the \(\alpha\)-cut of \(A\), denoted by \(^\alpha A\) is the crisp set

\[^\alpha A = \{ x \in X: A(x) \geq \alpha \}\]

(ii) the strong \(\alpha\)-cut of \(A\), denoted by \(^*A\) is the crisp set

\(^*A = \{ x \in X: A(x) > \alpha \}\)

1.2.7 Definition

Let \(A\) be a fuzzy set in \(X\).

(i) The support of \(A\), \(\text{supp}(A) = \{ x \in X: A(x) > 0 \} = ^0 A\)
(ii) The 1-cut, \(^1A\) is called the core of \(A\)

(iii) The height of \(A\), \(h(A) = \sup_{x \in X} A(x)\)

\(A\) is normal iff \(h(A) = 1\)

When \(X\) is finite, \(A\) is normal iff \(\exists x \in X\) such that \(A(x) = 1\)

(iv) When \(X\) is finite, the scalar cardinality of \(A\), \(|A| = \sum_{x \in X} A(x)\)

1.2.8 Definition

A fuzzy set \(A\) on \(\mathbb{R}\) (the set of real numbers) is convex iff \(^\alpha A\) is convex for every \(\alpha \in (0, 1]\). An equivalent definition of convexity is:

\(A\) is a convex fuzzy set in \(\mathbb{R}\) iff \(A(\lambda r + (1 - \lambda) t) \geq \min\{A(r), A(t)\}\)

for all \(r, t \in \mathbb{R}\) and all \(\lambda \in [0, 1]\)

1.2.9 Result

Let \(A, B \in \mathcal{F}(X)\). Then the following properties hold for all \(\alpha, \beta \in [0, 1]\)

(i) \(^\alpha(A \cap B) = ^\alpha A \cap ^\beta B\)

\(^\alpha(A \cup B) = ^\alpha A \cup ^\beta B\)

(ii) \(A \subseteq B\) iff \(^\alpha A \subseteq ^\beta B\ \forall \alpha \in [0, 1]\)
(iii) \( A = B \) iff \( A = B \quad \forall \alpha \in [0, 1] \)

Every fuzzy set can fully and uniquely be represented by its \( \alpha \)-cuts. This is shown in the following result.

For every \( A \in \mathcal{F}(X) \),

\[
A = \bigcup_{\alpha \in [0,1]} \alpha A \quad \text{where} \quad \alpha A(x) = \alpha \cdot A(x) \quad \text{for all} \quad x \in X \quad \text{and} \quad \alpha \in [0, 1]
\]

1.2.10 Definition (Extension Principle)

Let \( f: X \to Y, A \in \mathcal{F}(X) \), and \( B \in \mathcal{F}(Y) \)

Then the extension \( f: \mathcal{F}(X) \to \mathcal{F}(Y) \) is defined as follows

\( f(A) \) is a fuzzy set in \( Y \) defined by

\[
f(A)(y) = \begin{cases} 
\bigvee_{x \in X} A(x) & \text{if } f^{-1}(y) \neq \phi \\
0 & \text{if } f^{-1}(y) = \phi
\end{cases}
\]

\( f^{-1}(B) \) is a fuzzy set in \( X \) defined by

\[
f^{-1}(B)(x) = B(f(x)) \quad ; \quad x \in X
\]

1.2.11 Generalisation of extension principle [21]

Let \( X = X_1 \times X_2 \times \ldots \times X_r \) be a cartesian product of the universes \( X_1, \ldots, X_r \) and let \( A_i \) be a fuzzy set in \( X_i \); \( i = 1, \ldots, r \). The cartesian product \( A = A_1 \times A_2 \times \ldots \times A_r \) is defined as the fuzzy set
A(x_1, ..., x_r) = \min\{A_1(x_1), A_2(x_2), ..., A_r(x_r)\}

for all (x_1, ..., x_r) ∈ X_1 × ... × X_r

Let f be a mapping from X_1 × X_2 × ... × X_r to a universe Y.

Then f(A) is a fuzzy set in Y defined by

\[
f(A)(y) = \begin{cases} \bigvee_{(x_1, ..., x_r): y = f(x_1, ..., x_r)} \min\{A_1(x_1), ..., A_r(x_r)\} & \text{if } f^i(y) \neq \phi \\ 0 & \text{if } f^i(y) = \phi \end{cases}
\]

1.2.12 Result [45]

Let f: \(\mathbb{R}^n \rightarrow \mathbb{R}\) be an arbitrary crisp function and A ∈ \(\mathcal{F}(\mathbb{R}^n)\). Then

\[\alpha^* f(A) = f(\alpha^* A)\]

and \(\alpha f(A) \supseteq f(\alpha A)\) for all \(\alpha \in (0, 1]\)

If f is continuous and \(\alpha A\) is closed and bounded for any \(\alpha \in (0, 1]\), then

\[\alpha f(A) = f(\alpha A) \text{ for all } \alpha \in (0, 1]\]

1.2.13 Definition (Fuzzy Numbers)

In this thesis, fuzzy numbers are defined as follows: [45]

A fuzzy subset A of the real line \(\mathbb{R}\) is said to be a fuzzy number if
1.2.14 Notation

The set of all fuzzy numbers of $\mathbb{R}$ is denoted by $\mathcal{FN}(\mathbb{R})$ and the set of all fuzzy numbers whose supports are contained in $[0, 1]$ is denoted by $\mathcal{FN}([0, 1])$.

1.2.15 Operations on closed intervals

Let $\mathcal{I}(\mathbb{R})$ denote the set of all closed bounded intervals of $\mathbb{R}$.

For $[a_1, a_2], [b_1, b_2] \in \mathcal{I}(\mathbb{R})$,

(i) $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$

(ii) $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$

(iii) $[a_1, a_2] \cdot [b_1, b_2] = [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)]$

(iv) $[a_1, a_2] / [b_1, b_2] = [a_1, a_2]. \left[\frac{1}{b_2}, \frac{1}{b_1}\right]$ provided $0 \not\in [b_1, b_2]$

(v) $[a_1, a_2] \wedge [b_1, b_2] = [\min(a_1, b_1), \min(a_2, b_2)]$

(vi) $[a_1, a_2] \vee [b_1, b_2] = [\max(a_1, b_1), \max(a_2, b_2)]$

(vii) $[a_1, a_2] \leq [b_1, b_2]$ iff $a_1 \leq b_1$ and $a_2 \leq b_2$
1.2.16 Definition

Let $A$ and $B$ be two fuzzy numbers of $\mathbb{R}$. Then

$$A \leq B \text{ iff } \alpha A \leq \alpha B \quad \forall \alpha \in (0, 1]$$

$$A = B \text{ iff } \alpha A = \alpha B \quad \forall \alpha \in (0, 1]$$

1.2.17 Theorem [21]

If $A$ and $B$ are continuous fuzzy numbers whose membership functions are onto and $\ast$ is a continuous increasing binary operation, then the extension $A \ast B$ is a continuous fuzzy number with membership function

$$(A \ast B)(z) = \sup_{z=x,y} \min\{A(x), B(y)\}; \quad z \in \mathbb{R}$$

When $\ast$ is a continuous decreasing binary operation, then also the same results hold. When $\ast$ is such that, on $\mathbb{R}$

if $x_1 > y_1$ and $x_2 < y_2$, then $x_1 \ast x_2 > y_1 \ast y_2$, then the operation $\bot$ defined by

$$x_1 \bot x_2 = x_1 \ast (-x_2)$$

is increasing on $\mathbb{R}$. Hence the theorem applies to $\bot$ and hence to $\ast$
1.2.18 t-norm, implication and negation functions [43]

A. Definition

A t-norm \( t : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is such that

\[ t_1 \quad t(a, 1) = t(1, a) = a \text{ for all } a \in [0, 1] \]

\[ t_2 \quad t(a, b) = t(b, a) \text{ for all } a, b \in [0, 1] \]

\[ t_3 \quad t(a, t(b, c)) = t(t(a, b), c) \text{ for all } a, b, c \in [0, 1] \]

\[ t_4 \quad a \leq c \text{ and } b \leq d \text{ imply } t(a, b) \leq t(c, d) \]

For any t-norm \( t \), \( t(a, 0) = 0 \) for all \( a \in [0, 1] \) and \( t(1, 1) = 1 \)

B. Definition

A fuzzy implication \( i : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is such that

\[ i_1 \quad a \leq b \text{ implies } i(a, c) \geq i(b, c) \text{ for all } c \in [0, 1] \]

\[ i_2 \quad b \leq c \text{ implies } i(a, b) \leq i(a, c) \text{ for all } a \in [0, 1] \]

\[ i_3 \quad i(0, a) = 1 \text{ for all } a \in [0, 1] \]

\[ i_4 \quad i(a, 1) = 1 \text{ for all } a \in [0, 1] \]
\[ i(1, a) = a \text{ for all } a \in [0, 1] \]

In particular \( i(0, 0) = i(0, 1) = i(1, 1) = 1 \)

and \( i(1, 0) = 0 \)

C. Definition

A negation \( n : [0, 1] \rightarrow [0, 1] \) is such that

\( n \) is nonincreasing and satisfies

\( n(0) = 1 \) and \( n(1) = 0 \)

\( n \) is called strict iff it is strictly decreasing and continuous and is called strong iff

it is strict and an involution.

i.e., \( n(n(a)) = a \) for all \( a \in [0, 1] \)

Remark: The definitions of t-norm, implication and negation functions can be extended to any bounded lattice.

1.3 Evidence Theory

The following results and definitions are from [54, 83]. Throughout we assume

that \( X \) is a nonempty set and \( \mathcal{P}(X) \) - the crisp powerset of \( X \).
1.3.1 Definition (Fuzzy measure)

A fuzzy measure on \((X, \mathcal{P}(X))\) is a function

\[
\mu : \mathcal{P}(X) \to [0, 1]
\]
satisfying

\[
\mu_1. \quad \mu(\emptyset) = 0 \text{ and } \mu(X) = 1 \quad (\text{boundary condition})
\]

\[
\mu_2. \quad \text{for all } A, B \in \mathcal{P}(X),
\]

\[
A \subseteq B \text{ implies } \mu(A) \leq \mu(B) \quad (\text{monotonic condition})
\]

\[
\mu_3. \quad \text{for any increasing sequence}
\]

\[
A_1 \subseteq A_2 \subseteq \ldots \text{ where } A_i \in \mathcal{P}(X)
\]

\[
\lim_{i \to \infty} \mu(A_i) = \mu(\bigcup_{i=1}^{\infty} A_i) \quad (\text{continuity from below})
\]

\[
\mu_4. \quad \text{for any decreasing sequence}
\]

\[
A_1 \supseteq A_2 \supseteq \ldots \text{ where } A_i \in \mathcal{P}(X)
\]

\[
\lim_{i \to \infty} \mu(A_i) = \mu(\bigcap_{i=1}^{\infty} A_i) \quad (\text{continuity from above})
\]

1.3.2 Remark

When \(X\) is finite, the conditions \(\mu_3\) and \(\mu_4\) can be disregarded.
1.3.3 Note

Functions which satisfy $\mu_1$ and $\mu_2$ and either $\mu_3$ or $\mu_4$ are called semicontinuous fuzzy measures. They are either continuous from below or continuous from above.

Evidence Theory is based on two semicontinuous fuzzy measures: belief measures and plausibility measures.

1.3.4 Definition

A set function

$$Bel : \mathcal{P}(X) \to [0, 1]$$

is called a belief measure if

$B_1$. $Bel(\emptyset) = 0$

$B_2$. $Bel(X) = 1$

$B_3$. $Bel(A_1 \cup ... \cup A_n) \geq \sum_j Bel(A_j) - \sum_{j \neq k} Bel(A_j \cap A_k) + ... + (-1)^{n-1} Bel(A_1 \cap ... \cap A_n)$

where $\{A_1, ..., A_n\}$ is any finite subclass of $\mathcal{P}(X)$

$B_4$. $Bel$ is continuous from above.

When $X$ is finite, $B_4$ is redundant.
1.3.5 Result

Any belief measure is super additive (due to $B$,) and monotone

Super additivity: \( Bel(A_1 \cup A_2) \geq Bel(A_1) + Bel(A_2) \)

whenever \( A_1, A_2 \in \mathcal{P}(X) \) and \( (A_1 \cap A_2) = \emptyset \)

1.3.6 Definition

A set function

\[ PL : \mathcal{P}(X) \rightarrow [0, 1] \]

is called a plausibility measure if

\[ P_1. \quad PL(\emptyset) = 0 \]
\[ P_2. \quad PL(X) = 1 \]
\[ P_3. \quad PL((A_1 \cap \ldots \cap A_n) \leq \sum_{i=1}^{n} PL(A_i) - \sum_{i<j} PL(A_i \cup A_j) + \ldots + (-1)^{n+1} PL(A_1 \cup \ldots \cup A_n) \]

where \( \{A_1, \ldots, A_n\} \) is any finite subclass of \( \mathcal{P}(X) \).

\[ P_4. \quad PL \text{ is continuous from below.} \]

When \( X \) is finite, \( P_4 \) is redundant.

1.3.7 Result

Any plausibility measure is sub additive (due to \( P_3 \)) and monotone.
Sub additivity:

\[ PL(A_1 \cup A_2) \leq PL(A_1) + PL(A_2) \text{ whenever } A_1, A_2 \in \mathcal{P}(X) \]

1.3.8 Definition (basic probability assignment)

A set function

\[ m : \mathcal{P}(X) \rightarrow [0, 1] \]

is a normal basic probability assignment (bpa) function if

1. \[ m(\emptyset) = 0 \]

2. \[ \sum_{A \in \mathcal{P}(X)} m(A) = 1 \]

If we relax the condition \( m(\emptyset) = 0 \), we get subnormal basic probability assignment functions. Unless otherwise specified, throughout this thesis, by a basic probability assignment, we mean a normal basic probability assignment.

1.3.9 Note

\( m(A) \) can be interpreted as the degree of belief that a specific element of \( X \) belongs exactly to the set \( A \).

\( A \in \mathcal{P}(X) \) such that \( m(A) > 0 \) is called focal element. A body of evidence is defined by the pair \((\mathcal{F}, m)\) where \( \mathcal{F} = \{ A \in \mathcal{P}(X) : m(A) > 0 \} \) is a countable class and \( m \) the associated basic probability assignment.
Total ignorance is expressed by

\[ m(X) = 1 \text{ and } m(A) = 0 \text{ for all } A \neq X \]

Full certainty is expressed by

\[ m(\{x\}) = 1 \text{ for some } x \in X \text{ and } m(A) = 0 \text{ for all } A \neq \{x\}. \]

1.3.10 Definition

If \( m \) is a basic probability assignment on \( \mathcal{P}(X) \), then the set function

\[ Bel_m : \mathcal{P}(X) \rightarrow [0, 1] \]

determined by

\[ Bel_m (A) = \sum_{B \subseteq A} m(B) ; A \in \mathcal{P}(X) \]

is called a belief function on \((X, \mathcal{P}(X))\).

1.3.11 Note

\( Bel_m \) is a belief measure and is called the belief measure induced by \( m \).

\( Bel_m (A) \) is the degree of belief based on available evidence that a given element of \( X \) belongs to the set \( A \).
1.3.12 Result [83]

Let $X$ be finite. If a set function

$$\mu : \mathcal{P}(X) \rightarrow [0, 1]$$

satisfies

(i) $\mu(\emptyset) = 0$

(ii) $\mu(X) = 1$

(iii) $\mu(A_1 \cup \ldots \cup A_n) \geq \sum_j \mu(A_j) - \sum_{j,k} \mu(A_j \cap A_k) + \ldots + (-1)^{n+1} \mu(A_1 \cap \ldots \cap A_n)$

where $\{A_1, \ldots, A_n\}$ is any finite subclass of $\mathcal{P}(X)$, then the set function $m$ determined by

$$m(A) = \sum_{B \subseteq A} (-1)^{|A - B|} \mu(B) \quad ; A \in \mathcal{P}(X)$$

is a basic probability assignment and $\mu$ is just the belief measure induced from $m$. i.e., $\mu(A) = Bel_m(A)$

1.3.13 Definition

If $m$ is a basic probability assignment on $\mathcal{P}(X)$, then the set function

$$Pl_m : \mathcal{P}(X) \rightarrow [0, 1]$$

defined by

$$Pl_m(A) = \sum_{A \cap B = \emptyset} m(B) \quad ; A \in \mathcal{P}(X)$$

is called a plausibility function on $(X, \mathcal{P}(X))$
1.3.14 Note

$Pl_m$ is a plausibility measure and is called the *plausibility measure induced by* $m$.

1.3.15 Result

If $Bel_m$ and $Pl_m$ are the belief and plausibility measures induced by $m$ on $\mathcal{P}(X)$ respectively, then

(i) $Pl_m(A) = 1 - Bel_m(A^c)$

(ii) $Bel_m(A) \leq Pl_m(A) \quad \forall A \in \mathcal{P}(X)$

1.3.16 Definition

A basic probability assignment is called *consonant* iff its focal elements are nested (i.e., fully ordered by the inclusion relation of sets). The belief and plausibility measures induced by a consonant basic probability assignment are respectively known as *consonant belief* and *consonant plausibility measures*. 
1.4 Possibility Theory

The following results and definitions are from [54, 83]

Throughout we assume that $X$ is a nonempty set and $\mathcal{P}(X)$ the power set of $X$.

*Possibility Theory* is based on two dual semicontinuous fuzzy measures, called

possibility and necessity measures.

1.4.1 Definition

A normal possibility measure is a function

$$\text{Pos} : \mathcal{P}(X) \rightarrow [0, 1]$$

which satisfies the following requirements.

(i) $\text{Pos}(\emptyset) = 0$

(ii) $\text{Pos}(X) = 1$

(iii) $\text{Pos} \left( \bigcup_{i \in I} A_i \right) = \bigvee_{i \in I} \text{Pos}(A_i)$

for any family $\{A_i / A_i \in \mathcal{P}(X), i \in I\}$ where $I$ is an arbitrary index set.

If we relax the condition $\text{Pos}(X) = 1$ we get a subnormal possibility measure.

*Unless otherwise specified, throughout this thesis, possibility measures are normal possibility measures.*
1.4.2 Remark

Any possibility measure is a lower semicontinuous fuzzy measure.

1.4.3 Definition

A normal necessity measure \( 'Nec' \) is a function

\[ Nec : \mathcal{P}(X) \rightarrow [0, 1] \]

which satisfies the following requirements

(i) \( Nec(\emptyset) = 0 \)

(ii) \( Nec(X) = 1 \)

(iii) \( Nec\left( \bigcap \mathcal{A}_i \right) = \bigwedge_{i \in I} Nec(A_i) \)

for any family \( \{A_i / A_i \in \mathcal{P}(X), i \in I\} \) where \( I \) is an arbitrary index set.

If we relax the condition \( Nec(X) = 1 \), we get a subnormal necessity measure.

Unless otherwise specified, throughout this thesis, 'necessity measures are normal necessity measures.'

1.4.4 Remark

Any necessity measure is an upper semicontinuous fuzzy measure.
1.4.5 Remark

The above two formulations are dual in the sense that

\[ \text{Pos} (A) = 1 - \text{Nec} (A^c) \quad \text{for all } A \in \mathcal{P}(X) \]

1.4.6 Result

Possibility and necessity measures defined on \( \mathcal{P}(X) \) satisfy the following properties. For all \( A, B \in \mathcal{P}(X) \),

(i) \( \text{Pos} (A \cup B) = \max \{\text{Pos}(A), \text{Pos}(B)\} \)

(ii) \( \text{Nec} (A \cap B) = \min \{\text{Nec}(A), \text{Nec}(B)\} \)

(iii) \( \text{Pos} (A) < 1 \Rightarrow \text{Nec} (A) = 0 \)

(iv) \( \text{Nec} (A) > 0 \Rightarrow \text{Pos} (A) = 1 \)

(v) \( \text{Nec} (A) \leq \text{Pos} (A) \)

(vi) \( \text{Pos} (A) + \text{Pos} (A^c) \geq 1 \)

(vii) \( \text{Nec} (A) + \text{Nec} (A^c) \leq 1 \)

1.4.7 Definition

Any function \( r : X \rightarrow [0, 1] \) satisfying \( \bigvee_{x \in X} r(x) = 1 \)

is called a normal possibility distribution function.
If we relax the condition $\bigvee_{x \in X} r(x) = 1$, we get a subnormal possibility distribution function. *Unless otherwise specified, throughout this thesis, possibility distribution functions are normal possibility distribution functions.*

1.4.8 Result [83]

Any possibility distribution $r$ on $X$ uniquely characterises a possibility measure $Pos$, on $\mathcal{P}(X)$, via the formula

$$Pos(A) = \bigvee_{x \in A} r(x) \quad ; A \in \mathcal{P}(X)$$

To find the associated possibility distribution $r$ from the known possibility measure $Pos$, one can use the formula

$$r(x) = Pos(\{x\}), \quad x \in X$$

1.4.9 Theorem [83]

Let $X$ be a finite set. Then any possibility measure is a plausibility measure induced by a consonant basic probability assignment. Conversely, any plausibility measure induced by a consonant basic probability assignment is a possibility measure.

As a consequence the following result is obtained.

1.4.10 Corollary

If $X$ is finite, then any necessity measure is a belief measure induced by a consonant basic probability assignment.
1.4.11 Definition (Joint and Marginal possibility distribution functions)

The joint possibility distribution of the variables \( X \) and \( Y \) defined on the cartesian product \( X \times Y \) is denoted by \( r(x, y) \) and it represents the possibility that \( X = x \) and \( Y = y \) simultaneously.

If \( r \) is a joint possibility distribution on \( X \times Y \), then the marginal possibility distributions \( r_x \) and \( r_y \) are determined by the formulas

\[
\begin{align*}
  r_x(x) &= \sup_{y \in Y} r(x, y) \quad ; \ x \in X \\
  r_y(y) &= \sup_{x \in X} r(x, y) \quad ; \ y \in Y
\end{align*}
\]

1.4.12 Definition

The marginal distributions \( r_x \) and \( r_y \) of \( r \) are said to be noninteractive (in the possibilistic sense) if

\[
r(x, y) = \min \{ r_x(x), r_y(y) \} \quad \forall \ (x, y) \in X \times Y
\]

In this case \( r \) is said to be separable.

1.4.13 Note

The variables \( X \) and \( Y \) are said to be noninteractive, when all the joint possibility distributions under consideration on \( X \times Y \) are separable.
1.4.14 Measures of nonspecificity

Fuzzy measures provide a general framework for dealing with uncertainty due to ambiguity. Three types of ambiguity are recognizable in this framework. One of them is associated with the nonspecificity of the fuzzy measure, the other two results from the dissonance and confusion in characterising evidence.

The possibilistic measure of nonspecificity proposed by Higashi and Klir [37] which has become known under the name 'U-uncertainty' is defined by the function

\[ U(r) = \int_0^1 \log_2 |c(r, \alpha)| \, d\alpha \]

where \( r = (r(x) : x \in X) \) denotes a normalised possibility distribution and

\[ c(r, \alpha) = \{ x : r(x) \geq \alpha \} \]

Assuming \( |X| = n \) and if the possibility distribution \( r \) is such that

\[ r = (r_1, ..., r_n), r_i \geq r_{i+1} \text{ for all } i \in \mathbb{N}_n \text{ and } r_{n+1} = 0 \text{ by convention,} \]

the U-uncertainty of \( r \) can also be written in either of the following forms:

\[
U(r) = \sum_{i=2}^{n} r_i \left( \log_2 i - \log_2 (i - 1) \right)
\]

\[
= \sum_{i=1}^{n} (r_i - r_{i+1}) \log_2 i
\]
1.4.15 Sub additivity and Additivity properties of U

*Sub additivity:* When \( r_x \) and \( r_y \) are marginal possibility distributions which are calculated from the joint possibility distribution \( r \) on \( X \times Y \), then

\[
U(r) \leq U(r_x) + U(r_y)
\]

*Additivity:* When \( r_x \) and \( r_y \) are noninteractive i.e., if \( r(x, y) = \min \{r_x(x), r_y(y)\} \) for all \((x, y) \in X \times Y\), then

\[
U(r) = U(r_x) + U(r_y)
\]

1.4.16 Note

The generalisation of U-uncertainty to belief and plausibility measures is shown by Dubois and Prade [23]. If \( m \) is an arbitrary basic probability assignment on \( \mathcal{P}(X) \), the generalised U-uncertainty function \( N \) has the form

\[
N(m) = \sum_{A \in X} m(A) \log_2 |A|
\]